A New Formulation of Immiscible Compressible Two-Phase Flow in Porous Media Via the Concept of Global Pressure

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Scaling Up and Modeling for Transport and Flow in Porous Media

Outline

Compressible two-phase flow

We consider two-phase isothermal, compressible, immiscible flow through heterogeneous porous medium. For example, water and gas.

Assumptions:

- Incompressible fluid: water, $\rho_w = \text{const}$.
- Compressible fluid: gas, $\rho_g = c_g p_g$.
- Viscosities μ_w and μ_g are constant.
- No mass exchange between the phases;
- The temperature is constant;

Note that the assumptions on the form of mass densities are not essential.
Independent variables: water saturation S_w and gas pressure p_g ($p_w = p_g - p_c(S_w)$, $S_g = 1 - S_w$).

Flow equations

Mass conservation: for $\alpha \in \{w, g\}$,

$$\Phi \frac{\partial}{\partial t}(\rho_\alpha S_\alpha) + \mathrm{div}(\rho_\alpha \mathbf{q}_\alpha) = 0,$$

The Darcy-Muscat law: for $\alpha \in \{w, g\}$,

$$\mathbf{q}_\alpha = -\mathbb{K}\frac{kr_\alpha(S_\alpha)}{\mu_\alpha}(\nabla p_\alpha - \rho_\alpha \mathbf{g}),$$

Capillary law:

$$p_c(S_w) = p_g - p_w,$$

No void space.

$$S_w + S_g = 1.$$

Fractional flow formulation

Goal:

- Reformulate flow equations in a form giving less tight coupling between the two differential equations, allowing a sort of IMPES (implicit in pressure and explicit in saturation) numerical treatment.

There are two approaches:

1. Introduce total velocity: $\mathbf{Q}_t = \mathbf{q}_w + \mathbf{q}_g$: leads to non-conservative form of the equations
2. Introduce total flow: $\mathbf{Q}_t = \rho_w \mathbf{q}_w + \rho_g \mathbf{q}_g$: leads to conservative form of the equations.

We work with total flow.

Fractional flow formulation: equations

Total flow:

$$\mathbf{Q}_t = -\lambda(S_w, p_g)\mathbb{K}\left(\nabla p_g - f_w(S_w, p_g)\nabla p_c(S_w) - \bar{\rho}(S_w, p_g)\mathbf{g}\right),$$

Total mass conservation:

$$\Phi\frac{\partial}{\partial t}(S_w\rho_w + (1 - S_w)\rho_g(p_g)) + \text{div}\left(\mathbf{Q}_t\right) = 0,$$

Water mass conservation:

$$\Phi\rho_w\frac{\partial S_w}{\partial t} + \text{div}(f_w(S_w, p_g)\mathbf{Q}_t + \mathbb{K}\mathbf{g}b_g(S_w, p_g)) = \text{div}(\mathbb{K}a(S_w, p_g)\nabla S_w).$$

Fractional flow formulation: coefficients

phase mobilities $$\lambda_w(S_w) = \frac{kr_w(S_w)}{\mu_w}, \quad \lambda_g(S_w) = \frac{kr_g(S_w)}{\mu_g},$$

total mobility $$\lambda(S_w, p_g) = \rho_w \lambda_w(S_w) + \rho_g(p_g)\lambda_g(S_w),$$

water fractional flow $$f_w(S_w, p_g) = \frac{\rho_w \lambda_w(S_w)}{\lambda(S_w, p_g)},$$

mean density $$\bar{\rho}(S_w, p_g) = \frac{\lambda_w(S_w)\rho_w^2 + \lambda_g(S_w)\rho_g(p_g)^2}{\lambda(S_w, p_g)},$$

"gravity" coeff. $$b_g(S_w, p_g) = \rho_w \rho_g(p_g) \frac{\lambda_w(S_w)\lambda_g(S_w)}{\lambda(S_w, p_g)}(\rho_w - \rho_g(p_g)),$$

