Relaxation theorem and lower dimensional models in micropolar elasticity

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Abstract

In this paper we prove the relaxation theorem in micropolar elasticity and use it, together with the semicontinuity theorem, to justify lower-dimensional models of rods (and plates) by means of Γ-convergence starting from general energy functionals. The internal energy density is supposed to be continuous and satisfy some growth and coercivity conditions. In particular, we apply these results to derive a rod model starting from quadratic isotropic energy density function of a cylindrical three-dimensional micropolar body.

1 Introduction

In classical elasticity the motion of a material particle is fully described by a vector function called deformation function \( \varphi : \Omega \to \mathbb{R}^3 \). In contrast, in micropolar elasticity material particles undergo an additional micromotion, described by a rotation of material particles at a microscale \( \mathbf{R} : \Omega \to \text{SO}(3) \). Such generalized continua are introduced by the Cosserat brothers in [14]. Overview of micropolar elasticity, together with more general theories of microstretch and micromorphic continua is given in [18]. For physical relevance of micropolar (and micromorphic) elasticity in conjunction with finite elasto-plasticity and elastic metallic foams see [37, 40].

Existence theorems in linearized micropolar elasticity are well established and usually based on the uniform positivity of the internal energy density function (see [23] or [3]). A new approach has been taken by Neff in [25] considering the weakest possible, conformally invariant curvature expression (see [26, 43]). Only recently the first existence theorems for geometrically exact Cosserat models, based on convexity arguments, are given in [36]. In [37] existence of solutions is proved for zero Cosserat couple modulus (in this case the energy is not pointwise coercive), using the extended Korn inequality (see [32] and [45]). For generalized continua with microstructure the existence theorem is given in [28], where convexity of the internal energy density function in the derivative of the variable which describes microstructure is assumed (in micropolar elasticity that means convexity of energy in \( \nabla \mathbf{R} \)). These existence results in the case of micropolar elasticity are generalized to a more general constitutive behavior in [47] and [48].

Similarly as in the case of classical elasticity, three-dimensional micropolar elasticity can be used as a parent model for the derivation of models of rods, plates and shells (see [3], [4], [5], [19], [21], [24], [42] for the infinitesimal case and [34] and [39] for the finite strain case). Let us note that the Reissner-Mindlin plate model is obtained starting from three-dimensional linear micropolar elasticity in [3] and [42]. In [39] rigorous justification of the geometrically exact plate model via Γ-convergence is given even for zero Cosserat couple modulus (see [38] for the existence result in this case), using a specific energy density function.

In this paper we derive and justify lower-dimensional models of rods (and plates) starting from the three-dimensional equations of micropolar elasticity by means of Γ-convergence. We derive models for general objective energy density functions which satisfy a \( p, s \)-coercivity assumption and a less demanding \( p, s \)-growth assumption (see assumptions W2) and W3) in Section 4). The model for plates obtained here is the same as the model for the specific internal
energy density function from [39] if assumed that the energy density function is additionally coercive. Two main results needed for the general approach are the semicontinuity theorem from [48] and the relaxation theorem which we prove here using techniques from [15, chapter III]. The relaxation theorem is also important when we look for minimizing solutions of energy functionals which are not sequentially weakly lower semicontinuous (swlsc) (see [15, p.415]). Then the minimizers might not exist and we can conclude only that an infimizing sequence converges weakly to the solution of the relaxed problem (which has a minimum due to swlsc if the energy density function is coercive).

In the second section we state the necessary results from [47, 48]. In the third section we prove the relaxation theorem (Theorem 3.8). In the forth section we derive general models for rods and plates using the relaxation and the semicontinuity result. This is a standard procedure as in classical elasticity (see [27]). In the last chapter we apply the obtained results to an isotropic energy quadratic in strains. Since this energy is convex in strains the analysis is simple and direct and we do not have to calculate the (quasi)convex envelope of the minimum function. The obtained model is of the Antman-Cosserat classical rod type with some additional terms which disappear for the appropriate choice of constant ($\mu_c = \mu$).

Antman-Cosserat model is founded in the literature by the so called direct approach (see [7]) which takes the rod to be a one-dimensional directed medium. In that case one needs reasonable constitutive assumptions for such a medium (see [22] for the influence of the material symmetry on the constitutive laws and [8] for the influence of the choice of the base curve, see also [44] for some remarks on the objectivity demand for the constrained Cosserat rod).

Throughout the paper we use the notation $\| \cdot \|$ for a norm in the appropriate Euclidean space. As a rule lower subscript denotes the element of a sequence, upper subscript, e.g. $R_j$, denotes the $j$th column of the matrix $R$. Let $e_1, e_2, \ldots, e_m$ denote the canonical basis of $\mathbb{R}^m$. For a vector $v \in \mathbb{R}^3$ by $A_v$ we denote the skew–symmetric matrix with the axial vector $v$, i.e., $A_v x = v \times x$.

## 2 Micropolar elasticity, semicontinuity, quasiconvexity

Let $\Omega \subset \mathbb{R}^m, m = 1, 2, 3$ be an open bounded set with Lipschitz boundary. The strain energy functional of the homogeneous micropolar body with reference configuration $\overline{\Omega}$ is given by

$$ I(\varphi, \overline{R}) = \int_\Omega W(\nabla \varphi(x), \overline{R}(x), \nabla \overline{R}(x)) \, dx, $$

where $W$ is an energy density function (i.e., the volume density of the internal energy in the reference configuration). As $\overline{R}$ is a rotation the matrix $\partial_i \overline{R} \overline{R}^T$ is skew-symmetric. We denote its axial vector by $\omega^i$, i.e.,

$$ A_{\omega^i} = \partial_i \overline{R} \overline{R}^T, \quad i = 1, \ldots, m, $$

where the notation $A_{\omega^i}$ stands for the skew-symmetric matrix with the axial vector $\omega^i$. This definition is introduced in [48] and is the same one as in [46] and [47] since $\omega^i$ then satisfy

$$ \partial_i \overline{R} = \omega^i \times \overline{R} = A_{\omega^i} \overline{R}, \quad i = 1, \ldots, m, \quad (2.1) $$

where the vector product is taken with respect to the columns of $\overline{R}$. Then the vectors $\omega^i$ can also be expressed by

$$ \omega^i = \omega(\overline{R})^i = \frac{1}{2} \overline{R}^i \times \partial_i \overline{R}, \quad i = 1, \ldots, m, $$

where $\overline{R} = \left( \overline{R}^1 \overline{R}^2 \overline{R}^3 \right)$ and the summation convention is used. In the same manner we denote $\omega = \left( \omega^1 \cdots \omega^m \right)$. Now we change the dependance of the internal energy density
function and assume that the energy functional is given by

\[ I(\varphi, \mathbf{R}) = \int_{\Omega} W(\nabla \varphi(x), \mathbf{R}(x), \omega(x)) \, dx. \]

Motivation for this change is that, due to \( \mathbf{R} \) being rotation (pointwise it belongs to the three-dimensional manifold \( \text{SO}(3) \)) derivatives of \( \mathbf{R} \) are dependent (there are 27 of them). However, \( \omega \) have independent components and there is a one-to-one, purely algebraic, correspondence between \( (\mathbf{R}, \partial \mathbf{R}) \) and \( (\mathbf{R}, \omega) \) for \( \mathbf{R} \in \text{SO}(3) \). Note as well that there is an analogy between vector columns of \( \omega \) and angular velocity. The following Lemma 2.1 is essential for this change in the arguments of the internal energy density function. That all 27 \( \partial_i \mathbf{R} \) derivatives can be controlled by 9 independent components is obvious by geometry of \( \text{SO}(3) \). In [41] it is shown that \( \mathbf{R}^p \text{Curl} \mathbf{R} \) is isomorphic to \( \omega \) and suggested as a curvature measure. The reason why we work with \( \omega \) is the way oscillations of \( \mathbf{R} \) affect \( \omega \) (see Lemma 3.7. in [47]). The ambient set for rotations is denoted by

\[ W^{1,p}(\Omega, \text{SO}(3)) = \{ \mathbf{R} \in W^{1,p}(\Omega, \mathbb{R}^{3\times3}) : \mathbf{R}(x) \in \text{SO}(3) \text{ for a.e. } x \in \Omega \}. \]

The following lemma holds.

**Lemma 2.1** Let \( \Omega \subset \mathbb{R}^m \) be a bounded open set and \( p \in [1, \infty] \). Let \( \mathbf{R}_k, \mathbf{R} \in W^{1,p}(\Omega, \text{SO}(3)) \) and let \( \omega^k_i = \omega(\mathbf{R}_k)^i, \omega^i = \omega(\mathbf{R})^i \). Then \( \mathbf{R}_k \to \mathbf{R} \) in \( W^{1,p}(\Omega, \mathbb{R}^{3\times3}) \) if and only if

\[ \mathbf{R}_k \to \mathbf{R} \text{ in } L^p(\Omega, \mathbb{R}^{3\times3}) \quad \text{and} \quad \omega^k_i \to \omega^i \text{ in } L^p(\Omega, \mathbb{R}^3), \ i = 1, \ldots, m. \]

Moreover, the same holds for the weak convergence (weak * for \( p = \infty \)).

**Proof.** See Lemma 2.7 and Remark 2.8 from [47].

Note also that, since \( \mathbf{R} \) is bounded, we can control the norm of \( \nabla \mathbf{R} \) by the norm \( \omega \) and the opposite (i.e., there exist constants \( C_1, C_2 \), which depend on the vector and matrix norm, such that \( C_1 \| \partial_i \mathbf{R} \| \leq \| \omega^i \| \leq C_2 \| \partial_i \mathbf{R} \| \)).

An important question, e.g. for \( \Gamma \)-convergence, is whether the functional \( I \) is swlsc. In the case of classical elasticity the sequential weak lower semicontinuity of the total energy (under some additional conditions on the internal energy density function) is equivalent to quasiconvexity of the internal energy density function in \( \nabla \varphi \). This notion was introduced by Morrey (see [31]). Let us recall the definition of quasiconvex functions.

**Definition 2.2** The function \( f : \mathbb{R}^{n \times m} \to \mathbb{R} \) is quasiconvex if it is Borel measurable, bounded on compact sets and satisfies

\[ f(A) \leq \frac{1}{\text{meas}(D)} \int_D f(A + \nabla \chi(x)) \, dx \]

for every open bounded open set \( D \subset \mathbb{R}^m \) with Lipschitz boundary, for every \( A \in \mathbb{R}^{n \times m} \) and \( \chi \in W^{1,\infty}_0(D, \mathbb{R}^n) \).

In the last definition \( W^{1,\infty}_0(D, \mathbb{R}^n) \) is understood in the sense of Meyers, see [30], i.e., as a set of \( W^{1,\infty}(D, \mathbb{R}^n) \) functions with zero trace at the boundary; this is different from the closure of \( C_0^\infty(D, \mathbb{R}^n) \) in \( W^{1,\infty}(D, \mathbb{R}^n) \) norm.

One should also note that in the definition of quasiconvexity it is enough to demand the property for an arbitrary cube \( D \) (see [15, Remark 5.2., p. 157]).

In [47] we have proved the equivalence of the sequential weak lower semicontinuity of the total energy function \( I \) and the quasiconvexity of the energy density function in \( \nabla \varphi \) and \( \omega \) (the first and third variable) in the case of micropolar elasticity. Here we cite the necessity of quasiconvexity theorem.
Theorem 2.3 (necessity of quasiconvexity) Let \( \Omega \subset \mathbb{R}^m \) be an open bounded set, let \( f : \mathbb{R}^{3 \times m} \times \text{SO}(3) \times \mathbb{R}^{3 \times m} \to \mathbb{R} \) be continuous and let the functional defined by
\[
I(\varphi, \overline{R}) = \int_{\Omega} f(\nabla \varphi(x), \overline{R}(x), \omega(x)) \, dx
\]
be sequentially weakly lower semicontinuous, i.e., it satisfies the condition
\[
I(\varphi, \overline{R}) \leq \liminf_{k \to \infty} I(\varphi_k, \overline{R}_k) \quad (2.2)
\]
for every sequence \( ((\varphi_k, \overline{R}_k))_k \subset W^{1,\infty}(\Omega; \mathbb{R}^3) \times W^{1,\infty}(\Omega; \text{SO}(3)) \) that converges weak * to \((\varphi, \overline{R})\) in \( W^{1,\infty}(\Omega; \mathbb{R}^3) \times W^{1,\infty}(\Omega; \text{SO}(3)) \).

Then \( f \) is quasiconvex in the first and the last variable, i.e., \( f \) satisfies
\[
f(A, \overline{R}, B) \leq \frac{1}{\text{meas}(D)} \int_D f(A + \nabla \chi(x), \overline{R}, B + \nabla \psi(x)) \, dx
\]
for every open bounded set \( D \) with Lipschitz boundary, for every \( A, B \in \mathbb{R}^{3 \times m}, \overline{R} \in \text{SO}(3) \) and for every \( \chi, \psi \in W^{1,\infty}(D; \mathbb{R}^3) \).

Since weak * convergence is stronger than weak convergence in any \( W^{1,p} \) this theorem also implies that the quasiconvexity of the internal energy density function is necessary for sequential weak lower continuity with respect to \( W^{1,p} \). For the sufficiency theorem we cite [48] (Theorem 2.13 and Remark 2.14). This theorem is an improved version of the result from [47]. It is proved using techniques from [2].

Theorem 2.4 (sufficiency of quasiconvexity) Let \( p, s \in [1, \infty], \beta \geq 0 \) and let \( \Omega \subset \mathbb{R}^m \) be an arbitrary open bounded set. Let \( f : \mathbb{R}^{3 \times m} \times \text{SO}(3) \times \mathbb{R}^{3 \times m} \to \mathbb{R} \) be continuous, quasiconvex in the first and last variable and satisfy the growth condition

- if \( p, s \in [1, \infty) \)
  \[-\beta \leq f(A, B) \leq K_g(1 + \|A\|^s + \|B\|^p);\]
- if \( p = \infty, s \in [1, \infty) \)
  \[-\beta \leq f(A, B) \leq \gamma(\|B\|)(1 + \|A\|^s),\]
where \( \gamma \) is a continuous and increasing function;
- if \( p \in [1, \infty), s = \infty \)
  \[-\beta \leq f(A, B) \leq \gamma(\|A\|)(1 + \|B\|^p),\]
where \( \gamma \) is a continuous and increasing function;
- if \( p = \infty, s = \infty \)
  \[|f(A, B)| \leq \eta(\|A\|, \|B\|),\]
where \( \gamma \) is a continuous and increasing functions in each of its arguments (if \( f \) is continuous this is satisfied).

Let \( \varphi_k, \varphi \in W^{1,s}(\Omega, \mathbb{R}^3), \overline{R}_k, \overline{R} \in W^{1,p}(\Omega, \text{SO}(3)), \varphi_k \rightharpoonup \varphi \) weakly in \( W^{1,s}(\Omega, \mathbb{R}^3) \) and \( \overline{R}_k \rightharpoonup \overline{R} \) weakly in \( W^{1,p}(\Omega, \mathbb{R}^{3 \times 3}) \). Then we have
\[
I(\varphi, \overline{R}) \leq \liminf_{k \to \infty} I(\varphi_k, \overline{R}_k).
\]
3 Relaxation theorem

In this chapter we prove the relaxation theorem in two steps. In the first step we prove that for every \((\varphi, \overline{R})\) one can find a sequence \(I(\varphi_k, \overline{R}_k)\) that approaches \(\int_Q Qf(\nabla \varphi(x), \overline{R}(x), \omega(x)) \, dx\) (see Theorem 3.5 and Definition 3.1). It is easier to work with \(Q_r f\) since we can easily prove its continuity and also we can control the norm of weakly convergent sequences \((\varphi_k, \overline{R}_k)\) (see Theorem 3.5). In the second step the relaxation theorem follows easily by letting \(r \to \infty\) (see Theorem 3.8).

Definition 3.1 Let \(f : \mathbb{R}^{3 \times m} \times \text{SO}(3) \times \mathbb{R}^{3 \times m} \to \mathbb{R}\) be bounded below by \(-\beta\), bounded on compact sets and Borel measurable. Let \(D\) be an open bounded set with Lipschitz boundary and \(r \in (0, +\infty]\). We define

\[
Q_r f(A, \overline{R}, B) \quad = \quad \inf \left\{ \frac{1}{\text{meas}(D)} \int_D f(A + \nabla \phi, \overline{R}, B + \nabla \psi) : \phi, \psi \in W^{1,\infty}_0(D, \mathbb{R}^3), \right. \\
\left. \| \nabla \phi \|_{L^\infty} \leq r, \| \nabla \psi \|_{L^\infty} \leq r \right\}.
\]

Remark 3.2 For \(r = \infty\), by Dacorogna’s formula, we know that \(Q_\infty f\) is the quasiconvex envelope (in the first and last variable) of the function \(f\) (see [15, Theorem 6.9.]). Therefore we use the notation \(Q f\) instead of \(Q_\infty f\).

