The CB-norm approximation of generalized skew derivations by elementary operators

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Throughout, $R$ will be a semiprime ring, i.e. $R$ is an associative ring such that the zero-ideal is the intersection of prime ideals, or equivalently, if for every $a \in R$, $aRa = 0$ implies $a = 0$. 

Examples of semiprime rings:
- Any prime ring is obviously a semiprime ring.
- Any reduced ring is a semiprime ring.
- Any $J$-semisimple ring is semiprime. In particular, semisimple rings, von Neumann regular rings and $C^*$-algebras are all semiprime.
- Any direct product of semiprime rings is semiprime.
- Any matrix ring over a semiprime ring is a semiprime ring.
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- Any matrix ring over a semiprime ring is a semiprime ring.
An interesting class of additive maps $d : R \to R$ including both ring epimorphisms and generalized derivations is the class of \textit{generalized skew derivations}, that is, those satisfying

$$d(xy) = \delta(x)y + \sigma(x)d(y) \quad \forall x, y \in R,$$

for some map $\delta : R \to R$ and a ring epimorphism $\sigma : R \to R$. Since $R$ is semiprime, it is easy to see that the map $\delta$ is automatically additive and it is uniquely determined by $d$. Moreover, $\delta$ is a $\sigma$-\textit{derivation} (skew-derivation), i.e. $\delta$ satisfies

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$$\delta(xy) = \delta(x)y + \sigma(x)\delta(y) \quad \forall x \in R.$$

We decompose $d$ as

$$d = \delta + \rho,$$

where $\rho := d - \delta$. Note that $\rho$ is a left $R$-module $\sigma$-homomorphism, that is

$$\rho(xy) = \sigma(x)\rho(y) \quad \forall x \in R.$$
On the other hand, an attractive and fairly large class of additive maps \( \phi : R \to R \) is the class of \textbf{elementary operators}, that is, those that can be expressed as a finite sum

\[
\phi = \sum_i M_{a_i,b_i}
\]

of two-sided multiplications \( M_{a_i,b_i} : x \mapsto a_i xb_i \) such that \( a_i, b_i \) are elements of the multiplier ring \( M(R) \).

Motivated by the fact that elementary operators comprise both inner automorphisms \( x \mapsto pxp^{-1} \) and inner generalized derivations \( x \mapsto qx + xr \), we consider the following question:

\textbf{Problem 1.} Describe the form of generalized skew derivations that are simultaneously elementary operators.
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Motivated by the fact that elementary operators comprise both inner automorphisms \( x \mapsto pxp^{-1} \) and inner generalized derivations \( x \mapsto qx + xr \), we consider the following question:

**Problem 1.**

Describe the form of generalized skew derivations that are simultaneously elementary operators.
Recall that an **essentially defined double centralizer** on $R$ is a triple $(\mathcal{L}, \mathcal{R}, I)$, where $I$ is an essential ideal of $R$, $\mathcal{L} : I \rightarrow R$ is a left $R$-module homomorphism, $\mathcal{R} : I \rightarrow R$ is a right $R$-module homomorphism such that

$$\mathcal{L}(x)y = x\mathcal{R}(y) \quad \forall x, y \in I.$$
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One can form the **symmetric ring of quotients** $Q_s = Q_s(R)$ of $R$ which is characterized (up to isomorphism) by the following properties:

1. **(i)** $R$ is a subring of $Q_s$;
2. **(ii)** for any $q \in Q_s$ there is an essential ideal $I$ of $R$ such that $qI + lq \subseteq R$;
3. **(iii)** if $0 \neq q \in Q_s$ and $I$ is an essential ideal of $R$, then $qI \neq 0$ and $lq \neq 0$;
4. **(iv)** for any essentially defined double centralizer $(\mathcal{L}, \mathcal{R}, I)$ on $R$ there exists $q \in Q_s$ such that $\mathcal{L}(x) = qx$ and $\mathcal{R}(x) = xq$ for all $x \in I$. 
The center of $Q_s$ is called the **extended centroid** of $R$ and it is denoted by $C$. It is well known that $C$ is a von Neumann regular self-injective ring. Moreover, $C$ is a field if and only if $R$ is a prime ring.
The center of $Q_s$ is called the **extended centroid** of $R$ and it is denoted by $C$. It is well known that $C$ is a von Neumann regular self-injective ring. Moreover, $C$ is a field if and only if $R$ is a prime ring.