"diffusivity" coeff. $$a(S_w, p_g) = -\rho_w \rho_g(p_g) \frac{\lambda_w(S_w)\lambda_g(S_w)}{\lambda(S_w, p_g)} p_c'(S_w).$$

Decoupling of the system

In total flow we want to eliminate saturation gradient:

$$\mathbf{Q}_t = -\lambda(S_w, p_g)\mathbb{K}\left(\nabla p_g - f_w(S_w, p_g)p_c'(S_w)\nabla S_w - \bar{\rho}(S_w, p_g)\mathbf{g}\right),$$

- Idea: introduce a new pressure-like variable that will eliminate ∇S_w term (CHAVENT-JAFFRÉE: MATHEMATICAL MODELS AND FINITE ELEMENTS FOR RESERVOIR SIMULATION)
- Find a new pressure variable p, called global pressure, and a function $\omega(S_w, p)$ such that:

$$\nabla p_g - f_w(S_w, p_g)p_c'(S_w)\nabla S_w = \omega(S_w, p)\nabla p \tag{1}$$

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New global formulation

We introduce unknown function π such that

$$p_g = \pi(S_w, p),$$

where p is *global pressure* to be defined. Then (1) reads

$$\nabla p_g = \omega(S_w, p)\nabla p + f_w(S_w, \pi(S_w, p))p_c'(S_w)\nabla S_w,$$

or,

$$\frac{\partial \pi}{\partial S_w}(S_w, p)\nabla S_w + \frac{\partial \pi}{\partial p}(S_w, p)\nabla p = \omega(S_w, p)\nabla p + f_w(S_w, \pi(S_w, p))p_c'(S_w)\nabla S_w.$$

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$$= \omega(S_w, p)\nabla p + f_w(S_w, \pi(S_w, p))p_c'(S_w)\nabla S_w.$$

New global formulation

Since p and S_w are independent variables we must have:

$$\frac{\partial \pi}{\partial S_w}(S_w, p) = f_w(S_w, \pi(S_w, p))p_c'(S_w) \quad (2)$$

$$\frac{\partial \pi}{\partial p}(S_w, p) = \omega(S_w, p). \quad (3)$$

Conclusion:

1. To calculate $\pi(S_w, p)$ solve the Cauchy problem:

$$\begin{cases} \dfrac{d\pi(S,p)}{dS} = \dfrac{\rho_w \lambda_w(S) p_c'(S)}{\rho_w \lambda_w(S) + c_g \lambda_g(S)\pi(S,p)}, & 0 < S < 1 \\ \pi(1, p) = p. \end{cases}$$

2. Get $\omega(S_w, p)$ from (3).

New global formulation: Remarks

1. It is more natural to replace saturation S_w with capillary pressure $u = p_c(S_w)$ as an independent variable. Then:

$$\begin{cases} \dfrac{d\hat{\pi}(u,p)}{du} = \dfrac{\rho_w \hat{\lambda}_w(u)}{\rho_w \hat{\lambda}_w(u) + c_g \hat{\lambda}_g(u)\hat{\pi}(u,p)}, \quad u > 0 \\ \hat{\pi}(0,p) = p. \end{cases}$$

   and $\pi(S_w, p) = \hat{\pi}(p_c(S_w), p)$ [Hat denotes the change of variables.]

2. ω is strictly positive and less than 1:

$$\omega(S_w, p) = \exp\left(-\int_0^{p_c(S_w)} \frac{c_g \rho_w \hat{\lambda}_w(u)\hat{\lambda}_g(u)}{(\rho_w \hat{\lambda}_w(u) + c_g \hat{\lambda}_g(u)\hat{\pi}(u,p))^2}\, du \right),$$

3. $p \le \pi(S_w, p) \le p + p_c(S_w)$ and therefore $p_w \le p \le p_g$.