Remark 3.3 Definition 3.1 is independent of the domain \(D\). It can be easily checked by a standard argument in the quasiconvex analysis (homothety, translation and Vitali covering theorem—see the proof of Step 1 in the Theorem 6.9. p. 272 in [15]).

Lemma 3.4 Let \(f : \mathbb{R}^{3 \times m} \times \text{SO}(3) \times \mathbb{R}^{3 \times m} \to \mathbb{R}\) be continuous and bounded below by \(-\beta\). Let \(r \in (0, \infty)\). Then \(Q_r f\) is continuous and satisfies \(-\beta \leq Q_r f \leq f\).

Proof. Let \(D = (0,1)^m\). Let us choose a sequence \(((A_k, \overline{R}_k, B_k)) \subset \mathbb{R}^{3 \times m} \times \text{SO}(3) \times \mathbb{R}^{3 \times m}\) that converges to \((A, \overline{R}, B) \in \mathbb{R}^{3 \times m} \times \text{SO}(3) \times \mathbb{R}^{3 \times m}\). We have to show that for each \(\varepsilon \in (0,1)\) there exists \(k_0\) such that

\[
-\varepsilon < Q_r f(A_k, \overline{R}_k, B_k) - Q_r f(A, \overline{R}, B) < \varepsilon, \quad \forall k \geq k_0. \tag{3.1}
\]

For each \((A_k, \overline{R}_k, B_k)\) and \((A, \overline{R}, B)\), by the property of infimum, let us choose \(\phi_k, \psi_k, \phi, \psi \in W^{1,\infty}_0(\Omega; \mathbb{R}^3)\) such that their gradient is less or equal to \(r\) in infinity norm and

\[
\left| \int_D f(A_k + \nabla \phi_k(x), \overline{R}_k, B_k + \nabla \psi_k(x)) \, dx - Q_r f(A_k, \overline{R}_k, B_k) \right| < \varepsilon/4,
\]

\[
\left| \int_D f(A + \nabla \phi(x), \overline{R}, B + \nabla \psi(x)) \, dx - Q_r f(A, \overline{R}, B) \right| < \varepsilon/4.
\]

We calculate

\[
Q_r f(A_k, \overline{R}_k, B_k) - Q_r f(A, \overline{R}, B)
\]

\[
= \quad Q_r f(A_k, \overline{R}_k, B_k) - \int_D f(A_k + \nabla \phi_k(x), \overline{R}_k, B_k + \nabla \psi_k(x)) \, dx
\]

\[
\quad + \int_D \left[ f(A_k + \nabla \phi_k(x), \overline{R}_k, B_k + \nabla \psi_k(x)) - f(A_k + \nabla \phi_k(x), \overline{R}_k, B_k + \nabla \psi_k(x)) \right] dx
\]

\[
\quad + \int_D \left[ f(A_k + \nabla \phi_k(x), \overline{R}_k, B_k + \nabla \psi_k(x)) - f(A_k + \nabla \phi_k(x), \overline{R}_k, B_k + \nabla \psi_k(x)) \right] dx
\]

\[
\quad + \int_D f(A + \nabla \phi(x), \overline{R}, B + \nabla \psi(x)) \, dx - Q_r f(A, \overline{R}, B) = I_1 + I_2 + I_3 + I_4.
\]
By the choice of \( \phi, \psi, \phi_k, \psi_k \) we have \( |I_1|, |I_4| < \varepsilon / 4 \). Also by the definition of \( Q_r f \) and the property of \( \phi_k, \psi_k \) we have that \( I_2 < \varepsilon / 4 \). Using the uniform continuity of \( f \) on bounded sets we have that for \( k \) large enough \( |I_3| < \varepsilon / 4 \). This establishes the right inequality in (3.1). For the left inequality we estimate in the similar way the following expression

\[
Q_r f(a, \overline{R}, B) - Q_r f(a_k, \overline{R}_k, B_k)
\]

\[
= Q_r f(a, \overline{R}, B) - \int_D f(a + \nabla \phi(x), \overline{R}, B + \nabla \psi(x)) \, dx
\]

\[
+ \int_D [f(a + \nabla \phi(x), \overline{R}, B + \nabla \psi(x)) - f(a + \nabla \phi_k(x), \overline{R}, B + \nabla \psi_k(x))] \, dx
\]

\[
+ \int_D [f(a + \nabla \phi_k(x), \overline{R}, B + \nabla \psi_k(x)) - f(a_k + \nabla \phi_k(x), \overline{R}_k, B_k + \nabla \psi_k(x))] \, dx
\]

\[
+ \int_D f(a_k + \nabla \phi_k(x), \overline{R}_k, B_k + \nabla \psi_k(x)) \, dx - Q_r f(a_k, \overline{R}_k, B_k) = I_1 + I_2 + I_3 + I_4.
\]

We get \( |I_1|, |I_3|, |I_4| < \varepsilon / 4 \) and \( I_2 < \varepsilon / 4 \).

The last statement of the lemma is obvious. \( \square \)

**Theorem 3.5** Let \( \Omega \subset \mathbb{R}^m \) be an open bounded set, \( p, s \in [1, \infty) \), \( 1 < r < \infty \), \( \beta \geq 0 \) and \( f : \mathbb{R}^{3 \times n} \times SO(3) \times \mathbb{R}^{3 \times n} \to \mathbb{R} \) a continuous function which satisfies the growth condition from Theorem 2.4. Then for every \( r \in (1, \infty) \) and \( \varphi \in W^{1,s}(\Omega, \mathbb{R}^3) \), \( \overline{R} \in W^{1,p}(\Omega, SO(3)) \) we can choose a sequence \( (\varphi_k, \overline{R}_k) \in W^{1,s}(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, SO(3)) \) such that

a) \( \varphi_k \to \varphi \) weakly (weak * if \( s = +\infty \)) in \( W^{1,s}(\Omega, \mathbb{R}^3) \), \( \overline{R}_k \to \overline{R} \) weakly (weak * if \( p = +\infty \)) in \( W^{1,p}(\Omega, \mathbb{R}^3) \) and \( \varphi_k - \varphi \) in \( W^{1,s}_0(\Omega, \mathbb{R}^3) \), \( \overline{R}_k - \overline{R} \) in \( W^{1,p}_0(\Omega, \mathbb{R}^3) \),

b) \( \|\nabla \varphi_k(x)\| \leq \|\nabla \varphi(x)\| + r, \|\omega_k(x)\| \leq 2\|\omega(x)\| + r + 1 \) a.e. in \( \Omega \),
c) \( I(\varphi_k, \overline{R}_k) \to \int_\Omega Q_r f(\nabla \varphi, \overline{R}, \omega) \).

**Proof.** Let us first assume \( \Omega = (0, 1)^m \). Without loss of generality we assume \( \beta = 0 \) (otherwise we prove the statement for \( f + \beta \)). Let \( p, s \in [1, \infty) \) and \( r > 1 \). For \( p = \infty \) or \( s = \infty \) the proof is a slight modification of the proof below. Let us denote by \( \eta \) a continuous function such that \( \eta(0) = 0 \) and

\[
\int_B K_g(1 + \|\nabla \varphi(x)\|^s + \|\omega(x)\|^p + r^s + r^p + 1) \leq \eta(\text{meas}(B))
\]

for every measurable set \( B \) (\( K_g \) is the constant from the growth condition; we assume \( K_g > 1 \)). Let \( \varepsilon > 0 \). Then there exists \( \alpha(\varepsilon) \geq 1 \) (large enough) such that for

\[
E_\varepsilon = \{x \in \Omega : \|\omega(x)\|^p, \|\nabla \varphi(x)\|^s \leq \alpha(\varepsilon)\}
\]

one has

\[
\text{meas}(\Omega \setminus E_\varepsilon) < \varepsilon, \int_{\Omega \setminus E_\varepsilon} 2^{p+s} K_g(2\|\nabla \varphi(x)\|^s + 2\|\omega(x)\|^p + r^p + r^s + 1) \, dx < \varepsilon. \quad (3.2)
\]

By Lusin’s and Egoroff’s theorem there exists a compact set \( K_\varepsilon \subset \Omega \) such that (the representatives of) \( \overline{R}, \omega, \nabla \varphi \) are continuous on \( K_\varepsilon \) and

\[
\text{meas}(\Omega \setminus K_\varepsilon) < \frac{\varepsilon}{2^{p+s} K_g(1 + 2\alpha(\varepsilon) + r^p + r^s)}. \quad (3.3)
\]

For \( n \in \mathbb{N} \) let us divide \( \Omega \) in open cubes \( Q_{k,n}, k = 1, \ldots, (2^n)^m \) whose edges are parallel with the coordinate axes and are of length \( 2^{-n} \). Then \( \Omega = \cup_{k=1}^{(2^n)^m} Q_{k,n} \). For all \( k, n \) let us denote
\[ S_{\varepsilon,k,n} = K_{\varepsilon} \cap E_{\varepsilon} \cap Q_{k,n}. \] If \( \text{meas}(S_{\varepsilon,k,n}) \neq 0 \) we choose \( x_{k,n} \in K_{\varepsilon} \cap E_{\varepsilon} \cap Q_{k,n} \) such that \( \overline{R}(x_{k,n}) \) is a rotation. It is important to notice that the integral \( \int_{\Omega} Q_r f(\varphi(x), \overline{R}(x), \omega(x)) \, dx \) for \( n \) large enough is well approximated by the sum

\[
\frac{1}{(2^n)^m} \sum_{\text{meas}(S_{\varepsilon,k,n}) \neq 0} Q_r f(\nabla \varphi(x_{k,n}), \overline{R}(x_{k,n}), \omega(x_{k,n})).
\]

To see this let us write

\[
\left| \int_{\Omega} Q_r f(\nabla \varphi(x), \overline{R}(x), \omega(x)) \, dx - \frac{1}{(2^n)^m} \sum_{\text{meas}(S_{\varepsilon,k,n}) \neq 0} Q_r f(\nabla \varphi(x_{k,n}), \overline{R}(x_{k,n}), \omega(x_{k,n})) \right|
\]

\[
\leq \left| \int_{\Omega} Q_r f(\nabla \varphi(x), \overline{R}(x), \omega(x)) \, dx - \int_{K_{\varepsilon} \cap E_{\varepsilon}} Q_r f(\nabla \varphi(x), \overline{R}(x), \omega(x)) \, dx \right|
\]

\[
+ \left| \int_{K_{\varepsilon} \cap E_{\varepsilon}} Q_r f(\nabla \varphi(x), \overline{R}(x), \omega(x)) \, dx - \sum_{\text{meas}(S_{\varepsilon,k,n}) \neq 0} \text{meas}(S_{\varepsilon,k,n}) Q_r f(\nabla \varphi(x_{k,n}), \overline{R}(x_{k,n}), \omega(x_{k,n})) \right|
\]

\[
- \text{meas}(E_{\varepsilon} \cap Q_{k,n}) \sum_{\text{meas}(S_{\varepsilon,k,n}) \neq 0} Q_r f(\nabla \varphi(x_{k,n}), \overline{R}(x_{k,n}), \omega(x_{k,n}))
\]

\[
+ \left| \sum_{\text{meas}(S_{\varepsilon,k,n}) \neq 0} \text{meas}(E_{\varepsilon} \cap Q_{k,n}) Q_r f(\nabla \varphi(x_{k,n}), \overline{R}(x_{k,n}), \omega(x_{k,n})) \right|
\]

\[
- \frac{1}{(2^n)^m} \sum_{\text{meas}(S_{\varepsilon,k,n}) \neq 0} Q_r f(\nabla \varphi(x_{k,n}), \overline{R}(x_{k,n}), \omega(x_{k,n})) \right|
\]

\[ = J_1 + J_2 + J_3 + J_4. \]

Because of the growth condition, the way we chose \( E_{\varepsilon}, K_{\varepsilon} \) and the definition of \( \eta \) it holds \( J_1 \leq \eta(2\varepsilon) \). For \( n \) large enough because of the uniform continuity of \( \nabla \varphi, \overline{R}, \omega \) on \( K_{\varepsilon} \) and uniform continuity of \( Q_r f \) on bounded sets it holds \( J_2 \leq \varepsilon \) and because of the definition of \( K_{\varepsilon} \) we have \( J_3 \leq \varepsilon \). Also because of the definition of \( E_{\varepsilon} \) we have:

\[
J_4 \leq \int_{\Omega \setminus E_{\varepsilon}} K_g(1+2\alpha(\varepsilon)) \leq \int_{\Omega \setminus E_{\varepsilon}} K_g(1+2\|\nabla \varphi(x)\|^s + 2\|\omega(x)\|^p) < 2\varepsilon.
\]

Thus we have proved

\[
\left| \int_{\Omega} Q_r f(\nabla \varphi(x), \overline{R}(x), \omega(x)) \, dx \right|
\]

\[
- \frac{1}{(2^n)^m} \sum_{\text{meas}(S_{\varepsilon,k,n}) \neq 0} Q_r f(\nabla \varphi(x_{k,n}), \overline{R}(x_{k,n}), \omega(x_{k,n})) \right| \leq 4\varepsilon + \eta(2\varepsilon).
\]
Because of the uniform continuity of \( f \) on compact sets we can choose \( \delta(\varepsilon) > 0 \) small enough such that
\[
|f(A_1, B_1) - f(A_2, B_2)| \leq \varepsilon, \quad \forall A_1, A_2, B_1, B_2
\]
such that \(|A_1|, |B_1| \leq 2^s \cdot 3^p - 1(\alpha(\varepsilon) + r^p + r^s + 1), |A_1 - A_2| \leq \delta(\varepsilon), |B_1 - B_2| \leq \delta(\varepsilon), |R_1 - R_2| \leq \delta(\varepsilon)\).

Now we choose \( n \) large enough such that for all \( k \) we have
\[
\|\nabla \varphi(x_1) - \nabla \varphi(x_2)\| \leq \delta(\varepsilon), \|\omega(x_1) - \omega(x_2)\| \leq \delta(\varepsilon), \|R(x_1) - R(x_2)\| \leq \delta(\varepsilon),
\]
for all \( x_1, x_2 \in Q_{k,n} \cap K_{\varepsilon} \).

For such \( n \) and \( x_{k,n} \), by the definition of \( Q_r f \), we choose \( \phi_{\varepsilon,k,n}, \psi_{\varepsilon,k,n} \in W^{1,\infty}_0(Q_{k,n} ; \mathbb{R}^3) \) such that \( \|\nabla \phi_{\varepsilon,k,n}\|_{L^\infty(Q_{k,n})} \leq r, \|\nabla \psi_{\varepsilon,k,n}\|_{L^\infty(Q_{k,n})} \leq r \) and
\[
Q_r f(\varphi(x_{k,n}), R(x_{k,n}), \omega(x_{k,n})) \leq \frac{1}{\operatorname{meas}(Q_{k,n})} \int_{Q_{k,n}} f(\nabla \varphi(x_{k,n}) + \nabla \phi_{\varepsilon,k,n}, R(x_{k,n}), \omega(x_{k,n})) \leq Q_r f(\varphi(x_{k,n}), R(x_{k,n}), \omega(x_{k,n})) + \varepsilon.
\]

