On the homogenization of some double porosity models with periodic thin structures

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Abstract

Models describing global behavior of incompressible flow in fractured media are discussed. A fractured medium is regarded as a porous medium consisting of two superimposed continua, a continuous fracture system and a discontinuous system of medium-sized matrix blocks. We derive global behavior of fractured media versus different parameters such as the fracture thickness, the size of blocks and the ratio of the block permeability and the permeability of fissures, and oscillating source terms. The homogenization results are obtained by mean of the convergence in domains of asymptotically degenerating measure.

1 Setting of the problem

1.1 The geometry of the periodic medium

- $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ – a bounded connected domain with a periodic structure;
- $Y =]0, 1[^d$ – the reference cell of a fractured porous medium;
- we assume that Y is made up of two homogeneous porous media M^δ and F^δ corresponding to parties of the domain occupied by the matrix block and the fracture, respectively;
- we assume that M^δ is an open cube centered at the same point as Y with length equal to $(1 - \delta)$, where $0 < \delta < 1$;

Thus $Y = M^\delta \cup \Gamma^\delta_{m,f} \cup F^\delta$, where $\Gamma^\delta_{m,f}$ denotes the interface between the two media (see Figure 1). Then $\Omega = \Omega^{\varepsilon,\delta}_m \cup \Gamma^{\varepsilon,\delta}_{m,f} \cup \Omega^{\varepsilon,\delta}_f$, where $\Gamma^{\varepsilon,\delta}_{m,f} = \partial\Omega^{\varepsilon,\delta}_m \cap \partial\Omega^{\varepsilon,\delta}_f$ and the subscripts m and f refer to the matrix and fracture, respectively (see Figure 2). For the sake of simplicity, we will assume that $\partial\Omega \cap \Omega^{\varepsilon,\delta}_m = \emptyset$.

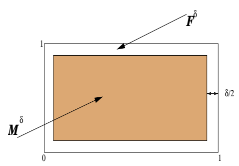

Figure 1: The reference cell Y.

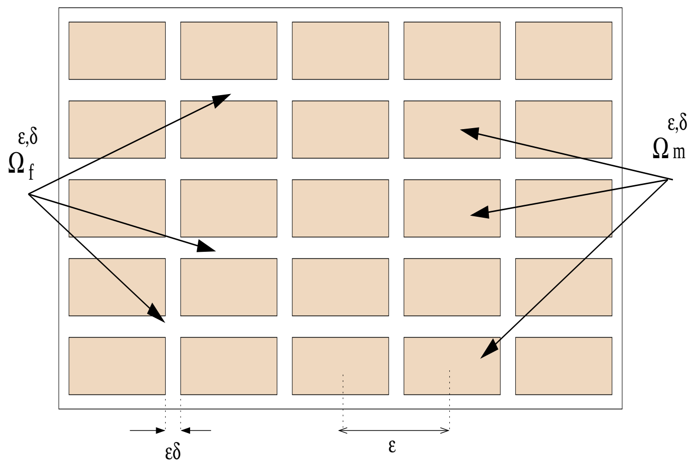

Figure 2: The periodic domain Ω.

1.2 Permeability and porosity of the porous medium

Now let us introduce the permeability coefficient and the porosity of the porous medium Ω. We set

$$K^{\varepsilon,\delta}(x) = k_m \, \mathbf{1}_m^{\varepsilon,\delta}(x) + k_f \, \mathbf{1}_f^{\varepsilon,\delta}(x) \quad \text{and} \quad \omega^{\varepsilon,\delta}(x) = \omega_m \, \mathbf{1}_m^{\varepsilon,\delta}(x) + \omega_f \, \mathbf{1}_f^{\varepsilon,\delta}(x), \qquad (1.1)$$

where k_f is the permeability or the hydraulic conductivity of fissures, k_m is the permeability or the hydraulic conductivity of blocks, ω_f is the porosity of fissures, ω_m is the porosity of blocks; $\mathbf{1}_f^{\varepsilon,\delta} = \mathbf{1}_f^{\varepsilon,\delta}(x)$ and $\mathbf{1}_m^{\varepsilon,\delta} = \mathbf{1}_m^{\varepsilon,\delta}(x)$ denote the (periodic) characteristic functions of the sets $\Omega_f^{\varepsilon,\delta}$ and $\Omega_m^{\varepsilon,\delta}$, respectively. Here $0 < k_f, k_m, \omega_f, \omega_m < +\infty$.

1.3 Assumptions

We make the following assumptions on permeabilities of fissures and blocks as well as on the source term.