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New global formulation: Flow equations

Total flow:

$$\mathbf{Q}_t = -\lambda^n(S_w, p)\mathbb{K}(\omega(S_w, p)\nabla p - \bar{\rho}^n(S_w, p)\mathbf{g}).$$

where the superscript n stands for **new**.

Total mass conservation:

$$\Phi \frac{\partial}{\partial t}(S_w \rho_w + c_g(1 - S_w)\pi(S_w, p)) + \mathrm{div}\mathbf{Q}_t = 0.$$

Water mass conservation:

$$\Phi \rho_w \frac{\partial S_w}{\partial t} + \mathrm{div}(f_w^n(S_w, p)\mathbf{Q}_t + \mathbb{K}\mathbf{g} b_g^n(S_w, p)) = \mathrm{div}(\mathbb{K} a^n(S_w, p)\nabla S_w).$$

New global formulation: Coefficients

New coefficients are obtained from the old ones by replacing gas pressure p_g by $\pi(S_w, p)$:

$$
\begin{array}{lr}
\text{total mobility} & \lambda^n(S_w, p) = \lambda(S_w, \pi(S_w, p)), \\
\text{water fractional flow} & f_w^n(S_w, p) = f_w(S_w, \pi(S_w, p)), \\
\text{mean density} & \bar{\rho}^n(S_w, p) = \bar{\rho}(S_w, \pi(S_w, p)), \\
\text{"gravity" coeff.} & b_g^n(S_w, p) = b_g(S_w, \pi(S_w, p)), \\
\text{"diffusivity" coeff.} & a^n(S_w, p) = a(S_w, \pi(S_w, p)).
\end{array}
$$

Simplified global formulation

The problem "Find a *mean pressure* p and a function $\omega(S_w, p)$ such that":

$$\nabla p_g - f_w(S_w, p_g) p_c'(S_w) \nabla S_w = \omega(S_w, p) \nabla p \quad (4)$$

can be solved by introducing a **simplification**:

- Gas density $\rho(p_g)$ can be replaced by $\rho(p)$ without introducing a significant error. Consequently, fractional flow function $f_w(S_w, p_g)$ can be replaced by $f_w(S_w, p)$.

Then, (4) reduces to:

$$\nabla p_g = \omega(S_w, p) \nabla p + f_w(S_w, p) p_c'(S_w) \nabla S_w.$$

Simplified global formulation

Now we can write solution:

$$p_g = p + \int_1^{S_w} f_w(s,p) p_c'(s)\, ds, \tag{5}$$

$$\omega(S_w, p) = 1 + \frac{\partial}{\partial p} \int_1^{S_w} f_w(s,p) p_c'(s)\, ds. \tag{6}$$

Note that (5) is nonlinear equation w.r.t. p to be solved.

Simplified global formulation: equations

The system written in unknowns p and S_w.

Total flow:

$$\mathbf{Q}_t = -\lambda(S_w, p)\mathbb{K}(\omega(S_w, p)\nabla p - \bar{\rho}(S_w, p)\mathbf{g}),$$

Total mass conservation:

$$\Phi \frac{\partial}{\partial t}(S_w \rho_w + c_g(1 - S_w)p) + \text{div}\mathbf{Q}_t = 0,$$

Water mass conservation:

$$\Phi \rho_w \frac{\partial S_w}{\partial t} + \text{div}(f_w(S_w, p)\mathbf{Q}_t + \mathbb{K}\mathbf{g} b_g(S_w, p)) = \text{div}(\mathbb{K} a(S_w, p)\nabla S_w).$$

Note that the equations are the same, only the coefficients are different.