For any subset $S \subseteq Q_s$ there exists a unique idempotent $\varepsilon(S)$ in $C$ such that

$$\text{ann}_{Q_s}(Q_s SQ_s) = (1 - \varepsilon(S))Q_s \quad \text{and} \quad \varepsilon(S)x = x \quad \forall x \in S,$$

where for $X \subset Q_S$, $\text{ann}_{Q_s}(X)$ denotes the (two-sided) annihilator of $X$ in $Q_s$. The idempotent $\varepsilon(S)$ is called the **central support** of $S$. Whenever $S = \{x\}$ for some $x \in Q_s$ we write $\varepsilon(x)$ for $\varepsilon(S)$. 
Theorem 1.1 (Eremita, G., Ilišević; 2014).

If a generalized $\sigma$-derivation $d$ of $R$ is also a generalized elementary operator, then there are elements $q, r \in Q_s$ such that $d(x) = qx + \sigma(x)r$ for all $x \in R$, and for every such pair of elements $(q, r)$ there is an invertible element $p \in Q_s^\times$ such that

$$\varepsilon(r)\sigma(x) = \varepsilon(r)pxp^{-1} \quad \forall x \in R.$$ 

In particular,

$$d(x) = qx + pxp^{-1}r \quad \forall x \in R.$$
Results

In a case when $R = A$ is a $C^*$-algebra we now that all elementary operators $\phi : A \to A$ are completely bounded, i.e.

$$\|\phi\|_{cb} := \sup_{n \in \mathbb{N}} \|\phi_n\| < \infty,$$

where for each $n$, $\phi_n$ is an induced map on $M_n(A)$; $\phi_n([a_{ij}]) = [\phi(a_{ij})]$. 
Results

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where for each $n$, $\phi_n$ is an induced map on $M_n(A)$; $\phi_n([a_{ij}]) = [\phi(a_{ij})]$. Moreover, we have the the following estimate for their cb-norm:

$$\left\| \sum_i M_{a_i,b_i} \right\|_{cb} \leq \left\| \sum_i a_i \otimes b_i \right\|_h,$$

where $\| \cdot \|_h$ is the Haagerup tensor norm on the algebraic tensor product $M(A) \otimes M(A)$, i.e.

$$\|u\|_h = \inf \left\{ \left\| \sum_i a_i a_i^* \right\|^{1/2} \left\| \sum_i b_i^* b_i \right\|^{1/2} : u = \sum_i a_i \otimes b_i \right\}.$$
Hence, if by $\mathcal{E}\ell(A)$ we denote the set of all elementary operators on $A$ and by $\text{CB}(A)$ the set of all completely bounded operators on $A$, inequality (1) implies that the mapping

$$(M(A) \otimes M(A), \| \cdot \|_h) \rightarrow (\mathcal{E}\ell(A), \| \cdot \|_{cb})$$

given by

$$\sum_i a_i \otimes b_i \mapsto \sum_i M_{a_i,b_i}.$$ 

is a well-defined contraction. Its continuous extension to the completed Haagerup tensor product $M(A) \otimes_h M(A)$ is known as a canonical contraction from $M(A) \otimes_h M(A)$ to $\text{CB}(A)$ and is denoted by $\Theta_A$. 
Theorem 2.1 (Mathieu).

\[ \Theta_A \text{ is isometric if and only if } A \text{ is a prime } C^*\text{-algebra.} \]
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\( \Theta_A \) is isometric if and only if \( A \) is a prime \( C^* \)-algebra.

Remark 2.2.

