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A Finite Volume Scheme for diffuson problems on general meshes ensuring monotony constraints

O. Angelini[1,2], C. Chavant[1,2] , E. Chenier[3] and R. Eymard[3]

[1]LaMSID - UMR EDF/CNRS 2832
[2]EDF R&D, AMA
[3]MSME, FRE3160 CNRS

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- Studies of unsteady and diphasic problems in porous medias.
- Need to respect the maximum principle:
  Let $f > 0$,
  Let $u(x)$ the solution of the following diffusion problem:

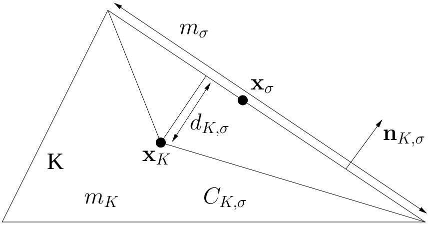
$$\begin{cases} \Delta u(x) = f & \text{on } \Omega \\ u(x) = u_0(x) & \text{on } \partial\Omega \end{cases}$$

  We suppose that $u_0 \in L^2(\Omega)$ so for all $x \in \Omega$

$$\min\left\{0, \inf_{\Omega} u_0\right\} \leq u(x) \leq \max\left\{0, \sup_{\Omega} u_0\right\}$$

- Let the following heterogeneous anisotropic diffusion problem:

$$\begin{cases} -\,div\,(\boldsymbol{\Lambda}(\mathbf{x})\nabla u(\mathbf{x})) & = f(\mathbf{x}) \quad \text{on } \Omega \\ u(\mathbf{x}) & = 0 \quad\quad \text{on } \partial\Omega \end{cases}$$



- $\boldsymbol{\Lambda}(\mathbf{x})$ can be a highly discontinuous function,
- The mesh can't be too flat.

- The weak formulation of the problem is:

$$\begin{cases} u \in H_0^1(\Omega), \\ \displaystyle\int_\Omega \boldsymbol{\Lambda}(\mathbf{x})\nabla u(\mathbf{x}).\nabla v(\mathbf{x})d\mathbf{x} = \int_\Omega f(\mathbf{x})v(\mathbf{x})d\mathbf{x}, \quad \forall v \in H_0^1(\Omega) \end{cases}$$

- The scheme consists in finding $u_{\mathcal{D}} \in X_{\mathcal{D},0}$ such that:

$$\boxed{\langle u_{\mathcal{D}}, v\rangle_{\mathcal{D},\alpha} = \int_\Omega f(\mathbf{x}) P_{\mathcal{M}}(v(\mathbf{x}))d\mathbf{x} \quad \forall v \in X_{\mathcal{D},0}}$$

$$\Leftrightarrow$$

$$\boxed{u_{\mathcal{D}} = argmin_{v\in X_{\mathcal{D},0}} J_{\mathcal{D},\alpha}(v)}$$

- With :

$$\begin{aligned} X_{\mathcal{D}} &= \{v = \left((v_K)_{K\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{E}}\right), v_K \in \mathbb{R}, v_\sigma \in \mathbb{R}\} \\ X_{\mathcal{D},0} &= \{u \in X_{\mathcal{D}}, u_\sigma = 0, \sigma \in \mathcal{E}_{ext}\} \\ J_{\mathcal{D},\alpha}(v) &= \frac{1}{2}\langle v, v\rangle_{\mathcal{D},\alpha} - \int_\Omega f(\mathbf{x}) P_{\mathcal{M}} u(\mathbf{x}) d\mathbf{x} \quad, \forall v \in X_{\mathcal{D},0} \end{aligned}$$

- Important issue : to find the expression of the bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{D},\alpha}$

- The symmetric and coercive bilinear form:

$$\langle u, v\rangle_{\mathcal{D},\alpha} = \sum_{K\in\mathcal{M}} \Big(m_K \nabla_K u . \mathbf{\Lambda}_K \nabla_K v + \alpha_K \sum_{\sigma\in\mathcal{E}_K} m_\sigma d_{K,\sigma} R_{K,\sigma}(u) R_{K,\sigma}(v) \mathbf{n}_{K,\sigma} . \mathbf{\Lambda}_K \mathbf{n}_{K,\sigma} \Big)$$