Simplified global formulation: Applicability

- Global pressure is well defined for all $p_g \geq 0$ and $S_w \in (0,1]$ such that $p_w = p_g - p_c(S_w) \geq 0$;
- $p_w \leq p \leq p_g$;
- $\omega(S_w, p) > 0$ holds only if certain additional condition is fullfield. For example: $p \geq p_{min} > 0$ and

$$\forall S_w \in (0,1], \quad \int_{S_w}^{1} \frac{c_g \rho_w \lambda_w(s) \lambda_g(s)}{(\rho_w \lambda_w(s) + c_g \lambda_g(s) p_{min})^2} |p_c'(s)|\, ds < 1.$$

Numerical comparison

The difference in the coefficients of new and simplified models depends on the difference between $\pi(S_w, p)$ and p.

$$0 \leq \pi(S_w, p) - p \leq \int_0^{p_c(S_w)} \frac{\rho_w \hat{\lambda}_w(u)}{\rho_w \hat{\lambda}_w(u) + c_g p \hat{\lambda}_g(u)} \, du$$

Usually permeability functions depend on dimensionless variable $v = u/P_r$, where P_r is some pressure constant (entry pressure in *Brooks-Corey* or *van Genuchten* models). Then we have:

$$0 \leq \pi(S_w, p) - p \leq P_r \int_0^{+\infty} \frac{\rho_w \hat{\lambda}_w(v)}{\rho_w \hat{\lambda}_w(v) + c_g p \hat{\lambda}_g(v)} \, dv$$

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Numerical comparison

As an example we show the difference in coefficients:

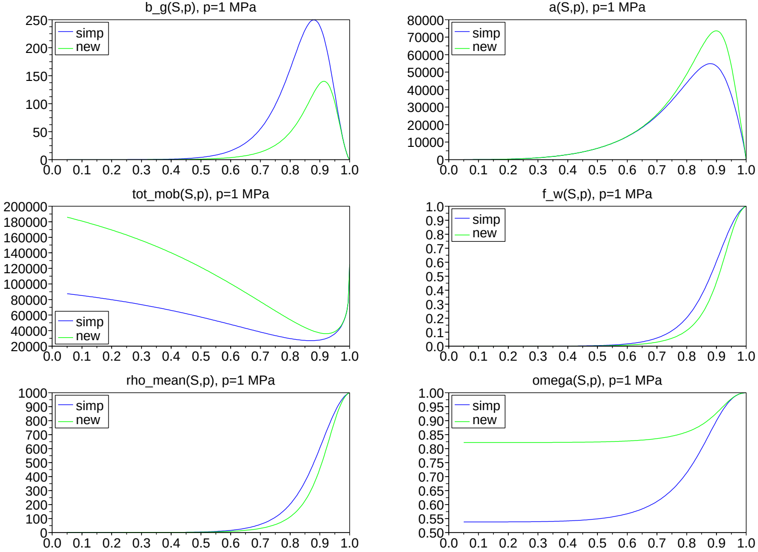
$$
\begin{aligned}
\lambda(S_w, p) \quad &\text{and} \quad \lambda(S_w, \pi(S_w, p)),\\
f_w(S_w, p) \quad &\text{and} \quad f_w(S_w, \pi(S_w, p)),\\
\bar{\rho}(S_w, p) \quad &\text{and} \quad \bar{\rho}(S_w, \pi(S_w, p)),\\
b_g(S_w, p) \quad &\text{and} \quad b_g(S_w, \pi(S_w, p)),\\
a(S_w, p) \quad &\text{and} \quad a(S_w, \pi(S_w, p)),\\
\omega^{simp}(S_w, p) \quad &\text{and} \quad \omega^{new}(S_w, p),
\end{aligned}
$$

for van Genuchten functions with parameters (Couplex test case):