(H.1) Porosities ω_f, ω_m of fissures and blocks are independent of ε, δ.

(H.2) The permeability of blocks is related to the permeability of fissures by $\mathbf{r}$, the permeability ratio:

$$k_m = \mathbf{r}\, k_f. \tag{1.2}$$

$\mathbf{r}$ is supposed to be small and then defined in the following way:

$$\mathbf{r} = (\varepsilon\delta)^\theta, \tag{1.3}$$

where $\theta > 0$ is a parameter.

(H.3) The source term is given by

$$f^{\varepsilon,\delta}(x) = (f_0 + f_m)(x)\mathbf{1}_f^{\varepsilon,\delta}(x) + f_m(x)\mathbf{1}_m^{\varepsilon,\delta}(x), \tag{1.4}$$

where $f_0 \in L^2(\Omega)$ and $f_m \in C^1(\overline{\Omega})$.

1.4 A model with two small parameters

We consider the following parabolic problem for the function $u^{\varepsilon,\delta} : Q \to \mathbb{R}$:

$$(\mathcal{P}_{\varepsilon,\delta}) \quad \begin{cases} \omega^{\varepsilon,\delta}(x) u_t^{\varepsilon,\delta} - \operatorname{div}\left(K^{\varepsilon,\delta}(x) \nabla u^{\varepsilon,\delta}\right) = f^{\varepsilon,\delta}(x) \quad \text{in } Q; \\ \nabla u^{\varepsilon,\delta} \cdot \vec{\nu} = 0 \quad \text{on } S_Q; \\ u^{\varepsilon,\delta}(0,x) = 0 \quad \text{in } \Omega, \end{cases} \tag{1.5}$$

where $Q =]0,T[\times \Omega$, $\vec{\nu}$ is the outward normal vector to Ω, $S_Q =]0,T[\times \partial\Omega$, $T > 0$ is given.

1.5 A model with one small parameter

The reference cell Y is represented as follows: $Y = M^\varepsilon \cup G^\varepsilon_{m,f} \cup F^\varepsilon$, where $G^\varepsilon_{m,f}$ denotes the interface between the two media and M^ε is the cube with length equal $(1 - \ell\varepsilon^{\frac{\alpha}{2}})$.

The porous medium Ω in this case is defined as follows: $\Omega = \Omega^\varepsilon_m \cup \Gamma^\varepsilon_{m,f} \cup \Omega^\varepsilon_f$, where $\Gamma^\varepsilon_{m,f} = \partial\Omega^\varepsilon_m \cap \partial\Omega^\varepsilon_f$. For simplicity, we will assume that $\partial\Omega \cap \Omega^\varepsilon_m = \emptyset$.

The permeability and porosity are given by:

$$K^\varepsilon(x) = k_f \mathbf{r}(\varepsilon)\, \mathbf{1}^\varepsilon_m(x) + k_f\, \mathbf{1}^\varepsilon_f(x) \quad \text{and} \quad \omega^\varepsilon(x) = \omega_m\, \mathbf{1}^\varepsilon_m(x) + \omega_f\, \mathbf{1}^\varepsilon_f(x), \tag{1.6}$$

where $\mathbf{1}^\varepsilon_f = \mathbf{1}^\varepsilon_f(x)$ and $\mathbf{1}^\varepsilon_m = \mathbf{1}^\varepsilon_m(x)$ denote characteristic periodic functions of the fissure set Ω^ε_f and the matrix system Ω^ε_m, respectively.

(H.4) *The source term is given by*

$$f^\varepsilon(x) = (f_0 + f_m)(x)\mathbf{1}^\varepsilon_f(x) + f_m(x)\mathbf{1}^\varepsilon_m(x) \quad \text{with } f_0 \in L^2(\Omega), f_m \in C^1(\overline{\Omega}). \tag{1.7}$$

$$(\mathcal{P}_\varepsilon) \quad \begin{cases} \omega^\varepsilon(x)u^\varepsilon_t - \operatorname{div}\left(K^\varepsilon(x)\nabla u^\varepsilon\right) = f^\varepsilon(x) \quad \text{in } Q; \\ \nabla u^\varepsilon(t,x)\cdot\vec{\nu} = 0 \quad \text{on } S_Q; \\ u^\varepsilon(0,x) = 0 \quad \text{in } \Omega. \end{cases} \tag{1.8}$$

1.6 The concepts of convergence

From now on $|\mathcal{O}|$ denotes the measure of the set $\mathcal{O}$.