If \( A \) is not a prime \( C^* \)-algebra, note that \( \Theta_A \) is not even injective.
**Theorem 2.1 (Mathieu).**

$\Theta_A$ is isometric if and only if $A$ is a prime $C^*$-algebra.

**Remark 2.2.**

If $A$ is not a prime $C^*$-algebra, note that $\Theta_A$ is not even injective.

Motivated by the observation that elementary operators on $C^*$-algebras are completely bounded we can consider the following problem:

**Problem 2.**

If $A$ is a $C^*$-algebra, describe all generalized skew derivations $d : A \to A$ that can be approximated in the cb-norm by elementary operators of $A$, that is, that lie in the cb-norm closure $\overline{E\ell(A)}_{cb}$. 
In the case when $A$ is a prime $C^*$-algebra we have the following result:

Theorem 2.3 (G.; preprint).

Let $A$ be a prime $C^*$-algebra and let $d : A \to A$ be a generalized $\sigma$-derivation. The following conditions are equivalent:

(i) $d \in \mathcal{E}\ell(A)_{cb}$.

(ii) Either $d$ is a left multiplication implemented by some element of $M(A)$ or there is an invertible element $a \in M(A)$ such that $\sigma = Ad(a)$ (so that $\sigma$ is an inner automorphism of $A$). In the latter case, if $d = \delta + \rho$ is a usual decomposition, then there are elements $b, c \in M(A)$ such that

$$
\delta(x) = cx - \sigma(x)c = cx - axa^{-1}c \quad \text{and} \quad \rho(x) = axa^{-1}b \quad \forall x \in A,
$$

so that

$$
d(x) = cx + axa^{-1}(b - c) \quad \forall x \in A.
$$
Remark 2.4.

Note that any left multiplier \( d = M_{c,1} \), where \( c \in M(A) \), is a generalized \( \sigma \)-derivation with respect to any ring epimorphism \( \sigma : A \to A \), since for any such \( \sigma \) we have \( d(x) = cx - \sigma(x)c + \sigma(x)c \). Hence, if \( \delta(x) = cx - \sigma(x)c \) and \( \rho(x) = \sigma(x)c \), then clearly \( \delta \) is a \( \sigma \)-derivation of \( A \) and \( \rho \) is a left \( \sigma \)-module homomorphism of \( A \) such that \( d = \delta + \rho \). Therefore in this case we cannot say anything about the epimorphism \( \sigma \) and corresponding maps \( \delta \) and \( \rho \).
Remark 2.4.

Note that any left multiplier $d = M_{c,1}$, where $c \in M(A)$, is a generalized $\sigma$-derivation with respect to any ring epimorphism $\sigma : A \to A$, since for any such $\sigma$ we have $d(x) = cx - \sigma(x)c + \sigma(x)c$. Hence, if $\delta(x) = cx - \sigma(x)c$ and $\rho(x) = \sigma(x)c$, then clearly $\delta$ is a $\sigma$-derivation of $A$ and $\rho$ is a left $\sigma$-module homomorphism of $A$ such that $d = \delta + \rho$.

Therefore in this case we cannot say anything about the epimorphism $\sigma$ and corresponding maps $\delta$ and $\rho$.

For the proof of Theorem 2.3 we will need some auxiliary results.

Lemma 2.5.

Let $A$ be a prime $C^*$-algebra. Suppose that $\sigma : A \to A$ is a ring epimorphism for which there exists a non-zero element $a \in M(A)$ such that

$$ax = \sigma(x)a \quad \forall x \in A. \quad (3)$$

Then $a \in M(A)^\times$, so that $\sigma = \text{Ad}(a)$ is an inner automorphism of $A$. 
Before proving Lemma 2.5, first recall that in a case when $R = A$ is a
$C^*$-algebra, $Q_s(A)$ has a natural structure as a unital complex $*$-algebra,
whose involution is positive definite. An element $q \in Q_s(A)$ is called
**bounded** if there is $\lambda \in \mathbb{R}_+$ such that $q^* q \leq \lambda 1$ in a sense that there is a
finite number of elements $q_1, \ldots, q_n \in Q_s(A)$ such that