We extend \( \phi_{k,n}, \psi_{k,n} \) by periodicity on \( \mathbb{R}^m \). We define sequences
\[
\varphi_{\nu,n}(x) = \begin{cases} \varphi(x), & \text{if } x \in Q_{k,n} \text{ s.t. } \operatorname{meas}(S_{\varepsilon,k,n}) = 0, \\ \varphi(x) + \frac{1}{2^s} \phi_{\varepsilon,k,n}(2^\nu x), & \text{if } x \in Q_{k,n} \text{ s.t. } \operatorname{meas}(S_{\varepsilon,k,n}) \neq 0, \end{cases}
\]
\[
\mathbf{R}_{\varepsilon,n}(x) = \begin{cases} \mathbf{R}(x), & \text{if } x \in Q_{k,n} \text{ s.t. } \operatorname{meas}(S_{\varepsilon,k,n}) = 0, \\ \exp(A \frac{1}{2^s} \psi_{\varepsilon,k,n}(2^\nu x)) \mathbf{R}(x), & \text{if } x \in Q_{k,n} \text{ s.t. } \operatorname{meas}(S_{\varepsilon,k,n}) \neq 0. \end{cases}
\]
It is obvious that
\[
\varphi_{\nu,n} \to \varphi, \quad \mathbf{R}_{\nu,n} \to \mathbf{R} \quad \text{in } L^\infty(\Omega)
\]
when \( \nu \to \infty \). It can be easily seen that
\[
\omega_{\varepsilon,n}(x) = \omega(x) + \sum_{k, \operatorname{meas}(S_{\varepsilon,k,n}) \neq 0} 1_{Q_{k,n}}(x) \nabla \psi_{\varepsilon,k,n}(2^\nu x) + r_{\varepsilon,n}(x),
\]
\[
\|\nabla \varphi_{\varepsilon,n}(x)\|^{p_s} \leq 2^{s-1}(\|\nabla \varphi(x)\|^{p_s} + r^s) \text{ a.e. in } \Omega,
\]
where for the matrices \( r_{\varepsilon,n} \) we have:
\[
\|r_{\varepsilon,n}(x)\| \leq \|r_{\varepsilon,1,n}(x)\| \|\omega(x)\| + \|r_{\varepsilon,2,n}(x)\| \text{ a.e. in } \Omega,
\]
\[
\|r_{\varepsilon,1,n}\|_{L^\infty(\Omega)} \to 0, \quad \|r_{\varepsilon,2,n}\|_{L^\infty(\Omega)} \to 0 \quad \text{for } \nu \to \infty.
\]
Because of the equalities (3.7) and (3.9) for \( \nu \) large enough we have
\[
\|\omega_{\varepsilon,n}(x)\|^p \leq 3^{p-1}(2\|\omega(x)\|^p + r^p + 1).
\]
Also
\[
\varphi_{\nu,n} \to \varphi \text{ weakly in } W^{1,s}(\Omega; \mathbb{R}^3), \quad \mathbf{R}_{\nu,n} \to \mathbf{R} \text{ weakly in } W^{1,p}(\Omega; \mathbb{R}^{3 \times 3}).
\]
Now we have
\[
\left| \int_\Omega f(\nabla \varphi_{\varepsilon,n}(x), \mathbf{R}_{\varepsilon,n}(x), \omega_{\varepsilon,n}(x)) \, dx \right| = \left| \int_\Omega Q_r f(\nabla \varphi(x), \mathbf{R}(x), \omega(x)) \, dx \right| \leq \left| \int_\Omega f(\nabla \varphi_{\varepsilon,n}(x), \mathbf{R}_{\varepsilon,n}(x), \omega_{\varepsilon,n}(x)) \, dx \right| \leq \left| \int_{K_{\varepsilon} \cap E_{\varepsilon}} f(\nabla \varphi_{\varepsilon,n}(x), \mathbf{R}_{\varepsilon,n}(x), \omega_{\varepsilon,n}(x)) \, dx \right|
\]
\[
+ \left| \int_{E_{\varepsilon}} f(\nabla \varphi_{\varepsilon,n}(x), \mathbf{R}_{\varepsilon,n}(x), \omega_{\varepsilon,n}(x)) \, dx \right|
\]
Because of (3.8) and (3.11) we have that $I_1 \leq 2^s \cdot 3^{p-1} \eta(2\varepsilon)$. As $\omega(x)$ is bounded on $K_\varepsilon \cap E_\varepsilon$ by $\alpha(\varepsilon)$ and that for $\nu$ large enough $\|r_{\varepsilon,n}\|_{L^\infty(K_\varepsilon)} \leq \delta(\varepsilon)$ (by (3.9) and (3.10)) it follows that $I_2 \leq \varepsilon$ for $\nu$ large enough. In the same way, using the fact that for $\nu$ large enough we have $\|\nabla \phi\|_{L^\infty(\Omega)} \leq \delta(\varepsilon)$, we accomplish $I_3 \leq \varepsilon$ and $I_4 \leq \varepsilon$. Also

\[
I_5 \leq \int_{\Omega \setminus K_\varepsilon} 2^{p+s} g(1 + r^p + r^s + 2\alpha(\varepsilon)) + \int_{\Omega \setminus E_\varepsilon} 2^{p+s} g(1 + r^p + r^s + 2\alpha(\varepsilon)) \\
\leq \varepsilon + \int_{\Omega \setminus E_\varepsilon} 2^{p+s} g(1 + r^p + r^s + 2\|\nabla \phi(x)\|^s + 2\|\omega(x)\|^p) \leq 2\varepsilon.
\]
Because of the way we chose \( \phi_{e,k,n}, \psi_{e,k,n} \) we have \( I_6 \leq \varepsilon \). In (3.4) we have already shown that \( I_7 \leq 4\varepsilon + \eta(2\varepsilon) \).

Using all of this we have that for an arbitrary \( \varepsilon > 0 \) we can choose \( \phi_e, \overline{R}_e \) such that:

\( a) \|\overline{R}_e - \overline{R}\|_{L^\infty(\Omega)} < \varepsilon, \|\phi_e - \phi\|_{L^\infty(\Omega)} < \varepsilon, \)

\( b) \|\nabla \phi_e(x)\| \leq \|\nabla \phi(x)\| + r, \|\omega_e(x)\| \leq 2\|\omega(x)\| + r + 1 \) a.e. in \( \Omega \),

\( c) |\int_{\Omega} f(\nabla \phi_e, \overline{R}_e, \omega_e) - \int_{\Omega} Q_r f(\nabla \phi, \overline{R}, \omega)| \leq 11\varepsilon + (2^s \cdot 3^p - 1 + 1)\eta(\varepsilon). \)

The construction of the desired sequence is now obvious (we take \( \varepsilon = \frac{1}{n} \)).

For an arbitrary open bounded set \( \Omega \) we argue as follows. From the properties of the Lebesgue integral we have that for an arbitrary \( \varepsilon > 0 \) we can choose a finite set of disjoint cubes \( D_k \subset \Omega, k = 1, \ldots, N \) and

\[
|\int_{\Omega} f(\nabla \phi_e, \overline{R}_e, \omega_e) - \sum_{k=1}^N \int_{D_k} f(\nabla \phi, \overline{R}, \omega)| \leq \varepsilon,
\]

\[
|\int_{\Omega} Q_r f(\nabla \phi_e, \overline{R}_e, \omega_e) - \sum_{k=1}^N \int_{D_k} Q_r f(\nabla \phi, \overline{R}, \omega)| \leq \varepsilon.
\]

Now we apply the construction from above on \( \frac{1}{n} \) and accompany all founded \( \phi_e^k \) and \( \overline{R}_e^k \) for different cubes in one \( \phi_e, \overline{R}_e \) (this is possible because on the boundary of \( D_k \) all the functions are equal to \( \phi, \) i.e., \( \overline{R} \)). Thus for an arbitrary \( \varepsilon > 0 \) we have found \( \phi_e \) and \( \overline{R}_e \) such that

\( a) \|\overline{R}_e - \overline{R}\|_{L^\infty(\Omega)} < \varepsilon, \|\phi_e - \phi\|_{L^\infty(\Omega)} < \varepsilon, \)

\( b) \|\nabla \phi_e(x)\| \leq \|\nabla \phi(x)\| + r, \|\omega_e(x)\| \leq 2\|\omega(x)\| + r + 1 \) a.e. in \( \Omega \),

\( c) |\int_{\Omega} f(\nabla \phi_e, \overline{R}_e, \omega_e) - \int_{\Omega} Q_r f(\nabla \phi, \overline{R}, \omega)| \leq 13\varepsilon + (2^s \cdot 3^p - 1 + 1)\eta(\varepsilon). \)

This completes the proof. \( \square \)

**Remark 3.6** In Theorem 3.5 we have chosen \( x_{k,n} \) and then \( \nabla \phi(x_{k,n}), \overline{R}(x_{k,n}), \omega(x_{k,n}) \). Of course this does not make sense if we know that \( \nabla \phi, \overline{R}, \omega \) are in fact classes of functions equal almost everywhere. It should be read that we have taken some special representants (continuous on \( K \)) whose existence is guarantied by the Lusin theorem and then have chosen \( \nabla \phi_{k,n} := \nabla \phi(x_{k,n}), \overline{R}_{k,n} := \overline{R}(x_{k,n}), \omega_{k,n} := \omega(x_{k,n}) \) (we must also choose \( x_{k,n} \) such that \( \overline{R}(x_{k,n}) \) is a rotation which is possible if \( \text{meas}(Q_{k,n} \cap K) > 0 \)). These values are good approximations of every representant in the class \( \phi, \) i.e., \( \overline{R}, \) i.e., \( \omega \) on the set \( Q_{k,n} \cap K \) in the sense that for \( n \) large enough and every \( k \) we have that \( \|\nabla \phi(x) - \nabla \phi_{k,n}\| \leq \varepsilon \) a.e. in \( Q_{k,n} \cap K, \) i.e., \( \|\overline{R}(x) - \overline{R}_{k,n}\| \leq \varepsilon \) a.e. in \( Q_{k,n} \cap K, \) i.e., \( \|\omega(x) - \omega_{k,n}\| \leq \varepsilon \) a.e. in \( Q_{k,n} \cap K. \)

**Definition 3.7** Let \( X \) be a topological space. For a function \( f : X \rightarrow \mathbb{R} \) we define its sequentially lower semicontinuous envelope \( \text{sclc} f \) by

\[
\text{sclc} \ f = \sup\{g \leq f | g \text{ sequentially lower semicontinuous}\}.
\]

It can be shown, directly from the definition, that supremum of an arbitrary family of sequentially lower semicontinuous functions is also sequentially lower semicontinuous and that a sum of sequentially lower semicontinuous functions is sequentially lower semicontinuous. It can be easily shown, using the diagonal procedure, that if the topology is metrizable we have

\[
\text{sclc} \ f(x) = \min\{\liminf_{n \rightarrow \infty} f(x_n) | x_n \rightarrow x\}.
\]
In general topological space, directly from the definition, it follows

\[ \text{slsc } f(x) \leq \inf \{ \liminf_{n \to \infty} f(x_n) | x_n \to x \}. \]

In general topological space we cannot use the diagonal procedure to show that the right hand side of the above inequality is sequentially lower semicontinuous (also the infimum need not to be attained).

**Theorem 3.8** Let \( \Omega \) be an arbitrary open bounded set and \( s, p \in [1, \infty] \), \( \beta \geq 0 \). Let \( f : \mathbb{R}^{3 \times m} \times \text{SO}(3) \times \mathbb{R}^{3 \times m} \to \mathbb{R} \) be a continuous function which satisfies the growth condition from Theorem 2.4. Let \( Qf \) be a continuous function as well. For \( \varphi \in W^{1,s}(\Omega, \mathbb{R}^3), \overline{R} \in W^{1,p}(\Omega, \text{SO}(3)) \) we define the functional \( I \) as before:

\[ I(\varphi, \overline{R}) = \int_{\Omega} f(\nabla \varphi(x), \overline{R}(x), \omega(x)) \, dx. \]

Let \( \text{slsc } I \) be a sequentially lower semicontinuous envelope of \( I \) in the weak topology of the space \( W^{1,s}(\Omega, \mathbb{R}^3) \times W^{1,p}(\Omega, \text{SO}(3)) \). Then we have

\[ \int_{\Omega} Qf(\nabla \varphi(x), \overline{R}(x), \omega(x)) \, dx = \text{slsc } I(\varphi, \overline{R}) \]

where

\[ \inf \{ \liminf_{k \to \infty} I(\varphi_k, \overline{R}_k, \Omega) : \varphi_k \rightharpoonup \varphi \text{ in } W^{1,s}(\Omega, \mathbb{R}^3), \overline{R}_k \to \overline{R} \text{ in } W^{1,p}(\Omega, \mathbb{R}^{3 \times 3}), \overline{R}_k \in \text{SO}(3) \}. \]

**Proof.** Let us define the functional

\[ F(\varphi, \overline{R}, \Omega) = \inf \{ \liminf_{k \to \infty} I(\varphi_k, \overline{R}_k, \Omega) : \varphi_k \rightharpoonup \varphi \text{ in } W^{1,s}(\Omega, \mathbb{R}^3), \overline{R}_k \to \overline{R} \text{ in } W^{1,p}(\Omega, \mathbb{R}^{3 \times 3}), \overline{R}_k \in \text{SO}(3) \}. \]

Because of Theorem 3.3. in [48] it is clear that \( \text{slsc } I(\varphi, \overline{R}) \geq \int_{\Omega} Qf(\nabla \varphi(x), \overline{R}(x), \omega(x)) \, dx \).

Also it is clear that

\( \text{slsc } I(\varphi, \overline{R}) \leq F(\varphi, \overline{R}). \)

Thus it is enough to prove that

\[ F(\varphi, \overline{R}) \leq \int_{\Omega} Qf(\nabla \varphi(x), \overline{R}(x), \omega(x)) \, dx. \]

Using Theorem 3.5 we have

\[ F(\varphi, \overline{R}) \leq \int_{\Omega} Qrf(\nabla \varphi(x), \overline{R}(x), \omega(x)) \, dx. \]

Using the Lebesgue dominated convergence theorem and the fact that \( Qrf \downarrow Qf \) letting \( r \to \infty \) we have the claim. \( \square \)

**Remark 3.9** If we assume in Theorem 3.8 that \( \Omega \) is with Lipschitz boundary and that \( \varphi = \varphi_0, \overline{R} = \overline{R}_0 \) on \( \partial \Omega \) where \( \varphi_0 \in W^{1,s}(\Omega, \mathbb{R}^3), \overline{R}_0 \in W^{1,p}(\Omega, \text{SO}(3)) \) then we have that

\[ \int_{\Omega} Qf(\nabla \varphi(x), \overline{R}(x), \omega(x)) \, dx \]

where

\[ \inf \{ \liminf_{k \to \infty} I(\varphi_k, \overline{R}_k, \Omega) : \varphi_k \rightharpoonup \varphi \text{ in } W^{1,s}(\Omega, \mathbb{R}^3), \overline{R}_k \to \overline{R} \text{ in } W^{1,p}(\Omega, \mathbb{R}^{3 \times 3}), \varphi_k = \varphi_0, \overline{R}_k = \overline{R}_0 \text{ on } \partial \Omega \}. \]
In the sequel we discuss continuity of $Qf$. A reasonable assumption on the internal energy density is that it is an objective function.

**Definition 3.10** We call the function $f : \mathbb{R}^{3 \times m} \times \text{SO}(3) \times \mathbb{R}^{3 \times m} \to \mathbb{R}$ objective if it satisfies

$$f(QA, Q\mathbf{R}, QB) = f(A, \mathbf{R}, B), \quad A, B \in \mathbb{R}^{3 \times m}, \ \mathbf{R}, Q \in \text{SO}(3).$$

**Proposition 3.11** Let $f : \mathbb{R}^{3 \times m} \times \text{SO}(3) \times \mathbb{R}^{3 \times m} \to \mathbb{R}$ be an objective function which is bounded below by $-\beta$ and bounded on compact sets. Then $Qf$ is a continuous objective function.

**Proof.** Plugging $Q = \mathbf{R}^T$ into the definition of an objective function and defining $\tilde{f}(A, B) := f(A, I, B)$ we have that

$$f(A, \mathbf{R}, B) = f(R^T A, I, R^T B) = \tilde{f}(R^T A, R^T B).$$

To prove objectivity and continuity of $Qf$ it is enough to prove that $Qf(A, \mathbf{R}, B) = Q\tilde{f}(R^T A, R^T B)$ (the objectivity is then obvious and the continuity follows from the fact that every quasiconvex function is locally Lipschitz and continuous, see [15, p.159]). We have

$$\inf \left\{ \frac{1}{\text{meas}(D)} \int_D f(A + \nabla \phi(x), \mathbf{R}, B + \nabla \psi(x)) \, dx : \phi, \psi \in W^{1,\infty}_0(D; \mathbb{R}^3) \right\}$$

$$= \inf \left\{ \frac{1}{\text{meas}(D)} \int_D \tilde{f}(R^T A + \nabla R^T \phi(x), \mathbf{R}^T B + \nabla \psi(x)) \, dx : \phi, \psi \in W^{1,\infty}_0(D; \mathbb{R}^3) \right\}$$

$$= \inf \left\{ \frac{1}{\text{meas}(D)} \int_D \tilde{f}(R^T A + \nabla \phi(x), R^T B + \nabla \psi(x)) \, dx : \phi, \psi \in W^{1,\infty}_0(D; \mathbb{R}^3) \right\}$$

$$= \inf \left\{ \frac{1}{\text{meas}(D)} \int_D \tilde{f}(R^T A, R^T B) \right\}$$

$$= Q\tilde{f}(R^T A, R^T B).$$

\[ \square \]

### 4 Justification of a model of plates and rods

In this section we derive and justify a model of nonlinear micropolar plates and rods via $\Gamma$-convergence (see [16]) using techniques from [27]. Derivation is performed in case of rods, but it easily extends to the case of plates. The following is a generalized Poincare’s inequality and can be found in [12, p. 281].

**Theorem 4.1** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and $p \in [1, \infty)$. Let $\Gamma_0$ be a da-measurable subset of $\partial \Omega$ with $\text{da} - \text{meas}(\Gamma_0) > 0$. Then there exists a constant $c_1 > 0$ such that

$$\int_{\Omega} |v|^p \, dx \leq c_1 \left( \int_{\Omega} |\nabla v|^p \, dx + \int_{\Gamma_0} v \, da \right)^{\frac{p}{p-1}}, \quad v \in W^{1,p}(\Omega).$$

The following lemma is a so-called measurable selection lemma (see [17]).