Definition 1.1 *Let* $\Omega = \Omega_m^{\varepsilon,\delta} \cup \Gamma_{m,f}^{\varepsilon,\delta} \cup \Omega_f^{\varepsilon,\delta}$ *with* $\lim_{\delta\to 0}\lim_{\varepsilon\to 0}|\Omega_f^{\varepsilon,\delta}| = 0$. *A sequence* $\{u^{\varepsilon,\delta}\} \subset L^2(\Omega_f^{\varepsilon,\delta})$ *is said to* $L_{\varepsilon,\delta}$*–converge to a function* $u \in L^2(\Omega)$ *if*

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{|\Omega_f^{\varepsilon,\delta}|}\|u^{\varepsilon,\delta} - u\|^2_{L^2(\Omega_f^{\varepsilon,\delta})} = 0.$$

Definition 1.2 *Let* $\Omega = \Omega_m^{\varepsilon} \cup \Gamma_{m,f}^{\varepsilon} \cup \Omega_f^{\varepsilon}$ *with* $\lim_{\varepsilon\to 0}|\Omega_f^{\varepsilon}| = 0$. *A sequence* $\{u^{\varepsilon}\} \subset L^2(\Omega_f^{\varepsilon})$ *is said to* L_{ε}*–converge to a function* $u \in L^2(\Omega)$ *if*

$$\lim_{\varepsilon\to 0}\frac{1}{|\Omega_f^{\varepsilon}|}\|u^{\varepsilon} - u\|^2_{L^2(\Omega_f^{\varepsilon})} = 0.$$

2 A double porosity model with thin fissures: $(\mathcal{P}_{\varepsilon,\delta})$ model

In this Section we assume that

$$\delta \to 0 \tag{2.1}$$

and study the asymptotic behavior of the solution of problem (1.5) first as $\varepsilon \to 0$ and then as $\delta \to 0$.

The measure of F^δ, $|F^\delta|$, for δ sufficiently small, is calculated as follows

$$|F^\delta| = \begin{cases} 2\,\delta + O(\delta^2) & \text{if } d = 2; \\ 3\,\delta + O(\delta^2) & \text{if } d = 3; \end{cases} \tag{2.2}$$

then the assumption (2.1) implies that

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} |\Omega_f^{\varepsilon,\delta}| = 0. \tag{2.3}$$

Notation:

$$u^{\varepsilon,\delta} = \begin{cases} \rho^{\varepsilon,\delta} & \text{in } \Omega_f^{\varepsilon,\delta}; \\ \sigma^{\varepsilon,\delta} & \text{in } \Omega_m^{\varepsilon,\delta}; \end{cases}$$

2.1 Effective equations for $(\mathcal{P}_{\varepsilon,\delta})$ when $\theta = 2$

We study the asymptotic behavior of $u^{\varepsilon,\delta}$ as $\varepsilon, \delta \longrightarrow 0$. We will show that, for any fixed δ, problem (1.5) admits (as $\varepsilon \longrightarrow 0$) a homogenization problem and that the homogenized solution converges, as $\delta \longrightarrow 0$, to the solution of the following effective problem:

$$\begin{cases} \omega_f \rho^*_t - K_f(d)\,\Delta\rho^* = (f_0 + f_m)(x) + \mathsf{S}(\rho^*) & \text{in } Q; \\ \nabla\rho^* \cdot \vec{\nu} = 0 & \text{on } S_Q; \\ \rho^*(0, x) = 0 & \text{in } \Omega, \end{cases} \tag{2.4}$$

where

$$K_f(d) = \begin{cases} k_f/2 & \text{if } d = 2; \\ 2k_f/3 & \text{if } d = 3; \end{cases} \tag{2.5}$$

and the additional source term $\mathsf{S}(\rho^*)$ is given by:

$$\mathsf{S}(\rho^*) = -\frac{2\sqrt{k_f \omega_m}}{\sqrt{\pi}} \int_0^t \frac{\rho^*_t(x,\tau)}{\sqrt{t-\tau}} d\tau + 4 f_m(x) \sqrt{\frac{t\, k_f}{\pi \omega_m}}. \tag{2.6}$$

The following convergence result is valid.