  - The discrete gradient:

$$\nabla_K u = \frac{1}{m_K} \sum_{\sigma\in\mathcal{E}_K} m_\sigma (u_\sigma - u_K) \mathbf{n}_{K,\sigma} \quad , \forall K \in \mathcal{M}, \forall u \in X_{\mathcal{D}}$$

  - $R_{K,\sigma}(u) = \dfrac{u_\sigma - u_K - \nabla_K u.(\mathbf{x}_\sigma - \mathbf{x}_K)}{d_{K,\sigma}} \quad , \forall K \in \mathcal{M}, \forall \sigma \in \mathcal{E}_K$

- Quality of the solution may depends on the choice of α
- An optimal choice of α exists: α will appear as the Lagrangian Multipliers of a monotony condition.
- $R_{K,\sigma}$: a measure of the local curvature of the discrete solution.

$$R_{K,\sigma}(u) = \frac{u_\sigma - u_K - \nabla_K u.(\mathbf{x}_\sigma - \mathbf{x}_K)}{d_{K,\sigma}}$$

We notice that:

$$\text{If} \quad \begin{cases} u \text{ linear function} \\ u_\sigma = u(x_\sigma) \\ u_K = u(x_K) \end{cases} \quad \text{then} \quad R_{K,\sigma}(u) = 0$$

- **Idea**: To build a constraint on $R_{K,\sigma}$ in order to decrease the oscillation,
- Practical the constraint is the following:
$G_K^{\mathcal{E}}(v) = \frac{1}{2} \sum_{\sigma \in \mathcal{E}_K} m_\sigma d_{K,\sigma} \mathbf{n}_{K,\sigma}.\mathbf{\Lambda}_K \mathbf{n}_{K,\sigma} R_{K,\sigma}^2(v) - m_K \varepsilon$
- We introduce a new constrained space:

$$X_{\mathcal{D},0}^{\mathcal{E}} = \left\{ v \in X_{\mathcal{D},0} \ , \ G_K^{\mathcal{E}}(v) \leq 0, \ \forall K \in \mathcal{M} \right\}$$

- The initial problem without constraint is:

$$\text{Find } u_{\mathcal{D}} \in X_{\mathcal{D},0} \text{ such as: } u_{\mathcal{D}} = argmin_{v \in X_{\mathcal{D},0}} J_{\mathcal{D},\alpha}(v)$$

- **The new constrained problem is :**

$$\boxed{\text{Find } u_{\mathcal{D}}^* \in X_{\mathcal{D},0}^{\varepsilon} \text{ such as: } u_{\mathcal{D}}^* = argmin_{v \in X_{\mathcal{D},0}^{\varepsilon}} J_{\mathcal{D},\beta}(v)}$$

Characterization of the solution of the constrained problem

Let $\beta = (\beta_K)_{K\in\mathcal{M}}$ be a family of strictly positive reals, let $\varepsilon > 0$.

Then there exists one and only one solution $u^*_{\mathcal{D}}$ to the problem with constraints, which satisfies:
there exists a family of non negative reals $\lambda^*_{\mathcal{D}} = (\lambda^*_{K,\mathcal{D}})_{K\in\mathcal{M}}$ such as $(u^*_{\mathcal{D}}, \lambda^*_{\mathcal{D}}) \in X^{\varepsilon}_{\mathcal{D},0} \times \mathbb{R}^{\mathcal{M}}_{+}$ is a saddle point of the function L :

$$L(v,\lambda) = J_\beta(v) + \sum_{K\in\mathcal{M}} \lambda_K G^{\varepsilon}_K(v)$$

and the so-called Kuhn and Tucker relations

$$\lambda^*_{K,\mathcal{D}} G^{\varepsilon}_K(u^*_{\mathcal{D}}) = 0 \quad , \forall K \in \mathcal{M}$$

The following relation holds:

$$\boxed{\langle u^*_{\mathcal{D}}, v\rangle_{\mathcal{D},(\beta+\lambda^*_{\mathcal{D}})} = \int_\Omega f(\mathbf{x}) \mathcal{P}_{\mathcal{M}} v(\mathbf{x}) d\mathbf{x}}$$

Theorem

- Hypothesis :
  - let $\mathcal{D}$ be a discretization of Ω,
  - let $\beta = (\beta_K)_{K\in\mathcal{M}}$ be a family of reals such that $\{\beta_K, K \in \mathcal{M}\} \subset [\underline{\beta}, \overline{\beta}]$,
  - let $\varepsilon_{\mathcal{D}} > 0$ be given,
  - let $u^*_{\mathcal{D}}$ be the unique solution of the constrained problem.