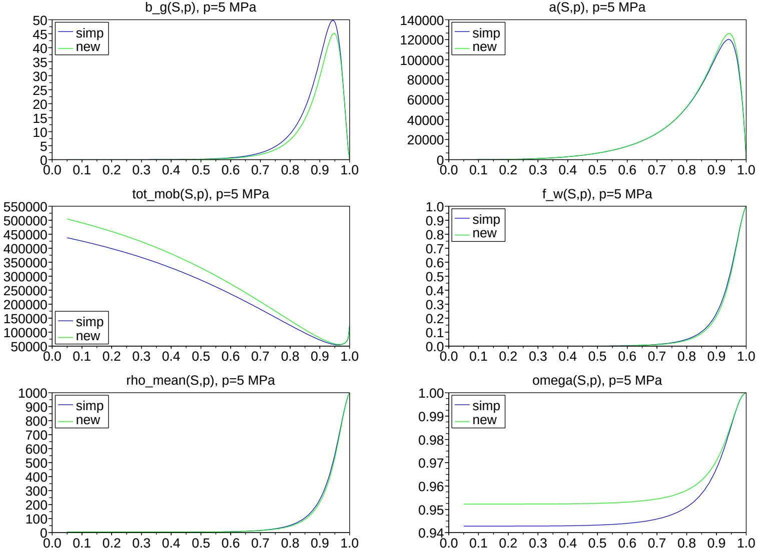
$$
n = 1.54,\ Pr = 2 \text{ MPa},\ \mu_w = 7.98 \cdot 10^{-3} \text{ Pas},\ \mu_g = 9 \cdot 10^{-6} \text{ Pas},
$$
$$
\rho_w = 10^3 \text{ kg/m3 and } c_g = 0.808.
$$

Notation: new = new model; simpl = simplified model.

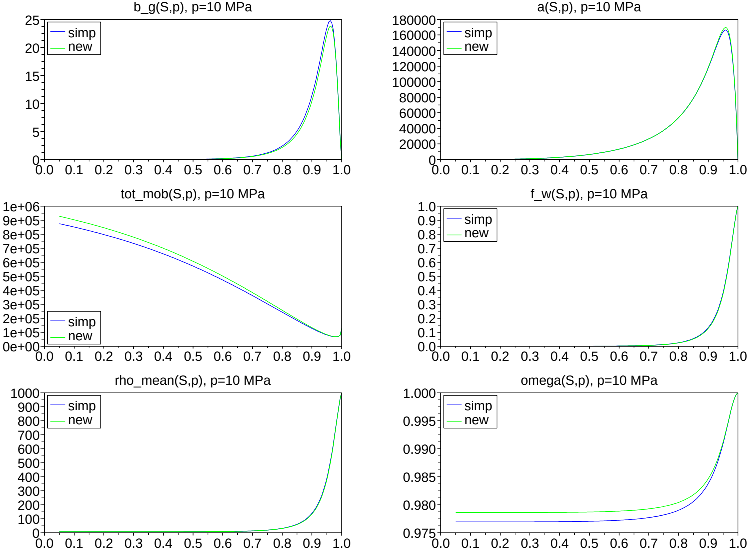
Coefficients at global pressure of 1 MPa



Coefficients at global pressure of 5 MPa



Coefficients at global pressure of 10 MPa



Conclusion

- In regimes with large global pressures compared to typical capillary pressure the difference in the coefficients is small.
- For small global pressures and relatively large capillary pressure the difference between new and simplified coefficients becomes significant.
- Replacing

$$\frac{\partial}{\partial t}(c_g(1 - S_w)\pi(S_w, p)) \quad \text{by} \quad \frac{\partial}{\partial t}(c_g(1 - S_w)p)$$

can have large influence on mass conservation even in for small capillary pressure and elevated global pressure.

Work in progress

- An extension to multiphase, multicomponent models is straightforward and it is at present in the course of study.
- The new formulation is well adapted for the mathematical analysis of the model. At present we study existence etc.
- The discretization of the model by a vertex-centered finite volume scheme is currently studied.

Reference: B. Amaziane, M. Jurak: *A new formulation of immiscible compressible two-phase flow in porous media*, C. R. A. S. Mécanique 336 (2008) 600-605.