Theorem 2.1 *Let $u^{\varepsilon,\delta} = \langle \rho^{\varepsilon,\delta}, \sigma^{\varepsilon,\delta} \rangle$ be the solution of (1.5) and let $\theta = 2$ in (1.3). Then, under assumptions* (H.1)–(H.3)*, for any $t \in]0, T[$,*

(I) *the function $\sigma^{\varepsilon,\delta}$, as well as the function $u^{\varepsilon,\delta}$, converges to $(t f_m(x))$, namely:*

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \left\|\omega_m \sigma^{\varepsilon,\delta}(t) - t\, f_m\right\|^2_{L^2(\Omega_m^{\varepsilon,\delta})} = \lim_{\delta\to 0}\lim_{\varepsilon\to 0} \left\|\omega^{\varepsilon,\delta} u^{\varepsilon,\delta}(t) - t\, f_m\right\|^2_{L^2(\Omega)} = 0; \tag{2.7}$$

(II) *the function $\rho^{\varepsilon,\delta}$ $L_{\varepsilon,\delta}$–converges to ρ^* the solution of a global model (2.4) with the additional source term (2.6) and the fracture porosity as effective porosity.*

2.2 Effective equations for $(\mathcal{P}_{\varepsilon,\delta})$ when $\theta > 2$

In this case the homogenized problem has the form:

$$\begin{cases} \omega_f \rho_t^* - K_f(d)\Delta\rho^* = (f_0 + f_m)(x) \quad \text{in } Q; \\ \nabla\rho^* \cdot \vec{\nu} = 0 \quad \text{on } S_Q; \\ \rho^*(0,x) = 0 \quad \text{in } \Omega, \end{cases} \tag{2.8}$$

where the coefficient $K_f(d)$ is defined in (2.5).

The following convergence result is valid.

Theorem 2.2 *Let $u^{\varepsilon,\delta} = \langle \rho^{\varepsilon,\delta}, \sigma^{\varepsilon,\delta} \rangle$ be the solution of (1.5) and let $\theta > 2$ in (1.3). Then, under assumptions* (H.1)–(H.3)*, for any $t \in]0,T[$,*

(I) *the function $\sigma^{\varepsilon,\delta}$, as well as the function $u^{\varepsilon,\delta}$, converges to $(tf_m(x))$, namely:*

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \left\|\omega_m \sigma^{\varepsilon,\delta}(t) - t\, f_m\right\|^2_{L^2(\Omega_m^{\varepsilon,\delta})} = \lim_{\delta\to 0}\lim_{\varepsilon\to 0} \left\|\omega^{\varepsilon,\delta} u^{\varepsilon,\delta}(t) - t\, f_m\right\|^2_{L^2(\Omega)} = 0; \tag{2.9}$$

(II) *the function $\rho^{\varepsilon,\delta}$ $L_{\varepsilon,\delta}$–converges to ρ^* the solution of the effective model (2.8).*

2.3 Effective equations for $(\mathcal{P}_{\varepsilon,\delta})$ when $0 < \theta < 2$

In this case the following convergence result is valid.

Theorem 2.3 *Let* $u^{\varepsilon,\delta} = \langle \rho^{\varepsilon,\delta}, \sigma^{\varepsilon,\delta} \rangle$ *be the solution of (1.5) and let* $\theta < 2$ *in (1.3). Then, under assumptions* (H.1)–(H.3)*, for any* $t \in]0, T[$,

(I) *the function* $\sigma^{\varepsilon,\delta}$*, as well as the function* $u^{\varepsilon,\delta}$*, converges to* $(t f_m(x))$*, namely:*

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \left\|\omega_m \sigma^{\varepsilon,\delta}(t) - t\, f_m\right\|^2_{L^2(\Omega_m^{\varepsilon,\delta})} = \lim_{\delta\to 0}\lim_{\varepsilon\to 0} \left\|\omega^{\varepsilon,\delta} u^{\varepsilon,\delta}(t) - t\, f_m\right\|^2_{L^2(\Omega)} = 0; \quad (2.10)$$

(II) *the function* $\rho^{\varepsilon,\delta}$ $L_{\varepsilon,\delta}$*–converges to* $(t\, \omega_m^{-1} f_m(x))$.

3 A double porosity model with thin fissures: $(\mathcal{P}_\varepsilon)$ model

In the previous Section, for the modeling of thin fissures two small parameters ε, δ were used. It was supposed that δ was a small parameter such that $0 < \varepsilon \ll \delta \ll 1$ and independent of ε. In this Section we relate the thickness of the fractured part δ to the relative size of a bloc ε as follows:

(H.5) *Fissures are thin with respect to the periodicity of blocks. Namely, δ is given by*

$$\delta = \delta(\varepsilon) = \ell\, \varepsilon^{\alpha/2-1}, \tag{3.1}$$

and the thickness of fissures is given by

$$\ell_\varepsilon = \varepsilon\delta(\varepsilon) = \ell\, \varepsilon^{\alpha/2}, \tag{3.2}$$

where $\alpha > 2$ and $\ell > 0$ is a constant.