- Then

$$u^*_{\mathcal{D}} \to u \text{ in } L^2(\Omega) \text{ as } h_{\mathcal{D}} \to 0 \text{ and } \frac{h_{\mathcal{D}}}{\sqrt{\varepsilon_{\mathcal{D}}}} \to 0$$
$$\nabla_{\mathcal{D}} u^*_{\mathcal{D}} \to \nabla u \text{ in } L^2(\Omega)$$

Elements of proof: Let $\phi \in C_c^\infty(\Omega)$, we get

$$\langle u_{\mathcal{D}}^*, P_{\mathcal{D}}\phi \rangle_{\mathcal{D},\beta} + T_1\left(u_{\mathcal{D}}^*, P_{\mathcal{D}}\phi\right)^2 = \int_\Omega f(\mathbf{x}) P_{\mathcal{M}} P_{\mathcal{D}}\phi(\mathbf{x}) d\mathbf{x}$$

With

$$T_1\left(w, v\right) = \sum_{K\in\mathcal{M}} \lambda_{\mathcal{D},K}^* \sum_{\sigma\in\mathcal{E}_K} m_\sigma d_{K,\sigma} R_{K,\sigma}(w) R_{K,\sigma}(v) \mathbf{n}_{K,\sigma}.\mathbf{\Lambda}_K \mathbf{n}_{K,\sigma}$$

- $\langle u_{\mathcal{D}}^*, P_{\mathcal{D}}\phi \rangle_{\mathcal{D},\beta}$ converges to $\int_\Omega \mathbf{\Lambda}(\mathbf{x})\nabla u(\mathbf{x}).\nabla\phi(\mathbf{x}) d\mathbf{x}$
- $\int_\Omega f(\mathbf{x}) P_{\mathcal{M}} P_{\mathcal{D}}\phi(\mathbf{x}) d\mathbf{x}$ converges to $\int_\Omega f(\mathbf{x})\phi(\mathbf{x}) d\mathbf{x}$
- Proof of T_1 tends to 0:
  - The Cauchy-Schwarz inequality,
  - The consistency of $R_{K,\sigma}(P_{\mathcal{D}}\phi)$
  - The estimate on the solution of the constrained scheme:

$$\|u_{\mathcal{D}}^*\|_{1,\mathcal{D}} \leq \frac{\|f\|_{L^2(\Omega)} C_1}{\alpha_0}$$

Theorem

We assume that $\mathbf{\Lambda}(\mathbf{x}) = \mathbf{Id}$. We assume also that the weak solution u satisfies $u \in C^2(\overline{\Omega})$ and we consider the same hypothesis as previously.

Then there exists C_2 depending only on $d, \Omega, \theta, \underline{\alpha}, \overline{\alpha}$ and u such that:

$$\|u^*_{\mathcal{D}} - P_{\mathcal{D}}(u)\|_{1,\mathcal{D}} \leq C_2 \left(\frac{h_{\mathcal{D}}}{\sqrt{\varepsilon_{\mathcal{D}}}} + h^2_{\mathcal{D}} \right)^{\frac{1}{2}}$$

there exists C_3 depending only on $d, \Omega, \theta, \underline{\alpha}, \overline{\alpha}$ and u such that:

$$\|P_{\mathcal{M}} u^*_{\mathcal{D}} - u\|_{L^2(\Omega)} \leq C_3 \left(\frac{h_{\mathcal{D}}}{\sqrt{\varepsilon_{\mathcal{D}}}} + h^2_{\mathcal{D}} \right)^{\frac{1}{2}}$$

and there exists C_4 depending only on $d, \Omega, \theta, \underline{\alpha}, \overline{\alpha}$ and u such that:

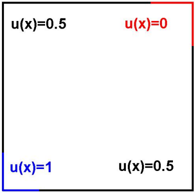
$$\|\nabla_{\mathcal{D}} u^*_{\mathcal{D}} - \nabla u\|_{L^2(\Omega)^d} \leq C_4 \left(\frac{h_{\mathcal{D}}}{\sqrt{\varepsilon_{\mathcal{D}}}} + h^2_{\mathcal{D}} \right)^{\frac{1}{2}}$$

Test 3 of the Benchmark on "discretization schemes for anisotropic diffusion problems on general grids"*: oblique flow

- Heterogeneous anisotropic tensor ($\theta = 40°$):

$$\Lambda = R_\theta \begin{pmatrix} 1 & 0 \\ 0 & 10^{-3} \end{pmatrix} R_\theta^{-1},$$

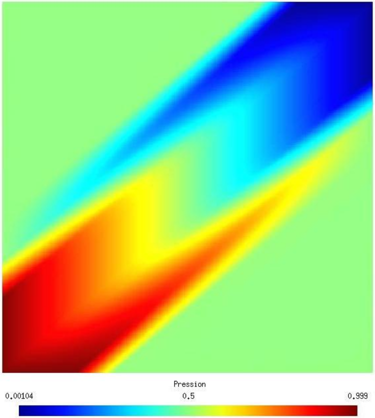
- Heterogeneous boundaries conditions are continuous and piecewise linear:



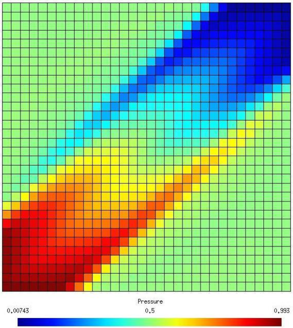
- Using the Uzawa's algorithm (the problem without constraint substituting α by $\beta + \lambda$)

* *Raphaele Herbin and Florence Hubert,2008*

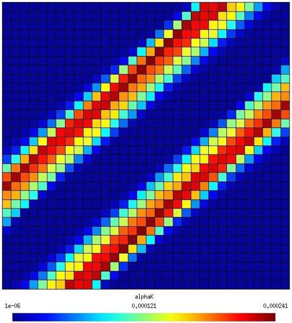
Values of the pressure using the non-constrained scheme on a grid with 65536 control volumes.



Results using the constrained scheme on a grid with 1024 control volumes.



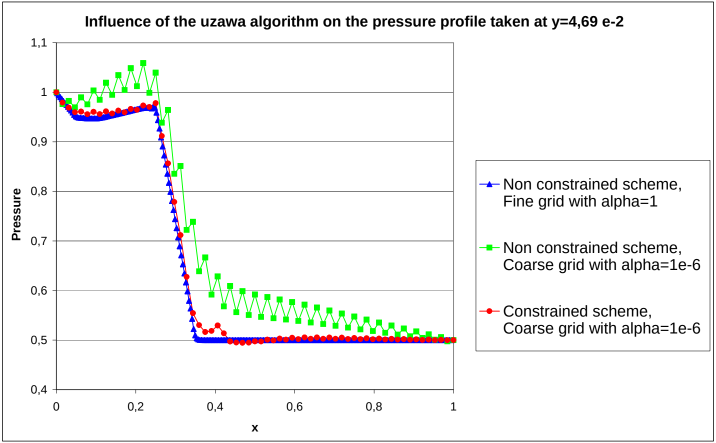
Values of the pressure



Values of $(\beta_K + \lambda_K^*)_{K\in\mathcal{M}}$

- Comparison with the solution on a fine grid shows an acceptable accuracy,
- The value of λ is increased only where it is needed.

Influence of the constraint on the solution



Profile at $y = 0.0469$ using a grid with 1024 control volumes.

- High decrease of pressure oscillations

- We propose a method which makes it possible to increase the monotony of the solution,
- We propose a mathematical analysis,
- Finally, we show some numerical results which are in agreement with the theoretical analysis.