$$q^* q + \sum_{i=1}^{n} q_i^* q_i = \lambda 1.$$
Before proving Lemma 2.5, first recall that in a case when \( R = A \) is a \( \mathrm{C}^\ast \)-algebra, \( Q_s(A) \) has a natural structure as a unital complex \( \ast \)-algebra, whose involution is positive definite. An element \( q \in Q_s(A) \) is called \textbf{bounded} if there is \( \lambda \in \mathbb{R}_+ \) such that \( q^\ast q \leq \lambda 1 \) in a sense that there is a finite number of elements \( q_1, \ldots, q_n \in Q_s(A) \) such that

\[
q^\ast q + \sum_{i=1}^{n} q_i^\ast q_i = \lambda 1.
\]

The set \( Q_b(A) \) of all bounded elements of \( Q_s(A) \) has a pre-\( \mathrm{C}^\ast \)-algebra structure with respect to the norm

\[
\| q \|^2 = \inf \{ \lambda \in \mathbb{R} : q^\ast q \leq \lambda 1 \},
\]

which clearly extends the norm of \( A \). One can easily check that an element \( q \in Q_s(A) \) is bounded if and only if it can be represented by a bounded (continuous) essentially defined double centralizer. We call \( Q_b(A) \) the \textbf{bounded symmetric algebra of quotients} of \( A \) and its completion \( M_{\mathrm{loc}}(A) \) the \textbf{local multiplier algebra} of \( A \). Note that \( M_{\mathrm{loc}}(A) \) has a structure of a \( \mathrm{C}^\ast \)-algebra as a completion of a pre-\( \mathrm{C}^\ast \)-algebra.
Proof of Lemma 2.5. First note that such an element $a$ cannot be a zero-divisor. Indeed, if there exists $x \in M(A)$ such that $ax = 0$ then for each $y \in A$ we have $ayx = \sigma(y)ax = 0$, so that $aAx = 0$. Since $A$ is prime, $A$ is an essential ideal of $M(A)$ and $a \neq 0$, we conclude that $x = 0$. Similarly, if $xa = 0$ then for all $y \in A$ we have $xya = x\sigma(y) = 0$. Since $\sigma$ is surjective, this is equivalent to $xAa = 0$, which again implies $x = 0$. 
Proof of Lemma 2.5. First note that such an element $a$ cannot be a zero-divisor. Indeed, if there exists $x \in M(A)$ such that $ax = 0$ then for each $y \in A$ we have $ayx = \sigma(y)ax = 0$, so that $aAx = 0$. Since $A$ is prime, $A$ is an essential ideal of $M(A)$ and $a \neq 0$, we conclude that $x = 0$. Similarly, if $xa = 0$ then for all $y \in A$ we have $xya = xa\sigma(y) = 0$. Since $\sigma$ is surjective, this is equivalent to $xAa = 0$, which again implies $x = 0$.

We now show that $a$ is invertible in $M(A)$. Since $M(A)$ is a unital $C^*$-subalgebra of $M_{\text{loc}}(A)$, we have $M(A)^\times = M(A) \cap M_{\text{loc}}(A)^\times$, so it suffices to show that $a$ is invertible in $M_{\text{loc}}(A)$. In order to do this, first note that $aA$ is a non-zero ideal of $A$, hence essential, since $A$ is prime. Indeed, since $\sigma$ is surjective, we have $A = \sigma(A)$, hence

$$AaA = \{\sigma(x)ay : x, y \in A\} = \{axy : x, y \in A\} \subseteq aA.$$ 

We define maps $\mathcal{L}, \mathcal{R} : aA \rightarrow A$ by

$$\mathcal{L}(ax) = \sigma(x) \quad \text{and} \quad \mathcal{R}(ax) = x.$$ 

That the maps $\mathcal{L}$ and $\mathcal{R}$ are well-defined follows from the fact that $a$ is not a (left) zero-divisor. It is easy to check that $(\mathcal{L}, \mathcal{R}, aI)$ is an essentially defined double centralizer on $A$. 