**Lemma 4.2** Let $\mathcal{K}$ be a compact metric space and let $\psi : \mathcal{K} \times \Omega \to \mathbb{R}$ be a mapping such that $\psi(k, .)$ is measurable for arbitrary $k \in \mathcal{K}$ and $\psi(., x)$ is a continuous mapping for arbitrary $x \in \Omega$. Then there exists a $\mathcal{K}$-valued measurable function $w : \Omega \to \mathcal{K}$ such that

$$\psi(w(x), x) = \min_{k \in \mathcal{K}} \psi(k, x).$$
In this section we start from a three-dimensional rod-like micropolar body and look for the $\Gamma$-limit of the energy functional when its thickness tends to 0. To be more precise for $\varepsilon > 0$ and $l > 0$ let $\Omega_\varepsilon = [0, l] \times \varepsilon S$, where $S \subset \mathbb{R}^2$ is an open, bounded set with Lipschitz boundary and with center of mass in the origin. In the sequel we suppose that the space of real $3 \times 3$ matrices $\mathbb{R}^{3 \times 3}$ is endowed with the usual Euclidean norm $\|A\| = \sqrt{\text{tr}(A^T A)}$. Let $p, s \in (1, \infty)$. We suppose that the energy density function $W : \mathbb{R}^{3 \times 3} \times SO(3) \times \mathbb{R}^{3 \times 3} \to \mathbb{R}$ is a continuous$^1$ objective function which satisfies conditions:

W1) $W \geq -\beta$,

W2) growth condition: there exists $K_g > 0$ such that

$$W(A, \overline{R}, B) \leq K_g(1 + \|A\|^s + \|B\|^p), \quad A, B \in \mathbb{R}^{3 \times 3}, \overline{R} \in SO(3),$$

W3) coercivity condition: there are $C_1 > 0, C_2 \subset \mathbb{R}$ such that

$$W(A, \overline{R}, B) \geq C_1(\|A\|^s + \|B\|^p) + C_2, \quad A, B \in \mathbb{R}^{3 \times 3}, \overline{R} \in SO(3).$$

Let us define $W_0 : \mathbb{R}^3 \times SO(3) \times \mathbb{R}^3 \to \mathbb{R}$ by

$$W_0(a^1, \overline{R}, b^1) = \min_{a^2, a^3, b^2, b^3 \in \mathbb{R}^3} W([a^1 a^2 a^3], \overline{R}, [b^1 b^2 b^3]);$$

(4.1)

here the minimum exists due to the continuity of $W$ and the coercivity assumption $W3$).

**Lemma 4.3** If $W$ is objective then $W_0$ is objective.

**Proof.** Since $W$ is objective we have that for all $Q \in SO(3)$ one has

$$W([Qa^1 Qa^2 Qa^3], Q\overline{R}, [Qb^1 Qb^2 Qb^3]) = W([a^1 a^2 a^3], \overline{R}, [b^1 b^2 b^3]).$$

Therefore, for all $Q \in SO(3)$ one has

$$W_0(Qa_1, Q\overline{R}, Qb_1) = \min_{a^2, a^3, b^2, b^3 \in \mathbb{R}^3} W([Qa^1 a^2 a^3], Q\overline{R}, [Qb^1 b^2 b^3])$$

$$= \min_{Qa^1, a^2, Qb^1, b^2 \in \mathbb{R}^3} W(QA, Q\overline{R}, QB)$$

$$= \min_{Qa^1, a^2, Qb^1, b^2 \in \mathbb{R}^3} W(A, \overline{R}, B)$$

$$= \min_{a^2, a^3, b^2, b^3 \in \mathbb{R}^3} W(A, \overline{R}, B) = W_0(a^1, \overline{R}, b^1).$$

Thus $W_0$ is objective. \hfill $\Box$

**Lemma 4.4** Let $W$ satisfies W1), W2) and W3), then $W_0$ satisfies

W01) $W_0 \geq -\beta$,

W02) growth condition:

$$W_0(a^1, \overline{R}, b^1) \leq K_g(1 + \|a^1\|^s + \|b^1\|^p), \quad a^1, b^1 \in \mathbb{R}^3, \overline{R} \in SO(3),$$

W03) coercivity condition:

$$W_0(a^1, \overline{R}, b^1) \geq C_1(\|a^1\|^s + \|b^1\|^p) + C_2, \quad a^1, b^1 \in \mathbb{R}^3, \overline{R} \in SO(3).$$

$^1$The more physical case in which $W(A, \overline{R}, B) = +\infty$ when $\text{det} A \leq 0$ and $W(A, \overline{R}, B) \to +\infty$ when $\text{det} A \to 0^+$ will not be treated here. In the classical elasticity such analysis is given in [9]. It is still open whether these results can be extended in the micropolar case. For the physically realistic models of energy density function and existence results in classical elasticity see [10].
Lemma 4.5 If $W : \mathbb{R}^{3×3} \times \text{SO}(3) \times \mathbb{R}^{3×3} \to \mathbb{R}$ is a continuous function which satisfies (4.1) for some $s, p > 0$ then the function $W_0$ defined by (4.1) is also continuous.

Proof. Let us take $a^1, a^2, b^1, b^2, k \in \mathbb{R}$ and $b^1, b^2, k \in \mathbb{R}$ for $k \in \mathbb{N}$ such that $a^1 \to a^1, R_k \to R, b^1, b^2, k \to 0$. We will prove $W_0(a^1, R_k, b^1) \to W_0(a^1, R, b^1)$.

Since $W$ is continuous and coercive we have that there exists a compact set $K \subset \mathbb{R}^3$ and $(a^1_k, a^2_k, b^1_k, b^2_k)_k \in K$ such that

$$W_0(a^1_k, R_k, b^1_k) = W(a^1_k, a^2_k, R_k, b^1_k, b^2_k, R_k),$$

$k \in \mathbb{N}$.

Since $K$ is a compact set, there exist vectors $\tilde{a}^1, \tilde{a}^2, \tilde{b}^1, \tilde{b}^2, \tilde{b}^3 \in \mathbb{R}^3$ and a subsequence of $(a^1_k, a^2_k, b^1_k, b^2_k)_k$, still denoted by $(a^1_k, a^2_k, b^1_k, b^2_k)_k$, such that

$$(a^1_k, a^2_k, a^3_k, b^1_k, b^2_k, b^3_k) \to (a^1, \tilde{a}^1, \tilde{a}^2, R, \tilde{R}, \tilde{b}^1, \tilde{b}^2, \tilde{b}^3).$$

By continuity of $W$ it follows

$$W_0(a^1_k, R_k, b^1_k) \to W(a^1, \tilde{a}^1, \tilde{a}^2, R, \tilde{R}, \tilde{b}^1, \tilde{b}^2, \tilde{b}^3).$$

To prove that $W(a^1, \tilde{a}^2, \tilde{a}^3, R, \tilde{R}, b^1, \tilde{b}^2, \tilde{b}^3) = W_0(a^1, R, b^1)$ let $a^2, a^3, b^2, b^3 \in \mathbb{R}^3$ be such that $W_0(a^1, R, b^1) = W(a^1, a^2, a^3, R, b^1, b^2, b^3)$. Taking the limit in the inequality

$$W(a^1_k, a^2_k, a^3_k, b^1_k, b^2_k, b^3_k) \leq W(a^1, a^2, a^3, R, b^1, b^2, b^3)$$

we obtain $W(a^1, \tilde{a}^2, \tilde{a}^3, R, \tilde{R}, b^1, \tilde{b}^2, \tilde{b}^3) \leq W_0(a^1, R, b^1)$. The other inequality is obvious by the definition of $W_0$.

For $\varepsilon > 0$ we assume that the three-dimensional micropolar body $\Omega_\varepsilon$ is submitted to the action of dead loads $\varphi^\varepsilon \in L^1(\Omega_\varepsilon; \mathbb{R}^3)$ where $t$ is such that $\frac{1}{s} + \frac{1}{p} = 1$. We assume that the external couples are 0 (which is not necessary for the subsequent analysis) and that the body is fixed on $\Gamma_\varepsilon = \{0, l\} \times \varepsilon S$ and force free at the remainder of the boundary $\Sigma_\varepsilon = (0, l) \times \varepsilon S$. Moreover, we suppose that the body is couple free at the whole boundary $\partial \Omega_\varepsilon$. Other boundary conditions are also possible. In particular, Dirichlet boundary conditions on $\Gamma_\varepsilon$ can be prescribed for microrotations (for more details see [39]). The equilibrium problem for the micropolar body $\Omega_\varepsilon$ may now be formulated as a minimization problem (see [33]):

$$\text{find } (\varphi^\varepsilon, R^\varepsilon) \in \Phi^\varepsilon \text{ such that } I^\varepsilon(\varphi^\varepsilon, R^\varepsilon) = \inf_{(\varphi, R) \in \Phi^\varepsilon} I^\varepsilon(\varphi, R),$$

where the total energy $I^\varepsilon$ is given by

$$I^\varepsilon(\varphi, R) = \int_{\Omega_\varepsilon} W(\nabla \varphi(x), R(x), \varphi(x)) \, dx - \int_{\Omega_\varepsilon} \varphi^\varepsilon \cdot \varphi \, dx$$

and the set of admissible deformations is

$$\Phi^\varepsilon = \{ (\varphi, R) \in W^{1,s}(\Omega_\varepsilon; \mathbb{R}^3) \times W^{1,p}(\Omega_\varepsilon; \text{SO}(3)) : \varphi(x) = x \text{ on } \Gamma_\varepsilon \}. $$

Let a diagonal minimizing sequence $(\varphi^\varepsilon, R^\varepsilon)$ for the family of energies $I^\varepsilon$ over the sets $\Phi^\varepsilon$ that satisfies

$$(\varphi^\varepsilon, R^\varepsilon) \in \Phi^\varepsilon, \quad I^\varepsilon(\varphi^\varepsilon, R^\varepsilon) \leq \inf_{(\varphi, R) \in \Phi^\varepsilon} I^\varepsilon(\varphi, R) + \varepsilon^2 h(\varepsilon)$$

be given; here $h(\varepsilon)$ is a positive function such that $h(\varepsilon) \to 0$ when $\varepsilon \to 0$. Such a sequence always exists (even if a minimization problem does not have a solution) and if a minimization problem has a solution, $(\varphi^\varepsilon, R^\varepsilon)$ may be chosen to be a minimizer (in that case $h(\varepsilon)$ can be chosen to be 0).
In order to obtain a model it is of crucial importance to specify the order of magnitude of the applied loads\(^2\). Namely, as noted in [27], it is always possible to stretch all thin cylinders $\Omega_\varepsilon$ into the same block, say $\Omega_1$, by applying sufficiently large forces. For such forces the limit behavior is obviously not the one we are looking for. We prescribe the order of the magnitude of $f^\varepsilon$ to be such that $\|f^\varepsilon\|_{L^1(\Omega,\mathbb{R}^3)} \leq C\varepsilon^2$. For example the weight of the body $f(x) = (0,0,-\rho g)^T$ is included in this analysis.

To perform the asymptotic analysis we rescale the problem on a domain independent of $\varepsilon$. Let $\Omega = \Omega_1$ and $\Gamma = \Gamma_1$ and define the rescaling operator $\Theta_\varepsilon$ by

$$(\Theta_\varepsilon \psi)(x_1, x_2, x_3) = \psi(x_1, \varepsilon x_2, \varepsilon x_3).$$

Let $\phi(\varepsilon) = \Theta_\varepsilon \phi$, $\Phi(\varepsilon) = \Theta_\varepsilon \Phi^*$ and $\phi_0(\varepsilon)(x) = (x_1, \varepsilon x_2, \varepsilon x_3)$. Note that all components of dependent variables $\phi^*$ and $\Phi^*$ are treated in the same way; we only scale the independent arguments. We accordingly rescale the energies by setting $I(\varepsilon)(\phi, \Phi) = \varepsilon^{-2} I(\Theta_\varepsilon^{-1}(\phi, \Phi))$, i.e.,

$$I(\varepsilon)(\phi, \Phi) = \int_\Omega W\left(\frac{\partial_1 \phi}{\varepsilon}, \frac{\partial_2 \phi}{\varepsilon}, \frac{\partial_3 \phi}{\varepsilon}\right) \Phi(\varepsilon, \omega^1, \omega^2, \omega^3) dx - \int_\Omega f(\varepsilon) \cdot \phi dx,$$

(4.3)

where $f(\varepsilon) = \Theta_\varepsilon f^\varepsilon$ and redefine the ambient set

$$\Phi(\varepsilon) = \{(\phi, \Phi) \in W^{1,s}(\Omega; \mathbb{R}^3) \times W^{1,p}(\Omega; \text{SO}(3)) : \phi = \phi_0(\varepsilon) \text{ on } \Gamma\}.$$

The set

$$\Phi^0 = \{(\phi, \Phi) \in W^{1,s}([0,\ell]; \mathbb{R}^3) \times W^{1,p}([0,\ell]; \text{SO}(3)) : \phi(0) = 0, \phi(\ell) = \ell e_1\}$$

will appear as the set of limit points of minimizing sequences and is canonically isomorphic to the set

$$\tilde{\Phi}^0 = \{(\phi, \Phi) \in W^{1,s}(\Omega; \mathbb{R}^3) \times W^{1,p}(\Omega; \text{SO}(3)) :$$

$$\phi(x) = x_1 e_1 \text{ on } \Gamma, \partial_2 \phi = 0, \partial_3 \phi = 0, \omega^2 = 0, \omega^3 = 0\}.$$

From (4.2) it is immediate that

$$I(\varepsilon)(\phi(\varepsilon), \Phi(\varepsilon)) \leq \inf_{(\phi, \Phi) \in \Phi(\varepsilon)} I(\varepsilon)(\phi, \Phi) + b(\varepsilon).$$

(4.4)

For simplicity we assume that $f(\varepsilon) = f$ (we can also assume that $f(\varepsilon) \to f$ in $L^1(\Omega; \mathbb{R}^3)$).

Now we want to compute the $\Gamma$-limit of the rescaled energies $I(\varepsilon)$. Since we want to work on a separable metric space, we extend the energies to $L^s(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{R}^{3 \times 3})$

$$\tilde{I}(\varepsilon)(\phi, \Phi) = \begin{cases} I(\varepsilon)(\phi, \Phi), & \text{if } (\phi, \Phi) \in \Phi(\varepsilon) \\ +\infty, & \text{otherwise} \end{cases}.$$

(4.5)

The main result of this section, $\Gamma$-convergence of $\tilde{I}(\varepsilon)$, is given in the following Theorem.

Theorem 4.6 The sequence $\tilde{I}(\varepsilon)$ $\Gamma$-converges for the strong topology of $L^s(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{R}^{3 \times 3})$ to $\tilde{I}_H(0)$ where

$$\tilde{I}_H(0)(\phi, \Phi) = \begin{cases} \text{meas}(S) \int_{[0,\ell]} \int_{\Omega} W_0(\partial_1 \phi, \Phi, \omega^1) dx_1 - \int_{[0,\ell]} F \cdot \phi dx_1, & \text{if } (\phi, \Phi) \in \Phi^0 \\ +\infty, & \text{otherwise} \end{cases}$$

(4.6)

where

$$F(x_1) = \int_S f(x_1, x_2, x_3) dx_2 dx_3.$$
In order to prove the theorem we use the following two theorems classical in the theory of Γ-convergence (see [11, p. 35,36]).

**Theorem 4.7** Let \((X,d)\) be a separable metric space and let \(f_j : X \to \mathbb{R}, j \in \mathbb{N}\). Then there is a subsequence \((f_{j_k})\) such that \(\Gamma\)-\(\lim\) \(f_{j_k}\) exists.

**Theorem 4.8** \(f_\infty = \Gamma\)-\(\lim\) \(f_j\) if and only if for every subsequence \((f_{j_k})\) there exists a further subsequence which \(\Gamma\)-converges to \(f_\infty\).

For clarity, we break the proof of Theorem 4.6 into a series of lemmas. We begin by extracting a \(\Gamma\)-convergent subsequence by Theorem 4.7 and call \(\bar{I}(0)\) its \(\Gamma\)-limit. The uniqueness of \(\bar{I}(0)\) will make the extraction of this subsequence superfluous a posteriori by Theorem 4.8.

**Lemma 4.9** Let \((v(\varepsilon), S(\varepsilon))\) be a sequence in \(L^s(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{R}^{3 \times 3})\), such that \(\bar{I}(\varepsilon)(v(\varepsilon), S(\varepsilon)) \leq C < \infty\) where \(C\) is independent of \(\varepsilon\). Then \((v(\varepsilon), S(\varepsilon))\) is uniformly bounded in \(W^{1,s}(\Omega; \mathbb{R}^3) \times W^{1,p}(\Omega; \mathbb{R}^{3 \times 3})\) and its limit points in the weak topology of this space belong to \(\Phi^0\).