(H.6) *The coefficient* $\mathbf{r}$ *in (1.2) is given by*

$$\mathbf{r} = \varepsilon^{\beta}, \tag{3.3}$$

where β is a positive parameter.

It is clear that the condition (3.1) imply that

$$\lim_{\varepsilon\to 0} |\Omega_f^{\varepsilon,\delta(\varepsilon)}| := \lim_{\varepsilon\to 0} |\Omega_f^{\varepsilon}| = 0. \tag{3.4}$$

3.1 Effective equations for $(\mathcal{P}_\varepsilon)$ model when $\alpha = \beta$

The homogenized problem is of the form:

$$\begin{cases} \omega_f \rho^*_t - K_f(d)\Delta\rho^* = (f_0 + f_m)(x) + \mathsf{S}(\rho^*) \quad \text{in } Q; \\ \nabla\rho^* \cdot \vec{\nu} = 0 \quad \text{on } S_Q; \\ \rho^*(0,x) = 0 \quad \text{in } \Omega. \end{cases} \tag{3.5}$$

where the coefficient $K_f(d)$ is defined in (2.5) and

$$\mathsf{S}(\rho^*) = -\frac{2\sqrt{k_f\omega_m}}{\sqrt{\pi}\ell}\int_0^t \frac{\rho^*_t(x,\tau)}{\sqrt{t-\tau}}\,d\tau + 4 f_m(x)\frac{1}{\ell}\sqrt{\frac{t\,k_f}{\pi\omega_m}}. \tag{3.6}$$

The following convergence result is valid.

Theorem 3.1 *Let $u^\varepsilon = \langle\rho^\varepsilon, \sigma^\varepsilon\rangle$ be the solution of (1.8) and let $\beta = \alpha$ in (3.3), where α, β are parameters defined in (3.1)–(3.3). Then, under assumptions* (H.1), (H.2), (H.4)–(H.6), *for any $t \in]0, T[$,*

(I) *the function σ^ε, as well as the function u^ε, converges to $(t f_m(x))$, namely:*

$$\lim_{\varepsilon\to 0}\|\omega_m\sigma^\varepsilon(t) - t\,f_m\|^2_{L^2(\Omega^\varepsilon_m)} = \lim_{\varepsilon\to 0}\|\omega^\varepsilon u^\varepsilon(t) - t\,f_m\|^2_{L^2(\Omega)} = 0; \tag{3.7}$$

(II) *the function ρ^ε L_ε–converges to ρ^* the solution of the global model (3.5) with the additional source term (3.6) and the fracture porosity as effective porosity.*

3.2 Effective equations for $(\mathcal{P}_\varepsilon)$ model when $2 < \alpha < \beta$

In this case the homogenized problem has the following form:

$$\begin{cases} \omega_f \rho_t^* - K_f(d)\Delta\rho^* = (f_0 + f_m)(x) \quad \text{in } Q; \\ \nabla\rho^* \cdot \vec{\nu} = 0 \quad \text{on } S_Q; \\ \rho^*(0,x) = 0 \quad \text{in } \Omega. \end{cases} \tag{3.8}$$

The following convergence result is valid.

Theorem 3.2 *Let $u^\varepsilon = \langle \rho^\varepsilon, \sigma^\varepsilon \rangle$ be the solution of (1.8) and let $\beta > \alpha$ in (3.3). Then, under assumptions* (H.1), (H.2), (H.4)–(H.6)*, for any $t \in]0,T[$,*

(I) *the function σ^ε, as well as the function u^ε, converges to $(t f_m(x))$, namely:*

$$\lim_{\varepsilon\to 0} \|\omega^\varepsilon \sigma^\varepsilon - t\, f_m\|^2_{L^2(\Omega^\varepsilon_m)} = \lim_{\varepsilon\to 0} \|\omega^\varepsilon u^\varepsilon - t\, f_m\|^2_{L^2(\Omega)} = 0; \tag{3.9}$$

(II) *the function ρ^ε L_ε–converges to ρ^* the solution of a single porosity model (3.8) with the fracture porosity as an effective porosity.*

3.3 Effective equations for ($\mathcal{P}_\varepsilon$) model when $0 < \beta < \alpha$

In this case the following convergence result is valid.

