Approximation by $\mathbb{P}1$ finite elements of second order linear elliptic equations in divergence form with right hand side in $L^1(\Omega)$

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The problem

Data: $\Omega \subset \mathbb{R}^N$ open, bounded

$A \in L^\infty(\Omega)^{N\times N}$, $A(x) \geqslant \alpha I$, $\alpha > 0$

$f \in L^1(\Omega)$

Find u such that: $\begin{cases} -\operatorname{div} A\,Du = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

Approximation by finite elements:

Find u_h such that: $u_h \in V_h$

$$\forall v_h \in V_h, \quad \int_\Omega A\,Du_h\,Dv_h = \int_\Omega f\,v_h$$

V_h = P1 finite elements

Existence of u_h: OK Convergence $u_h \to u$: ???

Convergence of u_n ?

- in what space?
- to what?

When $f \in L^1(\Omega)$, what is the meaning of the solution to

$$\begin{cases} -\operatorname{div} A Du = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad ?$$

not a weak solution $u \in H^1_0(\Omega)$ ...

... but a renormalized solution (P.L. Lions - Murat)

or an entropy solution (Benilan-Boccardo-Gallouet ... Vazquez)

or a solution by transposition) (Stampacchia)

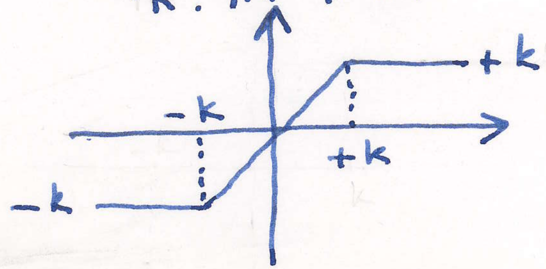
Definition of renormalized solution

When $f \in L^1(\Omega)$, u is a renormalized solution of

$$\begin{cases} -\operatorname{div} A D u = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

if:

$T_k : \mathbb{R} \to \mathbb{R}$



- $u \in L^1(\Omega)$
- $T_k(u) \in H^1_0(\Omega) \quad \forall k > 0$
- $\frac{1}{k} \int_\Omega |D T_k(u)|^2 \to 0$ if $k \to +\infty$
- $\forall k > 0$, $\forall S \in C^1_c(\mathbb{R})$ with $\operatorname{supp} S \subset [-k, +k]$
  $\forall v \in H^1_0(\Omega) \cap L^\infty(\Omega)$

$$\left\{ \int_\Omega A D T_k(u) D v \, S(u) + \int_\Omega A D T_k(u) D T_k(u) \, S'(u) \, v = \int_\Omega f \, S(u) \, v \right.$$

Theorem $\forall f \in L^1(\Omega)$

- there **exists** a renormalized solution u
- this solution is **unique**
- moreover $u \in W_0^{1,q}(\Omega) \quad \forall q < \frac{N}{N-1}$
- u **depends continuously** on f:

  if $f^\varepsilon \to f$ in $L^1(\Omega)$ strong, then

$$T_k(u^\varepsilon) \to T_k(u) \text{ in } H_0^1(\Omega) \text{ strong } \forall k > 0$$

$$\alpha \, \| T_k(u^\varepsilon - u) \|_{H_0^1(\Omega)} \leq k \, \| f^\varepsilon - f \|_{L^1(\Omega)} \quad \forall k \; \forall \varepsilon$$

$$\alpha \, \| u^\varepsilon - u \|_{W_0^{1,q}(\Omega)} \leq C(N, \Omega, q) \, \| f^\varepsilon - f \|_{L^1(\Omega)}$$

$$\forall q < \frac{N}{N-1} \quad \forall k \quad \forall \varepsilon$$

Our result: Theorem

- if $f \in L^1(\Omega)$
- if the triangulation is regular
- if V_h is the space of continuous $\mathbb{P}1$ finite elements
- if the discrete maximum principle holds

then the unique solution of $u_h \in V_h$

$$\forall v_h \in V_h \quad \int_\Omega A\, Du_h\, Dv_h = \int_\Omega f\, v_h$$

satisfies

$$u_h \to u \text{ in } W_0^{1,q}(\Omega) \text{ strong } \forall q < \frac{N}{N-1}$$

$$\Pi_h T_k(u_h) \to T_k(u) \text{ in } H_0^1(\Omega) \text{ strong } \forall k$$

where u is the renormalized solution

Regular triangulation

- $\mathcal{T}_h$ = finite number of closed triangles $(N=2)$, closed tetrahedra $(N=3)$, ...
- $\Omega_h =: \bigcup \{T : T \in \mathcal{T}_h\}$ $\qquad \Omega_h \subset \overline{\Omega}$
- $\forall K$ compact, $K \subset \Omega$, then K is absorbed by Ω_h
- $\forall T_1, T_2 \in \mathcal{T}_h$, then $\mathring{T}_1 \cap \mathring{T}_2 = \emptyset$
- conforming triangulation: every face of $T \in \mathcal{T}_h$ is either a face of $T' \in \mathcal{T}_h$ or a subset of $\partial\Omega_h$
- $h_T =:$ diameter of T, $\rho_T =:$ diameter of the inscribed ball
- $h =: \sup\{h_T : T \in \mathcal{T}_h\}$ $\quad h \to 0$
- $\exists\, \sigma$ such that $h_T \leq \sigma \rho_T \quad \forall T \in \mathcal{T}_h, \ \forall h$