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It is well known that all essentially defined double centralizers on prime C*-algebras are automatically continuous, so in particular there exists an element $b \in Q_b(A) \subseteq M_{loc}(A)$ such that

$$\sigma(x) = \mathcal{L}(ax) = axb = \sigma(x)ab \quad \text{and} \quad x = \mathcal{R}(ax) = bax$$

for all $x \in A$. Since $\sigma(A) = A$, this is equivalent to $A(1 - ab) = 0$ and $(1 - ba)A = 0$. Hence, $a$ is invertible in $M_{loc}(A)$ and $a^{-1} = b$. \qed
It is well known that all essentially defined double centralizers on prime \( C^* \)-algebras are automatically continuous, so in particular there exists an element \( b \in Q_b(A) \subseteq M_{\text{loc}}(A) \) such that

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Recall that a sequence \((a_n)\) in a \( C^* \)-algebra \( A \) such that the series \( \sum_{n=1}^{\infty} a_n^*a_n \) is norm convergent is said to be strongly independent if for every sequence \((\alpha_n) \in \ell^2\), equality \( \sum_{n=1}^{\infty} \alpha_n a_n = 0 \) implies \( \alpha_n = 0 \) for all \( n \).
It is well known that all essentially defined double centralizers on prime $C^*$-algebras are automatically continuous, so in particular there exists an element $b \in Q_b(A) \subseteq M_{\text{loc}}(A)$ such that

$$\sigma(x) = L(ax) = axb = \sigma(x)ab \quad \text{and} \quad x = R(ax) = bax$$

for all $x \in A$. Since $\sigma(A) = A$, this is equivalent to $A(1 - ab) = 0$ and $(1 - ba)A = 0$. Hence, $a$ is invertible in $M_{\text{loc}}(A)$ and $a^{-1} = b$. \hfill \Box

Recall that a sequence $(a_n)$ in a $C^*$-algebra $A$ such that the series $\sum_{n=1}^{\infty} a_n^*a_n$ is norm convergent is said to be strongly independent if for every sequence $(\alpha_n) \in \ell^2$, equality $\sum_{n=1}^{\infty} \alpha_n a_n = 0$ implies $\alpha_n = 0$ for all $n$.

**Remark 2.6.**

(i) Every tensor $t \in A \otimes_h A$ can be represented as convergent series $t = \sum_{n=1}^{\infty} a_n \otimes b_n$, where $(a_n)$ and $(b_n)$ are sequences in $A$ such that the series $\sum_{n=1}^{\infty} a_n a_n^*$ and $\sum_{n=1}^{\infty} b_n^* b_n$ are norm convergent. Moreover, the sequence $(b_n)$ can be chosen to be strongly independent.

(ii) If $t = \sum_{n=1}^{\infty} a_n \otimes b_n$ is a representation of $t$ as above, with $(b_n)$ strongly independent, then $t = 0$ if and only if $a_n = 0$, for all $n \in \mathbb{N}$. 

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**Corollary 2.7.**

Let $A$ be a prime $C^*$-algebra. If $(a_n)$, $(b_n)$ are sequences in $M(A)$ such that the series $\sum_{n=1}^{\infty} a_n a_n^*$ and $\sum_{n=1}^{\infty} b_n^* b_n$ are norm convergent, with $(b_n)$ strongly independent. If

$$\sum_{n=1}^{\infty} a_n x b_n = 0, \quad (4)$$

for all $x \in A$, then $a_n = 0$ for all $n \in \mathbb{N}$.
Corollary 2.7.

Let $A$ be a prime $C^*$-algebra. If $(a_n)$, $(b_n)$ are sequences in $M(A)$ such that the series $\sum_{n=1}^{\infty} a_n a_n^*$ and $\sum_{n=1}^{\infty} b_n^* b_n$ are norm convergent, with $(b_n)$ strongly independent. If

$$\sum_{n=1}^{\infty} a_n x b_n = 0,$$

(4)

for all $x \in A$, then $a_n = 0$ for all $n \in \mathbb{N}$.