Theorem 3.3 *Let* $u^\varepsilon = \langle \rho^\varepsilon, \sigma^\varepsilon \rangle$ *be the solution of (1.8) and let* $\beta < \alpha$ *in (3.3). Then, under assumptions* (H.1), (H.2), (H.4)–(H.6), *for any* $t \in]0, T[$,

(I) *the function* σ^ε*, as well as the function* u^ε*, converges to* $(t f_m(x))$*, namely:*

$$\lim_{\varepsilon \to 0} \|\omega_m \sigma^\varepsilon(t) - t\, f_m\|^2_{L^2(\Omega^\varepsilon_m)} = \lim_{\varepsilon \to 0} \|\omega^\varepsilon u^\varepsilon(t) - t\, f_m\|^2_{L^2(\Omega)} = 0; \tag{3.10}$$

(II) *the function* ρ^ε L_ε*–converges to* $(\omega_m^{-1} t f_m(x))$.

4 Homogenization of a single phase flow through a porous medium in a thin layer

Let Ω^ε be a rectangle in $\mathbf{R}^2$,

$$\Omega^\varepsilon = \Big(-\frac{\varepsilon}{2}, +\frac{\varepsilon}{2}\Big) \times (0, L).$$

We introduce a periodic structure in Ω^ε as follows. Denote by $\mathcal{Y}$ the reference cell

$$\mathcal{Y} = \Big(-\frac{1}{2}, +\frac{1}{2}\Big) \times (0, 1)$$

and by $\mathcal{F}^\delta$ the reference fracture part $\mathcal{F}^\delta = \big\{y \in \mathcal{Y}, \quad \operatorname{dist}(y, \partial\mathcal{Y}) < \frac{\delta}{2}\big\}$. The reference matrix bloc is then defined by $\mathcal{M}^\delta = \mathcal{Y} \setminus \overline{\mathcal{F}^\delta}$.

Assuming that L is an integer multiplier of ε: $L = N\varepsilon$, $N \in \mathbb{N}$, we define

$$\Omega_f^{\varepsilon,\delta} = \bigcup_{j=0}^{N-1} \varepsilon\Big(\mathcal{F}^\delta + (0, j)\Big), \qquad \Omega_m^{\varepsilon,\delta} = \bigcup_{j=0}^{N-1} \varepsilon\Big(\mathcal{M}^\delta + (0, j)\Big).$$

The flow in the matrix–fracture medium Ω^ε is described by:

$$\begin{cases} \omega^{\varepsilon,\delta}(x) u_t^{\varepsilon,\delta} - \operatorname{div}\big(K^{\varepsilon,\delta}(x)\nabla u^{\varepsilon,\delta}\big) = f^{\varepsilon,\delta}(x) \quad \text{in } (0,T)\times\Omega^\varepsilon; \\ \nabla u^{\varepsilon,\delta}\cdot\vec{\nu} = 0 \quad \text{on } (0,T)\times\partial\Omega^\varepsilon; \\ u^{\varepsilon,\delta}(0,x) = 0 \quad \text{in } \Omega^\varepsilon, \end{cases} \tag{4.1}$$

where

$$K^{\varepsilon,\delta}(x) = k_m\,(\varepsilon\delta)^2 \mathbf{1}_m^{\varepsilon,\delta}(x) + k_f\,\mathbf{1}_f^{\varepsilon,\delta}(x) \quad \text{and} \quad \omega^{\varepsilon,\delta}(x) = \omega_m\,\mathbf{1}_m^{\varepsilon,\delta}(x) + \omega_f\,\mathbf{1}_f^{\varepsilon,\delta}(x)$$

$$f^{\varepsilon,\delta}(x) = (f_0 + f_m)(x)\mathbf{1}_f^{\varepsilon,\delta}(x) + f_m(x)\mathbf{1}_m^{\varepsilon,\delta}(x)$$

Notation:

$$u^{\varepsilon,\delta} = \begin{cases} \rho^{\varepsilon,\delta} & \text{in } \Omega_f^{\varepsilon,\delta}; \\ \sigma^{\varepsilon,\delta} & \text{in } \Omega_m^{\varepsilon,\delta} \end{cases}$$

The goal of this section is to study the asymptotic behavior of $u^{\varepsilon,\delta}$ as $\varepsilon, \delta \to 0$.