$\mathbb{P}1$ finite elements

- $V_h = \{ v_h \in C^0(\bar{\Omega}),\quad v_h = 0 \text{ in } \Omega \setminus \mathring{\Omega}_h$
  $\qquad v_h|_T \in \mathbb{P}1(T) \quad \forall T \in \mathcal{T}_h \}$
- a_j vertices of $\mathcal{T}_h$ — interior vertices: $j \in I$
  boundary vertices: $j \in B$
- φ_i basis of V_h $\quad i \in I$

  $$\varphi_i(a_i) = 1, \quad \varphi_i(a_j) = 0 \quad \forall j \neq i, \; j \in I \cup B$$

  $$\forall v_h \in V_h, \quad v_h = \sum_{i \in I} v_h(a_i)\, \varphi_i$$
- Interpolation operator:

  $$\forall w \in C^0(\bar{\Omega}) \quad \Pi_h(w) = \sum_{i \in I} w(a_i)\, \varphi_i$$

Discrete maximum principle

Define Q_{ij} by $Q_{ij} = \int_\Omega A D\varphi_i D\varphi_j$

We will assume that

$$\forall i \in I \quad Q_{ii} - \sum_{\substack{j \in I \\ j \neq i}} |Q_{ij}| \geq 0$$

(almost equivalent to assume $Q_{ij} \leq 0 \quad \forall j \neq i$)

Then for every $f \in H^{-1}(\Omega)$, one has

$$\begin{cases} u_h \in V_h \\ \int_\Omega A Du_h Dv_h = \int_\Omega f v_h \end{cases}$$

satisfies $f \geq 0 \Rightarrow u_h \geq 0$

Examples of matrices and regular triangulations

for which the discrete maximum principle DMP holds

- $A = Id$ i.e. $-\operatorname{div} A(x) D = -\Delta$

  n_i the exterior normals to the faces

  DMP holds if $\forall T \in \mathcal{C}_h \quad n_i n_j \leq 0 \quad \forall i \neq j$

  i.e. every angle of every triangle is acute $(N=2)$

  every dihedral angle of every tetrahedron

  is acute $(N=3)$, ...

- Variant: $A(x) = a(x)\, Id \quad a(x) \in L^\infty(\Omega) \quad a(x) \geq \alpha > 0$

  DMP holds under the same condition

  (acute angles)

- $A \in \mathbb{R}^{N\times N}$ coercive (constant coefficients)

  making a change of variables transforms

  $-\operatorname{div} A D$ onto $-\Delta$

  Assume that the angles are acute after

  this change of variables $\Rightarrow$ DMP holds

- Variant: $A(x) = a(x)\, C$ $\quad a(x) \in L^\infty(\Omega)$, $a \geq \alpha > 0$

  $C \in \mathbb{R}^{N\times N}$ coercive

  DMP holds under the same condition

- $A(x) = a(x)\, C + E(x)$

  $a(x) \in L^\infty(\Omega)$, $a(x) \geq \alpha > 0$, $C \in \mathbb{R}^{N\times N}$ coercive

  $E(x) \in L^\infty(\Omega)^{N\times N}$ $\quad \|E\|_{L^\infty(\Omega)^{N\times N}}$ small

- $N=2$ if the matrix is $A(x)=\begin{pmatrix} a(x) & b(x) \\ b(x) & d(x) \end{pmatrix}$

we are able to build a regular triangulation for which DMP holds if

$$a(x) \geq \alpha, \quad d(x) \geq \alpha, \quad \alpha > 0, \quad |b(x)| \leq \inf(a(x), d(x))$$

while coerciveness of the matrix holds iff

$$\underline{\qquad}, \quad \underline{\qquad}, \quad \underline{\quad}, \quad |b(x)| \leq \sqrt{(a(x)-\alpha)(d(x)-\alpha)}$$

- In general for a given coercive matrix $A(x)$ we do not know whether it is possible to build a regular triangulation for which DMP holds

The proof: a priori estimate

$u_h \in V_h$, $\displaystyle\int_\Omega A\, Du_h\, Dv_h = \int_\Omega f\, v_h \quad \forall v_h \in V_h$

Take $v_h = u_h$? no! Take $v_h = T_k(u_h)$? no!