Proof. If $t := \sum_{n=1}^{\infty} a_n \otimes b_n \in M(A) \otimes_h M(A)$, then (4) is equivalent to $\Theta_A(t) = 0$. Since $A$ is prime, by Mathieu’s theorem $\Theta_A$ is isometric (hence injective), so $t = 0$. The claim now follows form Remark 2.6. \qed
Prposition 2.8.

Let $A$ be a prime C*-algebra and let $\sigma : A \to A$ be a ring epimorphism. If $\rho : A \to A$ is a non-zero left $A$-module $\sigma$-homomorphism, then the following conditions are equivalent:

(i) $\rho \in \overline{\mathcal{E}\ell(A)}_{cb}$.

(ii) There are elements $a, p \in M(A)$, with $a$ invertible and $p \neq 0$, such that $\sigma = \text{Ad}(a) \in \text{InnAut}(A)$ and

$$\rho(x) = \sigma(x)p = axa^{-1}p$$

for all $x \in A$. 
Prposition 2.8.

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$$\rho(x) = \sigma(x)p = axa^{-1}p$$

for all $x \in A$.

Proof. Since $A$ is prime, by Mathieu’s theorem the canonical contraction $\Theta_A : M(A) \otimes_h M(A)$ is isometric. In particular, the image of $\Theta_A$ is closed in the cb-norm so $\mathcal{E}\ell(A)_{cb}$ coincides with the image of $\Theta_A$. Hence, there is a tensor $t \in M(A) \otimes_h M(A)$ such that $\rho = \Theta_A(t)$. We can write

$$t = \sum_{n=1}^{\infty} a_n \otimes b_n,$$

where $(a_n)$ and $(b_n)$ are sequences in $M(A)$ such that the series $\sum_{n=1}^{\infty} a_n a_n^*$ and $\sum_{n=1}^{\infty} b_n^* b_n$ are norm convergent, with $(b_n)$
strongly independent. Since \( \rho(xy) = \sigma(x)\rho(y) \), we have

\[
\sum_{n=1}^{\infty} (a_n x - \sigma(x) a_n) y b_n = 0
\]

for all \( x, y \in A \). Corollary 2.7 implies

\[
a_n x = \sigma(x) a_n
\]

for all \( n \in \mathbb{N} \). Since \( \rho \) is non-zero, there is \( n_0 \in \mathbb{N} \) such that \( a_{n_0} \neq 0 \). By the previous Lemma \( a := a_{n_0} \) is invertible in \( M(A) \). Hence \( \sigma = \text{Ad}(a) \) is an inner automorphism of \( A \). Finally, if \( p := \sum_{n=1}^{\infty} a_n b_n \in M(A) \), using (5) we get

\[
\rho(x) = \sum_{n=1}^{\infty} a_n x b_n = \sigma(x) \left( \sum_{n=1}^{\infty} a_n b_n \right) = \sigma(x) p = ax a^{-1} p.
\]

As a direct consequence of Proposition 2.8 we get:
Corollary 2.9.

If $A$ is a prime $C^*$-algebra then every ring epimorphism $\sigma : A \rightarrow A$ that lies in $E_\ell(A)_{cb}$ must be an inner automorphism of $A$.
**Corollary 2.9.**

If $A$ is a prime $C^*$-algebra then every ring epimorphism $\sigma : A \to A$ that lies in $\mathcal{E}\ell(A)_{cb}$ must be an inner automorphism of $A$.