**Proof.** The coercivity of \(W\) (see W3) and the uniform boundedness of \(\bar{I}(\varepsilon)\) imply existence of a constant \(C_1\) such that

\[
\int_{\Omega} \left\| \partial_1 v(\varepsilon) \frac{\partial_2 v(\varepsilon)}{\varepsilon} - \partial_3 v(\varepsilon) \right\|^s dx + \int_{\Omega} \left\| \frac{\omega_3^2(\varepsilon)}{\varepsilon} \omega_3^3(\varepsilon) \right\|^p dx \leq C_1. \tag{4.7}
\]

Since for \(\varepsilon < 1\), \(\|z^1\| \leq \|z^2\| \leq \|z^3\|\), the estimate (4.7) implies

\[
\|\nabla v(\varepsilon)\|_{L^s(\Omega; \mathbb{R}^3)} \leq C_1, \quad \|\omega_3^2(\varepsilon)\|_{L^p(\Omega; \mathbb{R}^{3 \times 3})} \leq C_2.
\]

We can, using the fact that \(v(\varepsilon) = (x_1, x_2, x_3, x_4) = (\varphi_2(\varepsilon))\) on \(\Gamma\), apply Theorem 4.1 and obtain the uniform bound for \((v(\varepsilon), S(\varepsilon))\) in \(W^{1,s}(\Omega; \mathbb{R}^3) \times W^{1,p}(\Omega; \mathbb{R}^{3 \times 3})\) (recall that \(\|S(0)(x)\| = \sqrt{3}\) a.e. in \(\Omega\)). Let us denote by \((v(0), S(0))\) its weak limit. Since \(S(\varepsilon)(x)\) is a rotation for a.e. \(x \in \Omega\) and since the weak convergence in \(W^{1,p}(\Omega; \mathbb{R}^{3 \times 3})\) implies the strong in \(L^p(\Omega; \mathbb{R}^{3 \times 3})\), by taking a subsequence which converges a.e., we conclude that also \(S(0)(x)\) is a rotation for a.e. \(x \in \Omega\). On the other hand using \(\|z^1\| \leq |z^2| + |z^3|\) and (4.7) we conclude that there is a constant \(C_3\) such that

\[
\|\partial_2 v(\varepsilon)\| \leq C_3 \varepsilon, \quad \|\partial_3 v(\varepsilon)\| \leq C_3 \varepsilon, \quad \|\omega_3^2(\varepsilon)\| \leq C_3 \varepsilon, \quad \|\omega_3^3(\varepsilon)\| \leq C_3 \varepsilon.
\]

Therefore \(\partial_2 v(0) = \partial_3 v(0) = \omega_3^2(0) = \omega_3^3(0) = 0\) which also implies \(\partial_2 S(0) = \partial_3 S(0) = 0\) by (2.1). By the compactness of the embedding \(W^{1,p}(\Omega) \hookrightarrow L^p(\Gamma)\) we have that \(v(0)\) satisfies the boundary condition

\[
v(0)(x_1) = (x_1, 0, 0) \text{ for } x_1 = 0, \quad x_1 = l.
\]

Therefore \((v(0), S(0)) \in \Phi^0\). \(\Box\)

**Corollary 4.10** Let \((v, S) \in L^s(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{R}^{3 \times 3})\). Then \((v, S) \in \Phi^0\) if and only if \(\bar{I}(0)(v, S) < +\infty\).

**Proof.** By the definition of \(\Gamma\)-convergence we have that there exists a sequence \((v(\varepsilon), S(\varepsilon))\) such that \(\bar{I}(\varepsilon)(v(\varepsilon), S(\varepsilon)) \to \bar{I}(0)(v, S)\). If \(\bar{I}(0)(v, S) < +\infty\) then \(\bar{I}(\varepsilon)(v(\varepsilon), S(\varepsilon))\) is uniformly bounded (at least from some member further). Lemma 4.9 then implies \((v, S) \in \Phi^0\).

Let \((v, S) \in \Phi^0\) and \(v_0(\varepsilon) = (0, \varepsilon x_2, \varepsilon x_3)\). Then \(\lim_{\varepsilon \to 0} \bar{I}(\varepsilon)(v_0(\varepsilon) + v, S) < +\infty\) and \((v_0(\varepsilon) + v, S) \to (v, S)\) strongly in \(L^s(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{R}^{3 \times 3})\). Therefore \(\bar{I}(0)(v, S) < +\infty\). \(\Box\)

Let us first prove the lower estimate for the \(\Gamma\)-limit which can be viewed as the \(\lim\inf\)-inequality.
Proposition 4.11 For all \((\varphi, \overline{R}) \in \Phi^0\) one has

\[
\tilde{I}(0)(\varphi, \overline{R}) \geq \text{meas}(S) \int_0^1 QW_0(\partial_1 \varphi, \overline{R}, \omega^1) \, dx_1 - \int_0^l \mathcal{F} \cdot \varphi.
\]

Proof. Let \((\varphi, \overline{R}) \in \Phi^0\). Then \(\tilde{I}(0)(\varphi, \overline{R}) < +\infty\) by Corollary 4.10. By the definition of \(\Gamma\)-convergence there exists a sequence \((\varphi(\varepsilon), \overline{R}(\varepsilon))\) such that \((\varphi(\varepsilon), \overline{R}(\varepsilon)) \rightarrow (\varphi, \overline{R})\) strongly in \(L^s(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{R}^{3 \times 3})\) and \(I(\varepsilon)(\varphi(\varepsilon), \overline{R}(\varepsilon)) \rightarrow I(0)(\varphi, \overline{R})\). Then \((\varphi(\varepsilon), \overline{R}(\varepsilon)) \rightarrow (\varphi, \overline{R})\) weakly in \(W^{1,s}(\Omega; \mathbb{R}^3) \times W^{1,p}(\Omega; \mathbb{R}^{3 \times 3})\) on a subsequence by Lemma 4.9. It is clear that, as \(\varepsilon \to 0\),

\[
\int_\Omega f(\varepsilon) \cdot \varphi(\varepsilon) \, dx \to \int_0^l \mathcal{F} \cdot \varphi \, dx_1.
\]

By definition of \(W_0\) and of the quasiconvex envelope of \(W_0\), for the elastic energy we have that

\[
\int_\Omega W(\partial_1 \varphi(\varepsilon) \frac{\partial_2 \varphi(\varepsilon)}{\varepsilon} \frac{\partial_3 \varphi(\varepsilon)}{\varepsilon}, \overline{R}(\varepsilon), \omega^1(\varepsilon) \frac{\omega^2(\varepsilon)}{\varepsilon} \frac{\omega^3(\varepsilon)}{\varepsilon}) \, dx \\
\geq \int_\Omega W_0(\partial_1 \varphi(\varepsilon), \overline{R}(\varepsilon), \omega^1(\varepsilon)) \, dx \\
\geq \int_\Omega QW_0(\partial_1 \varphi(\varepsilon), \overline{R}(\varepsilon), \omega^1(\varepsilon)) \, dx.
\]

Let \(G : W^{1,s}(\Omega; \mathbb{R}^3) \times W^{1,p}(\Omega; \text{SO}(3)) \rightarrow \mathbb{R}\) be defined by

\[
G(v, s) = \int_\Omega QW_0(\partial_1 v, s, \omega^{1/2}) \, dx.
\]

Let us define the function \(c : \mathbb{R}^{3 \times 3} \times \text{SO}(3) \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}\) by

\[
c([a^1 a^2 a^3], \overline{R}, [b^1 b^2 b^3]) = QW_0(a^1, \overline{R}, b^1).
\]

Since \(QW_0\) is quasiconvex \(c\) is also quasiconvex. Indeed, let \(d = [0, 1] \times [0, 1] \times [0, 1]\). Consider any functions \(\phi, \psi \in C_0^\infty(D; \mathbb{R}^3)\). For all \((y, z) \in [0, 1]\) the functions \(\phi_{yz}(x_1) = \phi(x_1, y, z), \psi_{yz}(x_1) = \psi(x_1, y, z)\) belong to \(C_0^\infty(d; \mathbb{R}^3)\). Hence for all \(A, B \in \mathbb{R}^{3 \times 3}\)

\[
\int_D c(A + \nabla \phi, \overline{R}, B + \nabla \psi) = \int_D QW_0(a^1 + \partial_1 \phi, \overline{R}, b^1 + \partial_1 \psi) \, dx \\
\geq \int_{[0,1]^2} QW_0(a^1, \overline{R}, b^1) = c([a^1 a^2 a^3], \overline{R}, [b^1 b^2 b^3]).
\]

Since \(W\) is objective, continuous and satisfies assumptions W1), W2) and W3) the function \(W_0\) is also an objective continuous function which is bounded below and satisfies the growth and coercivity condition by Lemma 4.3, Lemma 4.4 and Lemma 4.5. Therefore, \(QW_0\) is an objective continuous function (by Proposition 3.11 that is bounded below and satisfies the growth condition \(W_0(2)\)). Then obviously \(c\) is a continuous, quasiconvex, bounded below function which satisfies the growth condition \(W_0(2)\). Therefore we may apply Theorem 2.4 to conclude that \(G\) is lower semicontinuous:

\[
\liminf_{\varepsilon \to 0} G(\varphi(\varepsilon), \overline{R}(\varepsilon)) \geq G(\varphi, \overline{R}).
\]

From that we conclude

\[
\liminf_{\varepsilon \to 0} \int_\Omega W(\partial_1 \varphi(\varepsilon) \frac{\partial_2 \varphi(\varepsilon)}{\varepsilon} \frac{\partial_3 \varphi(\varepsilon)}{\varepsilon}, \overline{R}(\varepsilon), [\omega^1(\varepsilon) \frac{\omega^2(\varepsilon)}{\varepsilon} \frac{\omega^3(\varepsilon)}{\varepsilon}]) \, dx \\
\geq \liminf_{\varepsilon \to 0} G(\varphi(\varepsilon), \overline{R}(\varepsilon)) \\
\geq G(\varphi, \overline{R}) = \text{meas}(S) \int_0^1 QW_0(\partial_1 \varphi, \overline{R}, \omega^1) \, dx_1
\]
and the proof is complete.

Let us now prove the upper estimate for the $\Gamma$-limit which can be viewed as the lim sup-inequality.

**Proposition 4.12** For all $(\varphi, \overline{R}) \in \Phi^0$ we have

$$
\overline{I}(0)(\varphi, \overline{R}) \leq \text{mes}(S) \int_{[0,l]} QW_0(\partial_1 \varphi, \overline{R}, \omega^1) \, dx_1 - \int_{[0,l]} F \cdot \varphi.
$$

(4.8)

**Proof.** Let us fix $(\varphi, \overline{R}) \in \Phi^0$ and $\theta, \eta, \phi, \psi \in C^0_c([0,l]; \mathbb{R}^3)$. For $0 < \varepsilon < 1$ we define

$$
\varphi(\varepsilon)(x) = \varphi(x_1) + (0, \varepsilon x_2, \varepsilon x_3) + \varepsilon x_2 \theta(x_1) + \varepsilon x_3 \eta(x_1),
$$

$$
\overline{R}(\varepsilon)(x) = \exp \left( \varepsilon A_{(x_2 \phi(x_1) + x_3 \psi(x_1))} \right) \overline{R}(x_1).
$$

This immediately implies that

$$
\partial_1 \varphi(\varepsilon) \to \partial_1 \varphi, \quad \frac{1}{\varepsilon} \partial_2 \varphi(\varepsilon) \to e_2 + \theta, \quad \frac{1}{\varepsilon} \partial_3 \varphi(\varepsilon) \to e_3 + \eta,
$$

all strongly in $L^\infty(\Omega; \mathbb{R}^3)$. Expanding $\exp \left( \varepsilon A_{(x_2 \phi(x_1) + x_3 \psi(x_1))} \right)$ we obtain the estimates

$$
\| \exp \left( \varepsilon A_{x_2 \phi(x_1) + x_3 \psi(x_1)} \right) - I \| \leq \varepsilon \| x_2 \| \phi(x_1) \| + x_3 \| \psi(x_1) \| + \| x_3 \| \psi(x_1) \|,
$$

$$
\| \partial_1 \exp \left( \varepsilon A_{x_2 \phi(x_1) + x_3 \psi(x_1)} \right) \| \leq \varepsilon \| x_2 \| \phi(x_1) \| + x_3 \| \psi(x_1) \| + \| x_3 \| \psi(x_1) \|,
$$

$$
\| \partial_2 \exp \left( \varepsilon A_{x_2 \phi(x_1) + x_3 \psi(x_1)} \right) - \frac{1}{\varepsilon} A_{\phi(x_1)} \| \leq \varepsilon \| \phi(x_1) \| + \| x_2 \| \phi(x_1) \| + x_3 \| \psi(x_1) \|,
$$

$$
\| \partial_3 \exp \left( \varepsilon A_{x_2 \phi(x_1) + x_3 \psi(x_1)} \right) - \frac{1}{\varepsilon} A_{\psi(x_1)} \| \leq \varepsilon \| \psi(x_1) \| + \| x_2 \| \phi(x_1) \| + x_3 \| \psi(x_1) \|.
$$

Therefore, there is a constant $K > 0$ independent of $x$ and $\varepsilon$ such that

$$
\| \overline{R}(\varepsilon)(x) - \overline{R}(x_1) \| \leq K \varepsilon,
$$

$$
\| \partial_1 \overline{R}(\varepsilon)(x) - \partial_1 \overline{R}(x_1) \| \leq K \varepsilon,
$$

$$
\| \frac{1}{\varepsilon} \partial_2 \overline{R}(\varepsilon)(x) - A_{\phi(x_1)} \overline{R}(x_1) \| \leq K \varepsilon,
$$

$$
\| \frac{1}{\varepsilon} \partial_3 \overline{R}(\varepsilon)(x) - A_{\psi(x_1)} \overline{R}(x_1) \| \leq K \varepsilon.
$$

Using these estimates we obtain

$$
\partial_1 \overline{R}(\varepsilon) \to \partial_1 \overline{R}, \quad \frac{1}{\varepsilon} \partial_2 \overline{R}(\varepsilon) \to A_{\phi} \overline{R}, \quad \frac{1}{\varepsilon} \partial_3 \overline{R}(\varepsilon) \to A_{\psi} \overline{R},
$$

all strongly in $L^p(\Omega; \mathbb{R}^{3 \times 3})$. Using the same argument as in Lemma 2.1 we conclude that

$$
\omega^1(\varepsilon) \to \omega^1, \quad \frac{1}{\varepsilon} \omega^2(\varepsilon) \to \phi, \quad \frac{1}{\varepsilon} \omega^3(\varepsilon) \to \psi,
$$

(4.10)

all strongly in $L^p(\Omega; \mathbb{R}^3)$. By an application of the Nemytsky operators (see [6, p.15]) using (4.9) and (4.10) we conclude

$$
\int_\Omega W \left( \| \partial_1 \varphi(\varepsilon) \| \frac{1}{\varepsilon} \partial_2 \varphi(\varepsilon), \frac{1}{\varepsilon} \partial_3 \varphi(\varepsilon), |\overline{R}(\varepsilon)|, \left[ \omega^1(\varepsilon) \frac{\omega^2(\varepsilon)}{\varepsilon}, \omega^3(\varepsilon) \frac{\omega^3(\varepsilon)}{\varepsilon} \right] \right) \, dx
$$

$$
\to \int_\Omega W(\partial_1 \varphi, e_2 + \theta, e_3 + \eta, \overline{R}, \omega^1, \phi, \psi) \, dx.
$$