References :

- *R. Eymard, R. Gallouët and R. Herbin, "A new finite volume scheme for anisotropic diffusion problems on general grids : convergence analysis",C.R.Acad.Sci.Paris, 2007*
- *R. Eymard, R. Gallouët and R. Herbin, "Discretization schemes for heterogeneous and anisotropic diffusion problems on general non conforming meshes", Submitted, 2008*

Conclusion

Thanks for your attention !!!!

6 Appendix

- Proof of the convergence
- Lagrange multipliers
- Uzawa's algorithm

Proof of the convergence

Let $\phi \in C_c^\infty(\Omega)$, we get

$$\langle u^*_{\mathcal{D}}, P_{\mathcal{D}}\phi\rangle_{\mathcal{D},\beta} + T_1\left(u^*_{\mathcal{D}}, P_{\mathcal{D}}\phi\right) = \int_\Omega f(\mathbf{x}) P_{\mathcal{M}} P_{\mathcal{D}}\phi(\mathbf{x}) d\mathbf{x}$$

With

$$T_1\left(w, v\right) = \sum_{K\in\mathcal{M}} \lambda^*_{\mathcal{D},K} \sum_{\sigma\in\mathcal{E}_K} m_\sigma d_{K,\sigma} R_{K,\sigma}(w) R_{K,\sigma}(v) \mathbf{n}_{K,\sigma}.\mathbf{\Lambda}_K \mathbf{n}_{K,\sigma}$$

We have that

$$\langle u^*_{\mathcal{D}}, P_{\mathcal{D}}\phi\rangle_{\mathcal{D},\beta} \text{ converges to } \int_\Omega \mathbf{\Lambda}(\mathbf{x})\nabla u(\mathbf{x}).\nabla\phi(\mathbf{x}) d\mathbf{x}$$

$$\int_\Omega f(\mathbf{x}) P_{\mathcal{M}} P_{\mathcal{D}}\phi(\mathbf{x}) d\mathbf{x} \text{ converges to } \int_\Omega f(\mathbf{x})\phi(\mathbf{x}) d\mathbf{x}$$

So we must prove that $T_1(u^*_{\mathcal{D}}, P_{\mathcal{D}}\phi)$ tends to 0.

- We apply the cauchy-Schwartz inequality :

$$T_1\left(u^*_{\mathcal{D}}, P_{\mathcal{D}}\phi\right)^2 \leq T_1\left(u^*_{\mathcal{D}}, u^*_{\mathcal{D}}\right) T_1\left(P_{\mathcal{D}}\phi, P_{\mathcal{D}}\phi\right)$$

- Thanks to the consistency of $R_{K,\sigma}$, we have that there exists C_5 depending only on d, θ and Ω such as :

$$|R_{K,\sigma}(P_{\mathcal{D}}\phi)| \leq C_5 h_{\mathcal{D}}$$

- So

$$T_1\left(P_{\mathcal{D}}\phi, P_{\mathcal{D}}\phi\right) \leq C_5^2 h_{\mathcal{D}}^2 \overline{\lambda} \sum_{K \in \mathcal{M}} \lambda^*_K m_K$$

- Thanks to the following estimate on the solution of the constrained scheme if $(u^*_{\mathcal{D}}, \lambda^*_{\mathcal{D}})$ is the saddle point

$$\sum_{K \in \mathcal{M}} \lambda^*_K m_K \leq \|f\|^2_{L^2(\Omega)} \frac{C_1^2}{2\alpha_0 \varepsilon}$$

- Thus we have:

$$T_1 \left(P_{\mathcal{D}}\phi, P_{\mathcal{D}}\phi\right)^2 \leq C_6 \frac{h_{\mathcal{D}}^2}{\varepsilon}$$

Hence, under the condition that $\dfrac{h_{\mathcal{D}}}{\sqrt{\varepsilon_{\mathcal{D}}}}$ tends to 0, we get that $T_1(u_{\mathcal{D}}, P_{\mathcal{D}}\phi)$ tends to 0 as well.