In the proof of Theorem 2.3 we will also use the following technical result:

**Lemma 2.10.**

Let $B$ be a unital $C^*$-algebra and let $f, g, h : B \to B$ be any functions with $f \neq 0$. Suppose that for all $x \in B$ we have the following equality

$$f(x) \otimes 1 = \sum_{n=1}^{\infty} (a_n g(x) + h(x) a_n) \otimes b_n$$  \hspace{1cm} (6)

of tensors in $B \otimes B$, where $(a_n)$ and $(b_n)$ are sequences in $B$ such that the series $\sum_{n=1}^{\infty} a_n a_n^*$ and $\sum_{n=1}^{\infty} b_n^* b_n$ are norm convergent, with $(b_n)$ strongly independent. Then there is a non-zero element $c \in B$ such that

$$f(x) = cg(x) + h(x)c \hspace{1cm} \forall x \in B.$$  \hspace{1cm} (7)
Proof. Choose $x_0 \in B$ such that $f(x_0) \neq 0$ and let $\varphi \in B^*$ be an arbitrary bounded linear functional such that $\varphi(f(x_0)) \neq 0$. If for $x = x_0$ we act on the equality (6) with the right slice map $R_\varphi : B \otimes_h B \to B$, $R_\varphi : a \otimes b \mapsto \varphi(a)b$, we obtain

$$\varphi(f(x_0)) \cdot 1 = \sum_{n=1}^{\infty} \varphi(a_ng(x_0) + h(x_0)a_n)b_n.$$  \hspace{1cm} (8)

For $n \in \mathbb{N}$ let

$$\alpha_n := \frac{\varphi(a_ng(x_0) + h(x_0)a_n)}{\varphi(\delta(x_0))}. $$

Note that $(\alpha_n) \in \ell^2$, since bounded linear functionals on $C^*$-algebras are completely bounded and the series

$$\sum_{n=1}^{\infty} (a_ng(x_0) + h(x_0)a_n)(a_ng(x_0) + h(x_0)a_n)^*$$

is norm convergent. Then (8) can be rewritten as $\sum_{n=1}^{\infty} \alpha_n b_n = 1$, so by (6) we have

$$\sum_{n=1}^{\infty} \left( \alpha_nf(x) - a_ng(x) - h(x)a_n \right) \otimes b_n = 0 \quad \forall x \in B.$$
Consequently, since \((b_n)\) is strongly independent, we conclude that

\[
\alpha_n f(x) = a_n g(x) + h(x) a_n
\]

for all \(n \in \mathbb{N}\) and \(x \in B\). Since \(\sum_{n=1}^{\infty} \alpha_n b_n = 1\), there is some \(n_0 \in \mathbb{N}\) such that \(\alpha_{n_0} \neq 0\). If \(c := (1/\alpha_{n_0}) a_{n_0}\), then the above equation is obviously equivalent to (7). Also, \(c \neq 0\) since \(f \neq 0\). \(\square\)

**Proof of Theorem 2.3.** (ii) \(\implies\) (i). This is trivial (see also Remark 2.4).
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**Proof of Theorem 2.3.** (ii) \(\implies\) (i). This is trivial (see also Remark 2.4). (ii) \(\implies\) (i). Assume that \(d \in \mathcal{E}(A)_{cb}\) and that \(d\) is not a left multiplier implemented by some element of \(M(A)\). In particular \(d \neq 0\). Using the same arguments as before, we see that there is a tensor \(t \in M(A) \otimes_h M(A)\) such that \(d = \Theta_A(t)\). We can write \(t = \sum_{n=1}^{\infty} a_n \otimes b_n\), where \((a_n)\) and \((b_n)\) are sequences in \(M(A)\) such that the series \(\sum_{n=1}^{\infty} a_n a_n^*\) and \(\sum_{n=1}^{\infty} b_n^* b_n\) are norm convergent, with \((b_n)\) strongly independent. Using the functional identity \(d(xy) = \delta(x)y + \sigma(x)d(y)\) we get

\[
\delta(x)y = \sum_{n=1}^{\infty} (a_n x - \sigma(x) a_n) y b_n,
\]
or equivalently

\[ \Theta_A(\delta(x) \otimes 1) = \Theta_A \left( \sum_{n=1}^{\infty} (a_n x - \sigma(x) a_n) \otimes b_n \right) \]  

(9)