Consequently,

$$
\tilde{I}(\varepsilon)(\varphi(\varepsilon), \overline{R}(\varepsilon)) \to \int_\Omega W(\partial_1 \varphi, e_2 + \theta, e_3 + \eta, \overline{R}, \omega^1, \phi, \psi) \, dx - \int_0^l F \cdot \varphi \, dx_1.
$$
Now using the definition of $\Gamma$-limit we have
\[
\tilde{I}(0)(\varphi, \overline{R}) \leq \inf_{\theta, \eta, \phi, \psi \in C^0_b([0,l]; \mathbb{R}^3)} \int_{\Omega} W(\partial_1 \varphi, e + \theta, e_3 + \eta, \overline{R}, \omega, \phi, \psi) \, dx - \int_0^l \mathcal{F} \cdot \varphi \, dx.
\]
By the density argument and an application of the Nemytsky operators we have that
\[
\inf_{\theta, \eta, \phi, \psi \in C^0_b([0,l]; \mathbb{R}^3)} \int_{\Omega} W(\partial_1 \varphi, e + \theta, e_3 + \eta, \overline{R}, \omega, \phi, \psi) \, dx
\]
\[
= \inf_{\theta, \eta \in L^1([0,l]; \mathbb{R}^3)} \int_{\Omega} W(\partial_1 \varphi, \theta, \eta, \overline{R}, \omega, \phi, \psi) \, dx.
\]
Let us define $g : [0, l] \times \mathbb{R}^3 \to \mathbb{R}$ by
\[
g(x, a^1, b^1, c^1, d^1) := W(\partial_1 \varphi(x), a^1, b^1, \overline{R}(x), \omega(x), c^1, d^1).
\]
By Lemma 4.2 there exist measurable functions\footnote{Lemma 4.2 requires compact sets. We can overcome this in the following way. Define $W_n : ([-n, n]^2 \times [0, l] \to \mathbb{R}$ by $W_n(a, b, c, d, x) = W(\partial_1 \varphi(x), a, b, c, d, x).$ By Lemma 4.2 we can choose measurable $a_n^1, b_n^1, c_n^1, d_n^1 : [0, l] \to [-n, n]$ such that $W_n(x) := W(\partial_1 \varphi(x), a_n^1(x), b_n^1(x), \overline{R}(x), \omega(x), c_n^1(x), d_n^1(x)) = \min_{a, b, c, d \in [-n, n]} W_n(a, b, c, d, x).$ Then define $a_1 = a_n^1, b_1 = b_n^1, c_1 = c_n^1, d_1 = d_n^1$ and $a_n = a_{n-1} + W_{n-1} - W_{n-1}$. Then, now take $\tilde{a}(x) = \lim_{n \to \infty} a_n(x), b(x) = \lim_{n \to \infty} b_n(x), c(x) = \lim_{n \to \infty} c_n(x), d(x) = \lim_{n \to \infty} d_n(x).$} $a, b, c, d : [0, l] \to \mathbb{R}^3$ such that
\[
W_0(\partial_1 \varphi, \overline{R}, \omega^1) = W(\partial_1 \varphi, a, b, \overline{R}, \omega, c, d).
\]
Due to the coercivity assumption for $W$ we have that $a, b \in L^4(\Omega; \mathbb{R}^3)$, $c \in L^p(\Omega; \mathbb{R}^3)$ and thus
\[
\inf_{\theta, \eta \in L^1([0,l]; \mathbb{R}^3)} \int_{\Omega} W(\partial_1 \varphi, \theta, \eta, \overline{R}, \omega, \phi, \psi) \, dx
\]
\[
\leq \int_{\Omega} W_0(\partial_1 \varphi, \overline{R}, \omega^1) \, dx = \text{meas}(S) \int_0^l W_0(\partial_1 \varphi, \overline{R}, \omega^1) \, dx.
\]
Therefore,
\[
\tilde{I}(0)(\varphi, \overline{R}) \leq \text{meas}(S) \int_0^l W_0(\partial_1 \varphi, \overline{R}, \omega^1) \, dx - \int_0^l \mathcal{F} \cdot \varphi \, dx.
\]
Now we apply Theorem 3.5 for $\Omega \equiv [0, l]$ and $f \equiv W_0$ to obtain, for each $r > 1$, a sequence $(\varphi^r_k, \overline{R}^r_k)_k$ which satisfies
a) $\varphi^r_k(0) = 0$, $\varphi^r_k(l) = l$,
b) $\varphi^r_k \to \varphi$, $\overline{R}^r_k \to \overline{R}$ weakly in $W^{1,1}([0,l]; \mathbb{R}^3)$, i.e., $W^{1,1}([0,l]; \mathbb{R}^3)$,
c) $\int_0^l W_0(\partial_1 \varphi^r_k, \overline{R}^r_k, \omega^1_k) \to \int_0^l Q \cdot W_0(\partial_1 \varphi, \overline{R}, \omega^1)$.

We extend $(\varphi^r_k, \overline{R}^r_k)$ naturally to $\tilde{\varphi}_0$ and use the lower semicontinuity of $\Gamma$-limit to conclude that for an arbitrary $r > 1$ we have
\[
\tilde{I}(0)(\varphi, \overline{R}) \leq \text{meas}(S) \int_0^l W_0(\partial_1 \varphi^r_k, \overline{R}^r_k, \omega^1_k) \, dx - \int_0^l \mathcal{F} \cdot \varphi \, dx.
\]
Using the arbitrariness of $r$ and taking $r \to \infty$ we have the claim.
\[\square\]
Remark 4.13 Since every quasiconvex function is also a rank-one convex (see [15, Chapter 5]), in case of rods, for \( \mathbf{R} \) fixed, the quasiconvex envelope is the same as the convex envelope, i.e., \( QW_0 = CW_0 \), where \( CW_0 \) is given by

\[
CW_0(\mathbf{a}^1, \mathbf{R}, \mathbf{b}^1) = \sup \{q_{\mathbf{R}}(\mathbf{a}^1, \mathbf{b}^1) | q_{\mathbf{R}} : \mathbb{R}^6 \to \mathbb{R}, q_{\mathbf{R}} \text{ convex and } q_{\mathbf{R}} \leq W_0(\cdot, \mathbf{R}, \cdot) \}
\]

\[
= \inf \left\{ \sum_{i=1}^{7} \lambda_i W_0(\mathbf{a}^1_i, \mathbf{R}, \mathbf{b}^1_i) | \lambda_i \in [0, 1], \sum_{i=1}^{7} \lambda_i = 1, \sum_{i=1}^{7} \lambda_i a^1_i = \mathbf{a}^1, \sum_{i=1}^{7} \lambda_i b^1_i = \mathbf{b}^1 \right\}.
\]

Namely, when dealing with matrices with just one raw or just one column, convexity and quasi-convexity coincide. In general this does not hold and thus we can not use this result for justifying the plate model by \( \Gamma \)-convergence. That is the reason why in Theorem 4.6 (and Proposition 4.11 and 4.12) we have used the notion of quasiconvexity. Taking the convex envelope of the minimum function is also done in classical elasticity when we justify the string model (see [1]).

The following is a standard in \( \Gamma \)-analysis. We use Theorem 4.6 to characterize the asymptotic behavior of diagonal minimizing sequences of rescaled deformations \( (\varphi(\epsilon), \mathbf{R}(\epsilon)) \) for the sequence of rescaled energies \( I(\epsilon) \) which are such that \( I(\epsilon)(\varphi(\epsilon), \mathbf{R}(\epsilon)) \leq \inf_{(\varphi, \mathbf{R}) \in \Phi(\epsilon)} I(\epsilon)(\varphi, \mathbf{R}) + h(\epsilon) \), where \( h(\epsilon) \) is a positive function such that \( h(\epsilon) \to 0 \) when \( \epsilon \to 0 \).

**Theorem 4.14** The sequence \( (\varphi(\epsilon), \mathbf{R}(\epsilon)) \) is relatively compact in \( W^{1,s}(\Omega; \mathbb{R}^3) \times W^{1,p}(\Omega; \mathbb{R}^{3 \times 3}) \). Its limit points belong to \( \Phi^0 \) and are solutions of the minimization problem for \( I(0) \) which is the energy of the micropolar rod defined on \( \Phi^0 \) and given by

\[
I(0)(\varphi, \mathbf{R}) = \text{meas}(S) \int_0^1 CW_0(\partial_1 \varphi, \mathbf{R}, \omega^1) \, dx - \int_0^1 \mathcal{F} \cdot \varphi.
\]

**Proof.** Using (4.4) for the minimizing sequence and Lemma 4.9 implies that the sequence \( (\varphi(\epsilon), \mathbf{R}(\epsilon)) \) is bounded in \( W^{1,s}(\Omega; \mathbb{R}^3) \times W^{1,p}(\Omega; \mathbb{R}^{3 \times 3}) \). Therefore its limit points belong to \( \Phi^0 \). Let \((\varphi, \mathbf{R})\) be a limit point for the strong topology of \( L^s(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{R}^{3 \times 3}) \) (such \((\varphi, \mathbf{R})\) is also a limit point for the weak topology of \( W^{1,s}(\Omega; \mathbb{R}^3) \times W^{1,p}(\Omega; \mathbb{R}^{3 \times 3}) \). Now we take subsequence such that \((\varphi(\epsilon), \mathbf{R}(\epsilon)) \to (\varphi, \mathbf{R})\) strongly in \( L^s(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{R}^{3 \times 3}) \).

Let \((v, S)\) be an arbitrary element of \( L^s(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{R}^{3 \times 3}) \) and consider a sequence \((v(\epsilon), S(\epsilon))\) in \( L^s(\Omega; \mathbb{R}^3) \times L^p(\Omega; \mathbb{R}^{3 \times 3}) \) such that

\[
(v(\epsilon), S(\epsilon)) \to (v, S), \quad \tilde{I}(\epsilon)(v(\epsilon), S(\epsilon)) \to \tilde{I}(0)(v, S).
\]

Such a sequence exists by the definition of \( \Gamma \)-convergence. Since \( \tilde{I}(\epsilon)(\varphi(\epsilon), \mathbf{R}(\epsilon)) \leq \tilde{I}(\epsilon)(v(\epsilon), S(\epsilon)) + h(\epsilon) \) it follows that

\[
\tilde{I}(0)(\varphi, \mathbf{R}) \leq \liminf_{\epsilon \to 0} \tilde{I}(\epsilon)(\varphi(\epsilon), \mathbf{R}(\epsilon)) \leq \liminf_{\epsilon \to 0} (\tilde{I}(\epsilon)(v(\epsilon), S(\epsilon)) + h(\epsilon)) = \tilde{I}(0)(v, S).
\]

Thus all limit points of the minimizing sequence are minimizers of \( \tilde{I}(0) \). \( \square \)

**Remark 4.15** It is classical in the theory of \( \Gamma \)-convergence that the total energy functional of the minimizing sequence converges as well, in the sense that

\[
I(\epsilon)(\varphi(\epsilon), \mathbf{R}(\epsilon)) \to I(0)(\varphi, \mathbf{R}).
\]

**Remark 4.16** As in classical elasticity even if the three-dimensional problem does not have any solution, i.e., the minimum point can not be attained, the limit problem always has solutions due to the quasiconvexity of the energy density function (see [48]).
Remark 4.17 The objectivity of $W$ is used (in Proposition 3.11) just to conclude that $QW$ is continuous. On the other hand we could use the objectivity assumption to simplify the proof.

If $W : \mathbb{R}^{3 \times 3} \times SO(3) \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ is objective then there exists $\tilde{W} : \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ such that

$$W(A, R, B) = \tilde{W}(R^T A, R^T B).$$

It can easily be shown that then

$$W_0(a^1, \tilde{R}, b^1) = \tilde{W}_0(R^T a^1, R^T b^1), \quad \forall a^1, b^1 \in \mathbb{R}^3, R \in SO(3)$$

provided that the left hand side of the equality exists (for example if $W$ is continuous and coercive). Here $\tilde{W}_0 : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$\tilde{W}_0(a^1, b^1) = \min_{a^2, a^3 \in \mathbb{R}^3} \tilde{W}([a^1 a^2 a^3], [b^1 b^2 b^3]).$$

If $W_0$ is bounded from below and satisfies the growth condition then $\tilde{W}_0$ is also bounded and satisfies the growth condition. Adapting the proof of Proposition 3.11 it follows

$$QW_0(a, R, b) = Q\tilde{W}_0(R^T a, R^T b), \quad \forall a, b \in \mathbb{R}^3, R \in SO(3).$$

Since quasiconvexification is done, in fact, on a matrix with one column and six rows we have that $Q\tilde{W}_0 = C\tilde{W}_0$, where $C\tilde{W}_0$ is the convex envelope of the function $\tilde{W}_0 : \mathbb{R}^6 \rightarrow \mathbb{R}$, see Remark 4.13.

Remark 4.18 In the same way we could derive the plate model, i.e., if we take $\Omega_{\varepsilon} = S \times [-\frac{1}{2}, \frac{1}{2}]$ where $S \subset \mathbb{R}^2$ bounded with Lipschitz boundary and if we rescale the $\varepsilon$-problems on the canonical domain $\Omega_1 = S \times [-\frac{1}{2}, \frac{1}{2}]$ and look for the $\Gamma$-limit, we will obtain that the $\Gamma$-limit is given by

$$\tilde{I}(0)(\varphi, \tilde{R}) = \left\{ \begin{array}{l l} \int_S QW_0(\partial_1 \varphi, \partial_2 \varphi, \tilde{R}, \omega^1, \omega^2) \: dx_1 dx_2 - \int_S \mathcal{F} \cdot \varphi \: dx_1 dx_2, & \text{if } (\varphi, \tilde{R}) \in \Phi^0, \\ +\infty, & \text{otherwise} \end{array} \right.$$}

where

$$W_0(a^1, a^2, \tilde{R}, b^1, b^2) = \min_{a^3 \in \mathbb{R}^3} W([a^1 a^2 a^3], [b^1 b^2 b^3]),$$

$$\Phi^0 = \{(\varphi, \tilde{R}) \in W^{1,s}(\omega; \mathbb{R}^3) \times W^{1,p}(\omega; SO(3)) : \varphi(x_1, x_2) = (x_1, x_2, 0), (x_1, x_2) \in \partial S, \mathcal{F}(x_1, x_2) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x_1, x_2, x_3) \: dx_3.\}$$

Note that in this case $QW_0 \neq C\tilde{W}_0$, in general. Also, if we do the analysis with the function $\tilde{W}_0$ we have that $Q\tilde{W}_0 \neq C\tilde{W}_0$, in general.

5 The micropolar rod model quadratic in strains

Now we apply the general theory from the previous section to derive a micropolar rod model in the case of an isotropic energy quadratic in strains (which remains nonlinear). For a given energy density $W$, by Theorem 4.6, we need to compute $C\tilde{W}_0$. Instead, using the setting of Remark 4.17 the energy density is given by $\tilde{W}$ and we need to compute $C\tilde{W}_0$.

We introduce the strain measures

\begin{equation}
\begin{aligned}
\mathbf{U} &= \mathbf{R}^T \nabla \varphi, \\
\Gamma &= \mathbf{R}^T \omega = \mathbf{R}^T \left( \omega^1 \omega^2 \omega^3 \right).
\end{aligned}
\end{equation}

\footnote{It can be shown that $\Gamma = (\frac{1}{2} \epsilon_{kmn} \partial_i \mathbf{R}^n \cdot \mathbf{R})_{k,l=1,2,3}$, as defined in [18], where $\epsilon_{kmn}$ is the permutation symbol.}
Remark 5.1 In the case \( \mu_c \) some specific physical situations (see [35, 37]). Note that the linearization of the energy with see [25], As usual the elasticity constants are taken to satisfy the pointwise uniform positivity conditions,  

\[
\tilde{W}(\mathbf{U}, \Gamma) = \mu \| \text{sym} (\mathbf{U} - \mathbf{I}) \|^2 + \mu_c \| \text{skew} (\mathbf{U} - \mathbf{I}) \|^2 + \frac{\lambda}{2} (\text{tr} (\mathbf{U} - \mathbf{I}))^2 + \frac{\gamma + \beta}{2} \| \text{sym} \Gamma \|^2 + \frac{\gamma - \beta}{2} \| \text{skew} \Gamma \|^2 + \frac{\alpha}{2} (\text{tr} \Gamma)^2. \tag{5.2}
\]

As usual the elasticity constants are taken to satisfy the pointwise uniform positivity conditions, see [25],  

\[
3\lambda + 2\mu > 0, \quad \mu > 0, \quad \mu_c > 0, \quad 3\alpha + \beta + \gamma > 0, \quad \gamma + \beta > 0, \quad \gamma - \beta > 0. \tag{5.3}
\]

**Remark 5.1** In the case \( \mu_c = 0 \) the energy loses pointwise coercivity in the deformation gradient \( \nabla \varphi \) and it is not clear whether the minimization problem is well posed (see [33]). The case \( \mu_c = 0 \) is interesting because it avoids some inherent problems when relating the model to some specific physical situations (see [35, 37]). Note that the linearization of the energy with \( \mu_c = 0 \) causes that the problem for displacements decouples from the problem of microrotations, i.e., we obtain the infinitesimal theory of classical elasticity, if microrotations remain free at the boundary.

Since the pointwise coerciveness is important for the conclusion that the weak limit depends on just one variable (and even to conclude that there exists a limit of the minimizers of energy functional) we continue our analysis for the energies with positive Cosserat couple modulus \( \mu_c > 0 \).

For elasticity constants satisfying (5.3) the quadratic energy given in (5.2) is uniformly convex in strains which simplifies the analysis because then \( \tilde{W}_0 \), defined in Remark 4.17, is a convex function, by Lemma 5.2, and thus its convex envelope is trivial \( \tilde{C}\tilde{W}_0 = \tilde{W}_0 \) (in fact the rod model in this case can be derived directly like the plate model in [39]). In the case of the plate model we have \( Q\tilde{W}_0 = \tilde{W}_0 \); since \( \tilde{W}_0 \) is also convex.

**Lemma 5.2** Let \( f : \mathbb{R}^n \to \mathbb{R} \cup \{ +\infty \} \) be a convex function bounded below by \(-\beta\) and let \( 0 \leq m \leq n \). Then the function defined by  

\[
f_0(x_1, \ldots, x_m) = \inf_{x_{m+1}, \ldots, x_n \in \mathbb{R}} f(x_1, \ldots, x_n)
\]

is also convex.