We show that for any fixed δ problem (4.1) admits homogenization (as $\varepsilon \longrightarrow 0$) and the homogenized solution converges, as $\delta \to 0$, to a solution of :

$$\begin{cases} \omega_f \rho_t^* - \frac{1}{2} k_f \frac{\partial^2 \rho^*}{\partial \xi^2} = (f_0 + f_m)(0,\xi) + \mathsf{S}(\rho^*) \quad \text{in } (0,T) \times (0,L); \\ \frac{\partial \rho^*}{\partial \xi}(t,0) = \frac{\partial \rho^*}{\partial \xi}(t,L) = 0 \quad \text{on } (0,T); \\ \rho^*(0,\xi) = 0 \quad \text{in } (0,L) \end{cases} \tag{4.2}$$

with an the additional source term

$$\mathsf{S}(\rho^*) = -\frac{2\sqrt{k_m \omega_m}}{\sqrt{\pi}} \int_0^t \frac{\rho_t^*(\tau,\xi)}{\sqrt{t-\tau}} d\tau + 4 f_m(0,\xi) \sqrt{\frac{t\, k_m}{\pi \omega_m}}.$$

Theorem 4.1 *Let $u^{\varepsilon,\delta} = < \rho^{\varepsilon,\delta}, \sigma^{\varepsilon,\delta} >$ be the solution of (4.1). Then, for any $t \in (0,T)$,*

(I) *the function $\sigma^{\varepsilon,\delta}$, as well as the function $u^{\varepsilon,\delta}$, converges to $(t f_m(x))$, namely:*

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{|\Omega^\varepsilon|}\left\|\omega^{\varepsilon,\delta}\sigma^{\varepsilon,\delta} - t\, f_m\right\|^2_{L^2(\Omega_m^{\varepsilon,\delta})} =$$

$$= \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{|\Omega^\varepsilon|}\left\|\omega^{\varepsilon,\delta}u^{\varepsilon,\delta} - t\, f_m\right\|^2_{L^2(\Omega^\varepsilon)} = 0; \tag{4.3}$$

(II) *the function $\rho^{\varepsilon,\delta}$ satisfies the limit relation*

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{|\Omega_f^{\varepsilon,\delta}|}\left\|\rho^{\varepsilon,\delta} - \rho^*\right\|^2_{L^2(\Omega_f^{\varepsilon,\delta})} = 0, \tag{4.4}$$

where $\rho^* = \rho(t,\xi)$ is a solution of (4.2).

(III) *For any $t \in (0,T)$, and any function $\phi = \phi(x)$ continuous in the vicinity of the segment $\{x \in \mathbf{R}^2 :\ x_1 = 0;\ 0 \le x_2 \le L\}$, it holds*

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{L}{|\Omega_f^{\varepsilon,\delta}|}\int_{\Omega^\varepsilon} k^{\varepsilon,\delta}(x)\nabla u^{\varepsilon,\delta}\phi(x)\,dx = \frac{k_f}{2}\int_0^L \vec{\mathbf{R}}^*(t,\xi)\phi(0,\xi)\,d\xi \tag{4.5}$$

with

$$\vec{\mathbf{R}}^*(t,\xi) = \left(0, \frac{\partial\rho^*}{\partial\xi}(t,\xi)\right).$$

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Appendix

5 The double porosity model

We introduce the notation:

$$u^\varepsilon = \begin{cases} \rho^\varepsilon & \text{in } \Omega_f^\varepsilon; \\ \sigma^\varepsilon & \text{in } \Omega_m^\varepsilon. \end{cases} \tag{5.6}$$

5.1 Effective model for $\theta = 2$

The homogenized problem has the following form:

$$\begin{cases} \omega^* \rho_t^* - \operatorname{div}(K^* \nabla \rho^*) = \frac{|F|}{|Y|}(f_0 + f_m)(x) + \mathsf{S}(\rho^*, \varrho) & \text{in } Q; \\ K^* \nabla \rho^* \cdot \vec{\nu} = 0 \quad \text{on } S_Q; \\ \rho^*(0, x) = 0 \quad \text{in } \Omega. \end{cases} \tag{5.7}$$

Here the effective porosity ω^* is scaled by the volume fraction of fractures $|F|/|Y|$:

$$\omega^* = \frac{|F|}{|Y|}\omega_f; \tag{5.8}$$

$K^* = \{k_{ij}^*\}$ is the effective permeability tensor defined by:

$$k_{ij}^* = \frac{1}{|Y|} \int_F k_f(\vec{e}_i + \nabla_y w_i) \cdot (\vec{e}_j + \nabla_y w_j) dy \tag{5.9}$$

where $\{\vec{e}_1, .., \vec{e}_d\}$ is the standard basis of $\mathbb{R}^d$, and w_i is the unique solution in the space $H^1_\#(F) \setminus \mathbb{R}$ of