The complete proof was made in :

- *R. Eymard, R. Gallouët and R. Herbin, "A new finite volume scheme for anisotropic diffusion problems on general grids : convergence analysis",C.R.Acad.Sci. Paris,2007*
- *R. Eymard, R. Gallouët and R. Herbin, "Discretization schemes for heterogeneous and anisotropic diffusion problems on general non conforming meshes,Submitted,2008*

Theorem (Lagrange multipliers 1/2)

Let:

- V *a finite dimensional euclidean space*
- K *the convex closed non empty subset of* V, *defined by*

$$K = \{v \in V,\ G_i(v) \leq 0, \quad \text{for} \ \ 1 \leq i \leq p\},$$

- $G_i \ : \ V \rightarrow \mathbb{R}$ *convex, continuously and differentiable*
- $J \ : \ V \rightarrow \mathbb{R}$ *strictly convex function such that* $\lim_{|u| \rightarrow \infty} J(u) = +\infty$
- $u^\star$ *the unique solution of the minimization problem*

$$u^\star = argmin_{u \in K} J(u)$$

Theorem (Lagrange multipliers 2/2)

Then:
$\exists\ \beta^\star$ *such as* $(u^\star, \beta^\star)$ *saddle point of* $\mathcal{L}\ :\ V \times I\!R^p \to I\!R$ *defined by*

$$\mathcal{L}(u,\beta) = J(u) + \sum_{i=1}^{p} \beta_i G_i(u)$$

Moreover, the so-called Kuhn and Tucker relations hold:

$$\left\{ \begin{array}{l} \nabla J(u^\star) + \sum_{i=1}^{p} \beta_i^\star \nabla G_i(u^\star) = 0, \\ \beta_i^\star G_i(u^\star) = 0,\ \forall i = 1, \dots, p, \end{array} \right. \quad (1)$$

are satisfied. Reciprocally, if there exists $(u^\star, \beta^\star)$ *such that relations (1) are satisfied, then* $u^\star = argmin_{u \in K} J(u)$ *and* $(u^\star, \beta^\star)$ *is a saddle point of* $\mathcal{L}$*.*

Uzawa's algorithm

The aim is to find an approximation of the solution u^* of the minimization problem.
Let $\rho > 0$, we define (u^n, β^n), $\forall i = 1, \ldots, p$, $\forall n \in \mathbb{N}$ by

$$\begin{aligned} u^n &= argmin_{u \in V} \mathcal{L}(u, \beta^n) \\ \beta_i^{(n+1)} &= \max(\beta_i^n + \rho G_i(u^n), 0) \end{aligned}$$

Theorem (Convergence of Uzawa's algorithm 1/2)

Let:

- V *a finite dimensional euclidean space*
- K *the convex closed non empty subset of* V, *defined by*

$$K = \{v \in V,\ G_i(v) \leq 0, \quad \textit{for}\ \ 1 \leq i \leq p\},$$

- $G_i \ :\ V \to I\!R$ *convex, continuously and differentiable*
- $J \ :\ V \to I\!R$ *continuously differentiable function such that there exists* $\alpha > 0$ *with*

$$(\nabla J(u) - \nabla J(v), u - v) \geq \alpha \|u - v\|^2,\ \forall u, v \in V, \qquad (2)$$

- $M = \max\{\sum_{i=1}^{p} \|\nabla G_i(u)\|^2,\ \|u\| \leq B\}$

We assume: $\exists B \geq 0$: $\forall \beta \in (I\!R_+)^p$, $\|argmin_{u \in V} \mathcal{L}(u, \beta)\| \leq B$.

Theorem (Convergence of Uzawa's algorithm 2/2)

Then for all $\rho \in (0, \frac{\alpha}{2M})$ *and for all* $\beta^{(0)} \in (\mathbb{R}_+)^p$, *the sequence defined by*

$$\begin{aligned} u^n &= argmin_{u \in V} \mathcal{L}(u, \beta^n) \\ \beta_i^{(n+1)} &= \max(\beta_i^n + \rho G_i(u^n), 0) \end{aligned}$$

is such that $(u^n)_{n \in \mathbb{N}}$ *converges to the solution* $u^\star$ *of*

$$u^\star = argmin_{u \in K} J(u)$$

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