**Proof.** Let us take an arbitrary \( (x^1_1, \ldots, x^1_m) \) and \( (x^2_1, \ldots, x^2_m) \) and \( \varepsilon > 0 \). There exist \( (x^1_{m+1}, \ldots, x^1_n) \) and \( (x^2_{m+1}, \ldots, x^2_n) \) such that  

\[
f(x^1_1, \ldots, x^1_n) \leq f_0(x^1_1, \ldots, x^1_m) + \varepsilon, \quad i = 1, 2.
\]

Now, for \( \lambda \in [0, 1] \) we have  

\[
f_0 (\lambda(x^1_1, \ldots, x^1_m) + (1 - \lambda)(x^2_1, \ldots, x^2_m)) \\
\leq f (\lambda(x^1_1, \ldots, x^1_m) + (1 - \lambda)(x^2_1, \ldots, x^2_m)) \\
\leq \lambda f(x^1_1, \ldots, x^1_m) + (1 - \lambda) f(x^2_1, \ldots, x^2_m) \\
\leq \lambda f_0(x^1_1, \ldots, x^1_m) + (1 - \lambda) f_0(x^2_1, \ldots, x^2_m) + \varepsilon.
\]

By the arbitrariness of \( \varepsilon \) we have the claim. \( \square \)

---

\footnote{Isotropic material should satisfy \( \tilde{W}(Q\mathbf{U}Q^T, Q\Gamma Q^T) = \tilde{W}(\mathbf{U}, \Gamma) \), for all \( Q \in \text{SO}(3) \) and \( \tilde{W}(\mathbf{U}, -\Gamma) = \tilde{W}(\mathbf{U}, \Gamma) \) (central symmetry).}
For given $\mathbf{a}^1, \mathbf{b}^1$ the unique minimizers of $\widetilde{W}$ are determined by direct calculation
\begin{equation}
\mathbf{a}_{\text{min}}^2 = \begin{bmatrix}
-\frac{\mu_0-\mu}{\mu_0+\mu}a_{21} \\
\frac{\lambda}{2(\lambda+\mu)}(a_{11} - 1) + 1
\end{bmatrix}, \quad \mathbf{a}_{\text{min}}^3 = \begin{bmatrix}
-\frac{\mu_0-\mu}{\mu_0+\mu}a_{31} \\
0
\end{bmatrix}, \quad \mathbf{b}_{\text{min}}^2 = \begin{bmatrix}
-\frac{\beta}{\gamma}b_{21} \\
\frac{\alpha}{2\alpha+\beta+\gamma}b_{11}
\end{bmatrix}, \quad \mathbf{b}_{\text{min}}^3 = \begin{bmatrix}
-\frac{\beta}{\gamma}b_{31} \\
0
\end{bmatrix},
\end{equation}
where $\mathbf{a}^1 = (a_{11}, a_{21}, a_{31})^T, \mathbf{b}^1 = (b_{11}, b_{21}, b_{31})^T$. Inserting the minimizers into $\widetilde{W}$ we obtain
\begin{equation}
\widetilde{W}_0(a^1, b^1) = \frac{1}{2} \mathcal{X}(a^1 - e^1) \cdot (a^1 - e^1) + \frac{1}{2} \mathcal{Y}b^1 \cdot b^1,
\end{equation}
where $\mathcal{X} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $\mathcal{Y} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ are linear operators given by
\begin{equation}
\mathcal{X} = \mu \begin{bmatrix}
\frac{3\lambda+2\mu}{\lambda+\mu} & 0 & 0 \\
0 & \frac{\mu_0+\mu}{\mu_0-\mu} & 0 \\
0 & 0 & \frac{2\mu_0+\mu}{\mu_0+\mu}
\end{bmatrix}, \quad \mathcal{Y} = (\beta + \gamma) \begin{bmatrix}
\frac{(\alpha+\beta+\gamma)}{(2\alpha+\beta+\gamma)} & 0 & 0 \\
0 & \frac{2\beta}{\gamma} & 0 \\
0 & 0 & \frac{\gamma-\beta}{\gamma}
\end{bmatrix}.
\end{equation}
Note that these linear operators will be used to give the constitutive law for the one-dimensional micropolar rod and are formally the same as in the case of the corresponding infinitesimal theory (see [3]). Also note that in the conformal case (see [43]), i.e., for $\gamma = \beta$ and $3\alpha + \beta + \gamma = 0$, the tensor $\mathcal{Y}$ is zero, so the curvature energy of the rod is absent. Note as well that this consideration is just formal as the conformal case does not comply with the assumptions (5.3). Now using Lemma 5.2 we obtain
\begin{equation}
Q\widetilde{W}_0(a^1, b^1) = \widetilde{W}_0(a^1, b^1) = \frac{1}{2} \mathcal{X}(a^1 - e^1) \cdot (a^1 - e^1) + \frac{1}{2} \mathcal{Y}b^1 \cdot b^1.
\end{equation}
Therefore, we only need to check the assumptions of Theorem 4.6 to obtain the following theorem.

**Theorem 5.3** Let $\Omega = [0, l] \times S$, where $S \subset \mathbb{R}^2$ is an open bounded set with Lipschitz boundary. Let $f \in L^2(\Omega, \mathbb{R}^3)$. Then the sequence of minimizers $(\varphi(\varepsilon), R(\varepsilon))$ of the functional
\begin{equation}
I(\varepsilon)(\varphi, R) = \int_\Omega (\varphi, R)^T [\partial_1 \frac{\partial_2 \varphi}{\varepsilon}, \partial_3 \frac{\partial_2 \varphi}{\varepsilon}, [\omega^1, \omega^2, \omega^3]] dx - \int_\Omega f \cdot \varphi dx,
\end{equation}
with $\widetilde{W}$ given by (5.2) and parameters which satisfy (5.3), in the space
\begin{equation}
\Phi(\varepsilon) = \{(\varphi, R) \in W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,2}(\Omega; \text{SO}(3)) : \varphi(x_1, x_2, x_3) = (x_1, \varepsilon x_2, \varepsilon x_3) \text{ on } \Gamma\}
\end{equation}
is bounded in $W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,2}(\Omega; \text{SO}(3))$. Its limit points are minimizers of the functional
\begin{equation}
I(0)(\varphi, R) = \text{meas}(S) \int_0^l (\varphi, R)^T [\partial_1 \varphi, R^T \omega^1] dx_1 - \int_0^l F \cdot \varphi dx
\end{equation}
in the space
\begin{equation}
\Phi^0 = \{(\varphi, R) \in W^{1,2}([0, l]; \mathbb{R}^3) \times W^{1,2}([0, l]; \text{SO}(3)) : \varphi(0) = 0, \varphi(l) = l \varepsilon_1\},
\end{equation}
where $\widetilde{W}_0$ is given by (5.6) and
\begin{equation}
F(x_1) = \int_0^l f(x_1, x_2, x_3) dx_2 dx_3.
\end{equation}
Moreover $I(\varepsilon)(\varphi(\varepsilon), R(\varepsilon)) \to \min_{(\varphi, R) \in \Phi^0} I(0)(\varphi, R)$. 23
Proof. For given parameters $\lambda, \mu, \mu_c, \alpha, \beta, \gamma$, let us take $c_1 < 2$ and $c_2 < 1$ such that $3\lambda + c_1\mu > 0$ and $3\alpha + c_2(\beta + \gamma) > 0$. Then $\tilde{W}$ satisfies
\[
\tilde{W}(\mathbf{U}, \mathbf{\Gamma}) \geq \frac{3\lambda + c_1\mu}{6} \operatorname{tr}(\mathbf{U} - \mathbf{I})^2 + \min \{ (2 - c_1)\frac{\mu}{2}, \mu_c \} ||\mathbf{U} - \mathbf{I}||^2 \\
+ \frac{3\alpha + c_2(\beta + \gamma)}{6} (\operatorname{tr} \mathbf{\Gamma})^2 + \min \left\{ (1 - c_2)\frac{\gamma + \beta}{2}, \frac{\gamma - \beta}{2} \right\} ||\mathbf{\Gamma}||^2.
\]

Therefore the function $W : \mathbb{R}^{3 \times 3} \times \text{SO}(3) \times \mathbb{R}^{3 \times 3} \to \mathbb{R}$ defined by

\[
W(\mathbf{A}, \mathbf{R}, \mathbf{B}) := \tilde{W}(\mathbf{R}^T \mathbf{A}, \mathbf{R}^T \mathbf{B})
\]
satisfies the conditions $W1), W2)$ and $W3)$ with $s = p = 2$ (recall that $||\mathbf{R}^T \mathbf{A} - \mathbf{I}|| = ||\mathbf{A} - \mathbf{R}||$, $||\mathbf{R}^T \mathbf{B}|| = ||\mathbf{B}||$, $||\mathbf{R}^T|| = ||\mathbf{R}|| = \sqrt{3}, \forall \mathbf{R} \in \text{SO}(3)$). Existence of minimizers of the functional $I(\varepsilon)$ is guaranteed by the coercivity and convexity in the variables $\nabla \varphi^\varepsilon, \omega^\varepsilon$ (see [33, 47, 48] and look for the solution on the physical domain $\Omega_\varepsilon$). Everything else is a consequence of Theorem 4.14, Remark 4.15, Remark 4.17 and Lemma 5.2. $\square$

Up to now we have proved that from any subsequence of minimizers $(\varphi(\varepsilon), \mathbf{R}(\varepsilon))$ of $I(\varepsilon)$ we can extract a further subsequence that satisfies
\[
\varphi(\varepsilon) \rightharpoonup \varphi \quad W^{1,2}(\Omega; \mathbb{R}^3), \quad \mathbf{R}(\varepsilon) \rightharpoonup \mathbf{R} \quad W^{1,2}(\Omega; \text{SO}(3)).
\]

Due to the coercivity property $W3)$ we can conclude even more. It will lead us to the first corrector of the limit $(\varphi, \mathbf{R})$. Note also that convexity of $\tilde{W}$ is essential for this.

**Proposition 5.4** Let the assumptions of Theorem 5.3 hold. If the sequence of the minimizers $(\varphi(\varepsilon), \mathbf{R}(\varepsilon))$ of $I(\varepsilon)$ weakly converges to $(\varphi, \mathbf{R})$ in $W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,2}(\Omega; \text{SO}(3))$ then we have
\[
\frac{\partial \varphi(\varepsilon)}{\varepsilon} \rightharpoonup \varphi^2 = \mathbf{R} \begin{bmatrix} -\frac{\mu - \mu_c}{\mu + \mu_c} R^2 \cdot \partial_1 \varphi \\ 0 \end{bmatrix},
\]
\[
\frac{\partial \varphi(\varepsilon)}{\varepsilon} \rightharpoonup \varphi^3 = \mathbf{R} \begin{bmatrix} -\frac{\mu - \mu_c}{\mu + \mu_c} R^3 \cdot \partial_1 \varphi \\ 0 \end{bmatrix},
\]
\[
\frac{\omega^2(\varepsilon)}{\varepsilon} \rightharpoonup \psi^2 = \mathbf{R} \begin{bmatrix} -\frac{\vartheta}{\gamma} R^2 \omega^1 \\ 0 \end{bmatrix},
\]
\[
\frac{\omega^3(\varepsilon)}{\varepsilon} \rightharpoonup \psi^3 = \mathbf{R} \begin{bmatrix} -\frac{\vartheta}{\gamma} R^3 \omega^1 \\ 0 \end{bmatrix},
\]
weakly in $L^2(\Omega; \mathbb{R}^3)$.

**Proof.** By the coercivity of $\tilde{W}$ (see the assumption $W3)$) the sequences $\frac{\partial \varphi(\varepsilon)}{\varepsilon}, \frac{\partial \varphi(\varepsilon)}{\varepsilon}, \frac{\omega^2(\varepsilon)}{\varepsilon}, \frac{\omega^3(\varepsilon)}{\varepsilon}$ are bounded in $L^2(\Omega; \mathbb{R}^3)$. Thus, there exist functions $\varphi^2, \varphi^3, \psi^2, \psi^3$ and a sequence in $(\varepsilon)_{\varepsilon > 0}$ (still denoted by $\varepsilon$) such that
\[
\frac{\partial \varphi_m(\varepsilon)}{\varepsilon} \rightharpoonup \varphi^2, \quad \frac{\partial \varphi_m(\varepsilon)}{\varepsilon} \rightharpoonup \varphi^3, \quad \frac{\omega^2(\varepsilon)}{\varepsilon} \rightharpoonup \psi^2, \quad \frac{\omega^3(\varepsilon)}{\varepsilon} \rightharpoonup \psi^3.
\]
weakly in \( L^2(\Omega; \mathbb{R}^3) \). Since \( \mathbf{R}(\varepsilon) \to \mathbf{R} \) strongly in \( L^2(\Omega; \mathbb{R}^9) \) we have that
\[
\mathbf{R}(\varepsilon)^T \frac{\partial \varphi(\varepsilon)}{\varepsilon} \to \mathbf{R}^T \hat{\varphi},
\]
\[
\mathbf{R}(\varepsilon)^T \frac{\partial^2 \varphi(\varepsilon)}{\varepsilon^2} \to \mathbf{R}^T \psi^2,
\]
\[
\mathbf{R}(\varepsilon)^T \frac{\partial^3 \varphi(\varepsilon)}{\varepsilon^3} \to \mathbf{R}^T \psi^3
\]
weakly in \( L^2(\Omega; \mathbb{R}^3) \). By the convexity of the function \( \tilde{W} \) (using the Mazur theorem) and the last claim in Theorem 5.3 we have that
\[
\int_\Omega \tilde{W}(\mathbf{R}^T [\partial_1 \varphi, \varphi^2, \varphi^3], \mathbf{R}^T [\omega^1, \psi^2, \psi^3])
\]
\[
\leq \lim_{\varepsilon \to 0} \int_\Omega \tilde{W}(\mathbf{R}(\varepsilon)^T [\partial_1 \varphi(\varepsilon) \frac{\partial^2 \varphi(\varepsilon)}{\varepsilon} \frac{\partial^3 \varphi(\varepsilon)}{\varepsilon^3}], \mathbf{R}(\varepsilon)^T [\omega^1(\varepsilon) \psi^2(\varepsilon) \psi^3(\varepsilon)])
\]
\[
= \int_\Omega \tilde{W_0}(\mathbf{R}^T \partial_1 \varphi, \mathbf{R}^T \omega^1).
\]
By the definition of \( \tilde{W_0} \) this is possible only if (see (5.4) and (5.5)):
\[
\mathbf{R}^T \varphi^2 = \begin{bmatrix}
- \frac{\mu_c - \mu}{\mu_c + \mu} R^2 \cdot \partial_1 \varphi & 0 \\
- \frac{\alpha}{2(\lambda + \mu)} R^1 \cdot \partial_1 \varphi - 1 & 1 
\end{bmatrix}, \quad \mathbf{R}^T \varphi^3 = \begin{bmatrix}
- \frac{\mu_c - \mu}{\mu_c + \mu} R^3 \cdot \partial_1 \varphi & 0 \\
- \frac{\alpha}{2(\lambda + \mu)} R^1 \cdot \partial_1 \varphi - 1 & 1 
\end{bmatrix},
\]
\[
\mathbf{R}^T \psi^2 = \begin{bmatrix}
- \frac{2}{\gamma} R^2 \omega^1 & 0 \\
- \frac{\alpha}{2\alpha + \beta + \gamma} R^1 \omega^1 & 0 
\end{bmatrix}, \quad \mathbf{R}^T \psi^3 = \begin{bmatrix}
- \frac{2}{\gamma} R^3 \omega^1 & 0 \\
- \frac{\alpha}{2\alpha + \beta + \gamma} R^1 \omega^1 & 0 
\end{bmatrix},
\]
a.e. in \( \Omega \). This gives the result. \( \Box \)

Note that in the conformal case, \( \gamma = \beta \) and \( 3\alpha + \beta + \gamma = 0 \), the first correctors do not depend on the micropolar coefficients.

As a consequence of this theorem we may conclude that for the function defined by
\[
\hat{\varphi}(\varepsilon) = \varphi + \varepsilon \varphi^1 = \varphi + \varepsilon(x_2 \varphi^2 + x_3 \varphi^3)
\]
and assumed additional regularity of \( \varphi \) (e.g., \( \varphi \in W^{2,2}([0, l]; \mathbb{R}^3) \)), the following convergence holds
\[
\nabla^\varepsilon (\varphi(\varepsilon) - \hat{\varphi}(\varepsilon)) \to 0 \text{ weakly in } L^2(\Omega; \mathbb{R}^3),
\]
where \( \nabla^\varepsilon = (\partial_1 \frac{1}{\varepsilon} \partial_2 \frac{1}{\varepsilon} \partial_3) \) (this convergence is more than \( \varphi(\varepsilon) \to \varphi \) in \( W^{1,2}(\Omega; \mathbb{R}^3) \)). This justifies the term correc for the function \( \varphi^1 \).

**Remark 5.5** By linearizing displacements, i.e., by introducing \( \varphi = \text{id} + u \) and \( \mathbf{R} \approx \mathbf{I} + \mathbf{A}_{\varphi} \), we obtain the same model as in [3] where the linear model is derived from three-dimensional linear micropolar elasticity. This commutation between the process of linearization and the process of reducing the dimension(s) by \( \Gamma \)-limit does not happen in classical elasticity for zero order loadings (see [13, 20]).