Take $v_h = \Pi_h T_k(u_h) \in V_h$

$$\int_\Omega A\, Du_h\, D\Pi_h T_k(u_h) = \int_\Omega f\, \Pi_h T_k(u_h)$$

$$\underbrace{\int_\Omega A\, D\Pi_h T_k(u_h)\, D\Pi_h T_k(u_h)}_{\text{coerciveness}\ \geq\ \alpha \int_\Omega |D\Pi_h T_k(u_h)|^2} + \int_\Omega A(Du_h - D\Pi_h T_k(u_h))\, D\Pi_h T_k(u_h) =$$

$$= \underbrace{\int_\Omega f\, \Pi_h T_k(u_h)}_{\leq\ \int_\Omega |f|\, k\ =\ k\, \|f\|_{L^1(\Omega)}}$$

?

Lemma If DMP holds, i.e. if

$$\forall i \qquad Q_{ii} - \sum_{j \neq i} |Q_{ij}| \geq 0$$

then

$$\int_\Omega A\left(Dv_h - D\Pi_h T_k(v_h)\right) D\Pi_h T_k(v_h) \geq 0 \qquad \forall k, \ \forall v_h \in V_h$$

(and almost conversely)

Compare with

$$\int_\Omega A\left(Dv - DT_k(v)\right) DT_k(v) = 0 \qquad \forall k, \ \forall v \in H^1(\Omega)$$

therefore a priori estimate

(1) $u_h \in V_h$ $\qquad \displaystyle\int_\Omega |D \Pi_h T_k(u_h)|^2 \leq k \, \frac{\|f\|_{L^1}}{\alpha}$

Lemma discrete version of Boccardo - Gallouet

If (1) holds, then

$$\|u_h\|_{W_0^{1,q}(\Omega)} \leq C(N, \Omega, q) \, \frac{\|f\|_{L^1}}{\alpha}$$

$$\forall q < \frac{N}{N-1}$$

Proof of $u_h \to u$ in $W_0^{1,q}(\Omega)$ strong (u renorm. sol.)

Let $f^\varepsilon \in L^2(\Omega)$ $\quad f^\varepsilon \to f$ in $L^1(\Omega)$ strong

Define u_h^ε by

$$u_h^\varepsilon \in V_h \qquad \int_\Omega A\, Du_h^\varepsilon\, Dv_h = \int_\Omega f^\varepsilon v_h \quad \forall v_h \in V_h$$

① Then for ε fixed $u_h^\varepsilon \to u^\varepsilon$ in $H_0^1(\Omega)$ strong

$$\Rightarrow \quad u_h^\varepsilon \to u^\varepsilon \text{ in } W_0^{1,q}(\Omega) \text{ strong}$$

② On the other hand

$$u_h^\varepsilon - u_h \in V_h \qquad \int_\Omega A\, D(u_h^\varepsilon - u_h)\, Dv_h = \int_\Omega (f^\varepsilon - f)\, v_h \quad \forall v_h \in V_h$$

Then (discrete Boccardo Gallouet)

$$\alpha \int_\Omega |D \Pi_h T_k (u_h^\varepsilon - u_h)|^2 \le k\, \|f^\varepsilon - f\|_{L^1}$$

$$\Rightarrow \quad \|u_h^\varepsilon - u_h\|_{W_0^{1,q}(\Omega)} \le C\, \frac{\|f^\varepsilon - f\|_{L^1(\Omega)}}{\alpha}$$

$$\|u_h - u\|_{W^{1,q}} \leq \|u_h - u_h^\varepsilon\|_{W^{1,q}} + \|u_h^\varepsilon - u^\varepsilon\|_{W^{1,q}} + \|u^\varepsilon - u\|_{W^{1,q}}$$

$$\leq C \frac{\|f^\varepsilon - f\|_{L^1}}{\alpha} \qquad \downarrow 0 \qquad \leq C \frac{\|f^\varepsilon - f\|_{L^1}}{\alpha}$$

discrete Boccardo Gallouet $\qquad$ ε fixed $\qquad$ renorm. sol. continuous Boccardo Gallouet

$$\Rightarrow u_h \to u \quad W_0^{1,q}(\Omega) \text{ strongly} \quad \forall q < \frac{N}{N-1} \qquad \text{QED}$$

Error estimate Take $f \in L^r(\Omega)$, $f^\varepsilon = T_{1/\varepsilon}(f)$, $r > 1$

then $\|f^\varepsilon - f\|_{L^1} \leq C \varepsilon^{r-1} \|f\|_{L^r}$

On the other hand, if A smooth, $N = 2$ or 3

$$\|u_h^\varepsilon - u^\varepsilon\|_{H_0^1} \leq C_A h \|f^\varepsilon\|_{L^2(\Omega)}$$

Taking $\varepsilon = \dfrac{h^{2/r}}{\|f\|_{L^r}}$ gives $\|u_h - u\|_{W_0^{1,q}} \leq C h^{2(1-\frac{1}{r})} \|f\|_{L^r}$