$$\begin{cases} -\Delta w_i = 0 \quad \text{in } F; \\ (\vec{e}_i + \nabla_y w_i) \cdot \vec{\nu} = 0 \quad \text{on } \partial M; \\ y \to w_i(y) \quad Y - \text{periodic}; \end{cases} \tag{5.10}$$

the additional source term is given by:

$$\mathsf{S}(\rho^*, \varrho) = -\omega_m \int_0^t \varrho_t(t-\tau)\rho_t^*(\tau, x) d\tau + \int_0^t \varrho_t(t-\tau) f_m(x) d\tau \tag{5.11}$$

with

$$\varrho(t) = \int_M \widetilde{U}(t,y)\, dy, \text{ where } \begin{cases} \omega_m \widetilde{U}_t - k_f \delta^2 \Delta_y \widetilde{U} = 0 \quad \text{in } (0,T) \times M; \\ \widetilde{U}(t,y) = 1 \text{ on } (0,T) \times \partial M \text{ and } \widetilde{U}(0,y) = 0 \text{ in } M. \end{cases} \tag{5.12}$$

Theorem 5.1 *Let $u^\varepsilon = \langle \rho^\varepsilon, \sigma^\varepsilon \rangle$ be the solution of (1.5) and let $\theta = 2$ in (1.3). Then for any $t \in]0, T[$, under assumptions* (H.1)–(H.3)*, the function ρ^ε converges in $L^2(\Omega_f^\varepsilon)$ to ρ^* the solution of the global model (5.7)–(5.12) with the effective porosity ω^* defined as $(|F|/|Y|)\,\omega_f$, the effective permeability tensor K^* defined in (5.9)–(5.10), and the additional source term (5.11).*

5.2 Effective model for $\theta > 2$

The homogenized problem has the form:

$$\begin{cases} \omega^* \rho_t^* - \operatorname{div}(K^* \nabla \rho^*) = \frac{|F|}{|Y|}(f_0 + f_m)(x) & \text{in } Q; \\ K^* \nabla \rho^* \cdot \vec{\nu} = 0 & \text{on } S_Q; \\ \rho^*(0, x) = 0 & \text{in } \Omega, \end{cases} \tag{5.13}$$

where the effective porosity ω^* is scaled by the volume fraction of fractures, i.e., $\omega^* = |F|/|Y|\omega_f$ and $K^* = \{k_{ij}^*\}$ is the effective permeability tensor defined by (5.9)–(5.10).

The following convergence result is valid.

Theorem 5.2 *Let $u^\varepsilon = \langle \rho^\varepsilon, \sigma^\varepsilon \rangle$ be the solution of (1.5) and let $\theta > 2$ in (1.3). Then for any $t \in]0, T[$, under assumptions* (H.1)–(H.3)*, the function ρ^ε converges in $L^2(\Omega_f^\varepsilon)$ to ρ^* the solution of a single porosity model (5.13) with the effective porosity $\omega^* = |F|/|Y|\omega_f$ and the permeability tensor K^* given by (5.9)–(5.10).*

5.3 Effective model for $\theta < 2$

The homogenized problem has the following form:

$$\begin{cases} \omega^* \rho_t^* - \operatorname{div}(K^* \nabla \rho^*) = \frac{|F|}{|Y|} f_0(x) + f_m(x) \quad \text{in } Q; \\ K^* \nabla \rho^* \cdot \vec{\nu} = 0 \quad \text{on } S_Q; \\ \rho^*(0, x) = 0, \quad \text{in} \quad \Omega, \end{cases} \tag{5.14}$$

where the effective porosity ω^* is the arithmetic average given by

$$\omega^* = \frac{|F|}{|Y|}\omega_f + \frac{|M|}{|Y|}\omega_m = \frac{|F|}{|Y|}\omega_f + \left(1 - \frac{|F|}{|Y|}\right)\omega_m \tag{5.15}$$

and $K^* = \{k_{ij}^*\}$ is the effective permeability tensor defined by (5.9)–(5.10).

The following convergence result is valid.

Theorem 5.3 *Let $u^\varepsilon = \langle \rho^\varepsilon, \sigma^\varepsilon \rangle$ be the solution of (1.5) and let $\theta < 2$ in (1.3). Then for any $t \in]0, T[$, under assumptions* (H.1)–(H.3), *the function ρ^ε converges in $L^2(\Omega_f^\varepsilon)$ to ρ^* the solution of a single porosity model (5.13) with the effective porosity ω^* given by (5.15) and the permeability tensor K^* given by (5.9)–(5.10).*