**Lemma 5.6** The minimization formulation of the micropolar rod model (5.8) implies the following variational formulation: find \( (\varphi, \mathbf{R}) \in \Phi^0 \) such that
\[
\text{meas}(S) \int_0^l \left( \mathbf{R} \nabla^T (\mathbf{R} \varphi - e_1) \cdot (v' + \varphi' \times w_1) + \mathbf{R} \nabla^T (\mathbf{R} \omega^1) \cdot w'_1 \right) dx_1
\]
\[
= \int_0^l \mathcal{F} \cdot v dx_1, \quad v \in H_0^1([0, l], \mathbb{R}^3), w_1 \in H^1([0, l], \mathbb{R}^3). \quad (5.10)
\]
**Proof.** Let us take \( v \in C_0^\infty([0, l], \mathbb{R}^3), w_1 \in C^\infty([0, l], \mathbb{R}^3) \). In order to prove the lemma we need to compute the Gateaux derivative of the energy functional in (5.8). For \( \varepsilon \in \mathbb{R} \) we set
\[
\hat{\varphi} = \varphi + \varepsilon v, \quad \hat{\mathbf{R}} = \varepsilon \omega \mathbf{R}.
\]
Let us denote, as usual, $A_{\omega} = \hat{R} \hat{R}^T$ and $A_{\omega} = \hat{R} \hat{R}^T$. Then
\[
\hat{R}^T (\varphi')^T = \hat{R}^T e^{-\varepsilon A_{w_1}} (\varphi' + \varepsilon v') = \hat{R}^T (I - \varepsilon A_{w_1}) (\varphi' + \varepsilon v')
= \hat{R}^T \varphi' + \varepsilon \hat{R}^T (v + \varphi' \times w_1) + O(\varepsilon^2),
\]
\[
A_{\hat{R}^T \omega_1} = \hat{R}^T A_{\omega_1} \hat{R} = \hat{R}^T \hat{R}' = \hat{R}^T e^{-\varepsilon A_{w_1}} ((\varepsilon A_{w_1})' \hat{R} + \varepsilon A_{w_1} \hat{R}')
= \hat{R}^T ((I - \varepsilon A_{w_1}) \varepsilon A_{w_1} + A_{\omega_1}) \hat{R} + O(\varepsilon^2)
= \varepsilon A_{\hat{R}^T w_1} + A_{\hat{R}^T \omega_1} + O(\varepsilon^2).
\]

Therefore using the density argument and the fact that $H^1_0([0,l]) \hookrightarrow C([0,l])$ we obtain the statement of the lemma.

\textbf{Remark 5.7} With the notation
\[ n = \hat{R} \lambda (\hat{R}^T \varphi' - e_1), \quad m = \hat{R} \lambda (\hat{R}^T \omega^1). \tag{5.11} \]
the variational equation (5.10) implies the differential form of the balance of forces and moments
\[ n' + f_a = 0, \quad m' + \varphi' \times n = 0, \tag{5.12} \]
where $f_a = \frac{1}{\text{meas}(S)} \mathcal{F}$. These equations are given on the reference domain; $n$ is the resultant contact force and $m$ is the resultant contact couple. The equations (5.11) are constitutive equations of the model. Moreover, the weak formulation (5.10) implies the boundary conditions
\[ m(0) = m(l) = 0, \quad \varphi(0) = 0, \quad \varphi(l) = l e_1. \tag{5.13} \]
Now, the differential formulation (5.11), (5.12) and (5.13) is formally equivalent to the weak formulation (5.10).

The obtained model described in the Lemma 5.6 and Remark 5.7 is the same one as the Antman-Cosserat rod model (see [7, Chapter VII]) if we identify the columns of the microrotation respectively with the directors $d_3, d_1, d_2$. The physical interpretation of the (macroscopic) directors $d_1, d_2$ is that they are the unit vectors which describe the deformation of the cross-section of the rod (and $d_3 = d_1 \times d_2$). On the other hand the microrotation matrix does not have clear macroscopic interpretation. In fact from the Proposition 5.4 and (5.9) we see that $\varphi$ models the deformation of the middle curve of the rod, while the first corrector $\varphi^1 = \varepsilon x_2 \phi^2 + \varepsilon x_3 \phi^3$ gives the leading order of the deformation of the cross-sections. Therefore we set
\[ d_2 = \frac{\phi^2}{\| \phi^2 \|}, \quad d_3 = \frac{\phi^3}{\| \phi^3 \|}, \quad d_1 = d_2 \times d_3 \]
as directors and express the model in the variables $\varphi, d_1, d_2, d_3$. As in [46] we also linearize the model about the (nonlinear) strains to obtain the final macroscopic model (i.e., we neglect all terms which are quadratic in components of $\hat{R}^T \varphi' - e_1, \hat{R}^T \omega^1$ and quadratic in derivatives of these components). Therefore
\[
d_1 = \frac{\mu_c - \mu}{\mu_c + \mu} (\hat{R}^T \cdot \varphi') \hat{R}^2 + \hat{R}^2 + \text{h.o.t.},
\]
\[
d_2 = \frac{\mu_c - \mu}{\mu_c + \mu} (\hat{R}^T \cdot \varphi') \hat{R}^3 + \hat{R}^3 + \text{h.o.t.},
\]
\[
d_3 = \hat{R}^2 - \frac{\mu_c - \mu}{\mu_c + \mu} (\hat{R}^2 \cdot \varphi') \hat{R}^2 - \frac{\mu_c - \mu}{\mu_c + \mu} (\hat{R}^2 \cdot \varphi') \hat{R}^2 + \text{h.o.t.},
\]

26
The internal energy density of the micropolar rod is then finally given by (see (5.6) and (5.8))

\[
d_1 \cdot \varphi' = \frac{2\mu_c}{\mu_c + \mu}(R^2 \cdot \varphi') + \text{h.o.t.},
\]

\[
d_2 \cdot \varphi' = \frac{2\mu_c}{\mu_c + \mu}(R^3 \cdot \varphi') + \text{h.o.t.},
\]

\[
d_3 \cdot \varphi' = R^1 \cdot \varphi' + \text{h.o.t.}
\]

From this we have (using the fact that \((d_1, d_2, d_3)\) is an orthonormal basis)

\[
R^1 = \frac{\mu_c - \mu}{2\mu_c} (d_1 \cdot \varphi') d_1 + \frac{\mu_c - \mu}{2\mu_c} (d_2 \cdot \varphi') d_2 + d_3 + \text{h.o.t.,}
\]

\[
R^2 = d_1 - \frac{\mu_c - \mu}{2\mu_c} (d_1 \cdot \varphi') d_3 + \text{h.o.t.,}
\]

\[
R^3 = d_2 - \frac{\mu_c - \mu}{2\mu_c} (d_2 \cdot \varphi') d_3 + \text{h.o.t.}
\]

Now we easily obtain

\[
\langle R^1 \rangle' = \frac{\mu_c - \mu}{2\mu_c} (d_1 \cdot \varphi')' d_1 + \frac{\mu_c - \mu}{2\mu_c} (d_2 \cdot \varphi')' d_2 + d_3' + \text{h.o.t.,}
\]

\[
\langle R^2 \rangle' = d_1' - \frac{\mu_c - \mu}{2\mu_c} (d_1 \cdot \varphi')' d_3 + \text{h.o.t.,}
\]

\[
\langle R^3 \rangle' = d_2' - \frac{\mu_c - \mu}{2\mu_c} (d_2 \cdot \varphi')' d_3 + \text{h.o.t.}
\]

Since we know \(\Gamma_{11} = R^2 \cdot R^3, \Gamma_{21} = R^3 \cdot R^1, \Gamma_{31} = R^1 \cdot R^2\) we easily deduce

\[
\Gamma_{11} = d_1' \cdot d_2 + \text{h.o.t.,}
\]

\[
\Gamma_{21} = d_2' \cdot d_3 - \frac{\mu_c - \mu}{2\mu_c} (d_2 \cdot \varphi')' + \text{h.o.t.,}
\]

\[
\Gamma_{31} = \frac{\mu_c - \mu}{2\mu_c} (d_1 \cdot \varphi')' - d_1' \cdot d_3 + \text{h.o.t.}
\]

Now we follow Antman (see [7]) and introduce the same strain measures. Because \(D = [d_1 \, d_2 \, d_3]\) is an orthogonal matrix the matrix \(D'D^T\) is skew-symmetric. Therefore there is a vector function \(w\) such that \(D' = A_w D\). Let \(w_j = w \cdot d_j, j = 1, 2, 3\). Then \(w_j = \frac{\epsilon_{jkl}}{2} d_k \cdot d_l\), where \(\epsilon_{jkl}\) is the permutation symbol. We also introduce \(v_j = \varphi' \cdot d_j\). The variables \(v_1\) and \(v_2\) measure shear of the vectors \(d_1\) and \(d_2\) with respect to \(\varphi'\), the variable \(v_3\) measure stretch, the variables \(w_1, w_2\) measure flexure and the variable \(w_3\) measures twist. Then

\[
\begin{bmatrix}
\Gamma_{11} \\
\Gamma_{21} \\
\Gamma_{31}
\end{bmatrix} =
\begin{bmatrix}
w_3 \\
w_1 \\
w_2
\end{bmatrix} + \frac{\mu_c - \mu}{2\mu_c} \begin{bmatrix} 0 \\ -v_2' \\ v_1'
\end{bmatrix} + \text{h.o.t.}
\]

Equations (5.14) are then

\[
\begin{bmatrix}
R^1 \cdot \varphi' \\
R^2 \cdot \varphi' \\
R^3 \cdot \varphi'
\end{bmatrix} = \begin{bmatrix} v_3 \\
\frac{\mu_c + \mu}{2\mu_c} v_1 \\
\frac{\mu_c + \mu}{2\mu_c} v_2
\end{bmatrix} + \text{h.o.t.}
\]

The internal energy density of the micropolar rod is then finally given by (see (5.6) and (5.8))

\[
\tilde{W}_m(v_1, v_2, v_3, v_1', v_2', w_1, w_2, w_3) = \frac{\mu}{2} \frac{3\lambda}{\lambda + \mu} (v_3 - 1)^2 + \frac{\mu(\mu + \mu_c)}{4\mu_c} v_1^2 + \frac{\mu(\mu + \mu_c)}{4\mu_c} v_2^2
\]

\[27\]
energies which are unbounded for \( \det \nabla \varphi \to 0^+ \)? For the general approach one would need to analyze the semicontinuity and the relaxation theorem for these energies (see \[9\] and the references therein for the classical elasticity). The other question concerns the hierarchy of models (i.e., what happens if we apply the loads which are of higher order, see \[20\] for classical elasticity).

Also, the analysis given here is not suitable for some special interesting choices of the material parameters. The first one is the case \( \mu_c = 0 \) (for the physical relevance see \[35\]). Let us just mention here that in this case the linearization of three-dimensional problem gives the classical linearized elasticity (for displacements). Since the quadratic, isotropic energy density function looses its pointwise coercivity for \( \mu_c = 0 \) one can not even obtain the boundedness (and thus the weak convergence) of the infimizing sequence. We feel that the hierarchy of models would

\[
+ \frac{(\beta + \gamma)(3\alpha + \beta + \gamma)}{2(2\alpha + \beta + \gamma)} w_3^2 + \frac{\gamma^2 - \beta^2}{2\gamma} \left( w_1 - \frac{\mu_c - \mu}{2\mu_c} v'_2 \right)^2 \\
+ \frac{\gamma^2 - \beta^2}{2\gamma} \left( w_2 + \frac{\mu_c - \mu}{2\mu_c} v'_1 \right)^2 + \text{h.o.t.}
\]

**Remark 5.8** If we write the resultant contact force \( \mathbf{n} \) and the resultant contact couple \( \mathbf{m} \) (see Remark 5.7) in the base \( \mathbf{d}_i, \mathbf{d}_2, \mathbf{d}_3 \) \( (m_i = \mathbf{m} \cdot \mathbf{d}_i, n_i = \mathbf{n} \cdot \mathbf{d}_i) \) from \( (5.11) \) we obtain

\[
n_1 = \mu v_1 + \text{h.o.t.}, \quad n_2 = \mu v_2 + \text{h.o.t.}, \quad n_3 = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} (v_3 - 1) + \text{h.o.t.}
\]

\[
m_1 = \frac{\gamma^2 - \beta^2}{\gamma} \left( w_1 - \frac{\mu_c - \mu}{2\mu_c} v'_2 \right) + \text{h.o.t.}, \quad m_2 = \frac{\gamma^2 - \beta^2}{\gamma} \left( w_2 + \frac{\mu_c - \mu}{2\mu_c} v'_1 \right) + \text{h.o.t.}
\]

\[
m_3 = \frac{(\beta + \gamma)(3\alpha + \beta + \gamma)}{2\alpha + \beta + \gamma} w_3 + \text{h.o.t.}
\]

Note that the constitutive equations in \([7, \text{Chapter VIII}]\) do not suppose any dependence on \( v'_1, v'_2 \). In the conformal case (see \[43\]) the contact couple \( \mathbf{m} \) is formally equal to zero, so the corresponding rod model is of the second order.

## 6 Discussion and conclusions

In order to use the classical approach (see \([27]\)) in justifying lower dimensional models for general energy density functions by means of \( \Gamma \)-convergence one needs two results: the relaxation theorem and the semicontinuity theorem. The second result is already proved by the authors in \([48]\). The first result we proved in this paper for energy density functions that are continuous and satisfy the \( p, s \)-growth condition. The relaxed functional preserves its integral shape and is given by quasiconvexification of the energy density function in the first and the last variable together (this is in analogy to classical elasticity). Moreover, for the derivation of the model the \( p, s \)-coercivity assumption on the energy density function is additionally needed.

For the applications, it is interesting to specialize the obtained models to the quadratic, isotropic energy density function. The plate model for this energy is already obtained by means of \( \Gamma \)-convergence in \([39]\) and by formal asymptotic approach in \([46]\). Since the energy density function is convex in strains, the analysis is simplified and one does not have to do the relaxation of the minimum function. The obtained rod model, after rewriting it in macroscopic unknowns and after appropriate approximation is done, is of the form of the classical Antman-Cosserat rod model, see \([7]\). Only for the special choice of the Cosserat couple modulus \( \mu_c = \mu \) microrotation and directors of the obtained model and the Antman-Cosserat rod model coincide (intuitively this is expected in general). For the choice of the parameter \( \mu_c \neq \mu \) some additional terms compared to the Antman-Cosserat rod model are obtained.

Several open questions remain. Can the (general) approach be given for the more realistic energies which are unbounded for \( \det \nabla \varphi \to 0^+ \)? For the general approach one would need to analyze the semicontinuity and the relaxation theorem for these energies (see \[9\] and the references therein for the classical elasticity). The other question concerns the hierarchy of models (i.e., what happens if we apply the loads which are of higher order, see \[20\] for classical elasticity).

\[
\gamma = \frac{\lambda}{\lambda + 2\mu}, \quad \alpha = \frac{3\lambda - 2\mu}{6\mu}, \quad c = \frac{\mu - \sigma c}{2\mu_c} = 0
\]

\[
1 = \frac{\gamma}{\lambda + 2\mu}, \quad 2 = \frac{\alpha c}{3\lambda - 2\mu}, \quad 3 = \frac{c}{\mu - \sigma c}
\]

\[
\gamma = \frac{\lambda}{\lambda + 2\mu}, \quad \alpha = \frac{3\lambda - 2\mu}{6\mu}, \quad c = \frac{\mu - \sigma c}{2\mu_c} = 0
\]

\[
1 = \frac{\gamma}{\lambda + 2\mu}, \quad 2 = \frac{\alpha c}{3\lambda - 2\mu}, \quad 3 = \frac{c}{\mu - \sigma c}
\]

\[
\gamma = \frac{\lambda}{\lambda + 2\mu}, \quad \alpha = \frac{3\lambda - 2\mu}{6\mu}, \quad c = \frac{\mu - \sigma c}{2\mu_c} = 0
\]

\[
1 = \frac{\gamma}{\lambda + 2\mu}, \quad 2 = \frac{\alpha c}{3\lambda - 2\mu}, \quad 3 = \frac{c}{\mu - \sigma c}
\]

\[
\gamma = \frac{\lambda}{\lambda + 2\mu}, \quad \alpha = \frac{3\lambda - 2\mu}{6\mu}, \quad c = \frac{\mu - \sigma c}{2\mu_c} = 0
\]

\[
1 = \frac{\gamma}{\lambda + 2\mu}, \quad 2 = \frac{\alpha c}{3\lambda - 2\mu}, \quad 3 = \frac{c}{\mu - \sigma c}
\]
be especially interesting in this case. Loss of coercivity of the energy density function also happens for the choices of parameters suggested in [43, 26] (i.e., the symmetric case $\gamma = \beta$ and the conformal case $\gamma = \beta$, $\alpha = -\frac{2}{3}\gamma$). All of these cases have a well-posed three-dimensional linearized theory (although the linear elliptic form is not pointwise coercive) and are physically relevant (see [25]).

References