Since \( \Theta_A \) is isometric (hence injective), (9) is equivalent to the equality

\[ \delta(x) \otimes 1 = \sum_{n=1}^{\infty} (a_n x - \sigma(x) a_n) \otimes b_n \]

of tensors in \( M(A) \otimes_h M(A) \) for all \( x \in A \). If \( \delta = 0 \), then \( d \) must be a non-zero left \( A \)-module \( \sigma \)-homomorphism of \( A \) so the claim follows from Proposition 2.8. If \( \delta \neq 0 \), Lemma 2.10 implies that there is a non-zero element \( c \in M(A) \) such that

\[ \delta(x) = cx - \sigma(x)c \]

for all \( x \in A \).
If we make a usual decomposition \( d = \delta + \rho \), the map \( \rho' : A \to A \) defined by
\[
\rho'(x) := \rho(x) - \sigma(x)c = d(x) - cx
\]
is obviously a left \( A \)-module \( \sigma \)-homomorphism of \( A \) that lies in \( \mathcal{E}_\ell(A)_{cb} \) (since \( d \) does). Since, by assumption, \( d \) is not a left multiplier, \( \rho' \) is non-zero. Hence, by Proposition 2.8 there are elements \( a, p \in M(A) \) with \( a \) invertible and \( p \neq 0 \) such that \( \sigma = \text{Ad}(a) \) and
\[
\rho'(x) = \sigma(x)p = axa^{-1}p
\]
for all \( x \in A \). In particular, if we put \( b := p + c \) we get (2). \( \square \)
Final remarks

In my previous papers, I showed that any derivation of a $C^*$-algebra $A$ with $\text{Prim}(A)$ Hausdorff that lies in $\mathcal{E}l(A)_{cb}$ must be an inner derivation. This result was further extended for unital $C^*$-algebras whose every Glimm ideal is prime.
Final remarks

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In light of this, it is natural to ask if one can extend Corollary 2.9 in its original form (and consequently Theorem 2.3) for a similar class of $C^*$-algebras. However, this will no longer be possible. In fact, there are unital $C^*$-algebras $A$ of the form $C(X, M_n(\mathbb{C}))$, where $X$ is a suitable compact Hausdorff space, that admit outer automorphisms which are simultaneously elementary operators on $A$. 
Final remarks

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First recall that if $A$ is a unital homogeneous $C^*$-algebra with $X = \text{Prim}(A)$, there is an exact sequence

$$0 \longrightarrow \text{InnAut}^\ast(A) \longrightarrow \text{Aut}^\ast_{C(X)}(A) \xrightarrow{\eta} \check{H}^2(X; \mathbb{Z})$$

of abelian groups and that the image of $\eta$ is contained in the torsion subgroup of $\check{H}^2(X; \mathbb{Z})$. 
Example 3.1.

For $n \geq 2$ let $X_n$ be the projective unitary group $PU(n) = U(n)/S^1$. It is well known that $\tilde{H}^2(X_n; \mathbb{Z}) \cong \mathbb{Z}_n$, so that $\tilde{H}^2(X_n; \mathbb{Z})$ is a torsion group. Using this fact and some additional calculations, Kadison and Ringrose showed that for $A_n = C(X_n, M_n(\mathbb{C}))$ we have $\text{InnAut}^*(A) \subseteq \text{Aut}_C^*(X)(A)$. 

On the other hand, $\text{Aut}_{C(X_n)}(A_n) \subseteq \mathcal{E}\ell(A_n)$. This follows from the fact that for any homogeneous $C^*$-algebra $A$ with $X = \text{Prim}(A)$, any bounded $C_0(X)$-linear map $\phi : A \to A$ preserves all two-sided ideals of $A$ (i.e. $\phi(I) \subseteq I$ for any such ideal $I$). But, by Magajna’s theorem, any such map on $A$ is an elementary operator on $A$. 
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Problem 3.

Is (some variant of) Theorem 2.3 true for all boundedly centrally closed \( C^* \)-algebras (in particular for \( AW^* \)-algebras)?