Noncommutative branched coverings

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- *A is equipped with an involution, i.e. a map $* : A \to A$, $a \mapsto a^*$ satisfying the properties:*

$$(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*, \quad (ab)^* = b^* a^*, \quad \text{and} \quad (a^*)^* = a,$$

for all $a, b \in A$ and $\alpha, \beta \in \mathbb{C}$.

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  for all $a, b \in A$ *and* $\alpha, \beta \in \mathbb{C}$.
- *Norm* $\|\cdot\|$ *satisfies the* C^*-**identity**, *i.e.*

$$\|a^* a\| = \|a\|^2$$

  for all $a \in A$.

Remark

The C^*-identity is a very strong requirement. For instance, together with the spectral radius formula, it implies that the C^*-norm is uniquely determined by the algebraic structure: For all $a \in A$ we have

$$\|a\|^2 = \|a^*a\| = \sup\{|\lambda| \ : \ \lambda \in \sigma(a^*a)\},$$

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In the category of C^*-algebras, the natural morphisms are the $*$-**homomorphisms**, i.e. the algebra homomorphisms which preserve the involution. They are automatically contractive.

Commutative example

Let X be a CH (compact Hausdorff) space and let $C(X)$ be the set of all continuous complex-valued functions on X. Then $C(X)$ becomes a C^*-algebra with respect to the pointwise operations, involution $f^*(x) := \overline{f(x)}$, and max-norm $\|f\|_\infty := \sup\{|f(x)| \, : \, x \in X\}$. Obviously, $C(X)$ is a unital commutative C^*-algebra.

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- For each $a \in A$ let $\hat{a} : X \to \mathbb{C}$ be a function defined by $\hat{a}(\chi) := \chi(a)$. Then $\hat{a} \in C(X)$ and $\hat{a}(X) = \sigma(a)$ for all $a \in A$.

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- For each $a \in A$ let $\hat{a} : X \to \mathbb{C}$ be a function defined by $\hat{a}(\chi) := \chi(a)$. Then $\hat{a} \in C(X)$ and $\hat{a}(X) = \sigma(a)$ for all $a \in A$.
- The **Gelfand transform** of A is a map $\mathcal{G}_A : A \to C(X)$ defined by $\mathcal{G}(a) := \hat{a}$.

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  as follows:
  - ▷ The functor C sends a CH space X to the unital commutative C^*-algebra $C(X)$, and a continuous function $F : X \to Y$ to the unital $*$-homomorphism $C(F) : C(Y) \to C(X)$, $C(F)(f) := f \circ F$.

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  - ▷ The functor X sends a unital commutative C^*-algebra A to the space of characters $X(A)$, and a unital $*$-homomorphism $\phi : A \to B$ to the continuous function $X(\phi) : X(B) \to X(A)$, $X(\phi)(\chi) := \chi \circ \phi$.

Commutative Gelfand-Naimark theorem, 1943

$X \circ C \cong \mathrm{id}_{\mathbf{CH}}$ i $C \circ X \cong \mathrm{id}_{\mathbf{UCC}^*}$ (natural isomorphism of functors). In particular, the categories **CH** and **UCC*** are dual.

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In other words: By passing from the space X the function algebra $C(X)$, no information is lost. In fact, X can be recovered from $C(X)$.

Thus, topological properties of X can be translated into algebraic properties of $C(X)$, and vice versa, so the theory of C^*-algebras is often thought of as **noncommutative topology**.

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- If A C^*-algebra and X is a CH space, then $C(X, A)$ becomes a C^*-algebra with respect to the pointwise operations and max-norm.

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- If A C^*-algebra and X is a CH space, then $C(X, A)$ becomes a C^*-algebra with respect to the pointwise operations and max-norm.
- To every locally compact group G, one can associate a C^*-algebra $C^*(G)$. Everything about the representation theory of G is encoded in $C^*(G)$.
- The category of C^*-algebras is closed under the formation of direct products, direct sums, extensions, direct limits, pullbacks, pushouts, (some) tensor products, etc.

Definition

A **representation** *of a C^*-algebra A is a $*$-homomorphism $\pi : A \to \mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. A representation π is said to be* **irreducible** *if it has no nontrivial closed invariant subspaces (i.e. if $\mathcal{K}$ is a closed subspace of $\mathcal{H}$ such that $\pi(A)\mathcal{K} \subseteq \mathcal{K}$, then $\mathcal{K} = \{0\}$ or $\mathcal{K} = \mathcal{H}$).*

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Remark

Because of the previous theorem, C^*-algebras can be concretely defined to be norm closed self-adjoint subalgebras of bounded operators on some Hilbert space $\mathcal{H}$.

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- *The* **primitive spectrum** *of A is the set $\mathrm{Prim}(A)$ of primitive ideals of A equipped with the* **Jacobson topology***: If S is a set of primitive ideals, its closure is*

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Example

If $A = C(X)$, let $C_x(X) := \{f \in C(X) \ : \ f(x) = 0\}$ $(x \in X)$. Then $\mathrm{Prim}(C(X)) = \{C_x(X) \ : \ x \in X\}$. Moreover, the correspondence $x \mapsto C_x(X)$ defines a homeomorphism between X and $\mathrm{Prim}(C(X))$.

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Dauns-Hofmann theorem, 1968

Let A be a unital C^*-algebra. For each $P \in \mathrm{Prim}(A)$, let $q_P : A \to A/P$ be the quotient map. Then there is a $*$-isomorphism Φ_A of $C(\mathrm{Prim}(A))$ onto the center $Z(A)$ of A such that

$$q_P(\Phi_A(f)a) = f(P)q_P(a)$$

for all $f \in C(\mathrm{Prim}(A))$, $a \in A$ and $P \in \mathrm{Prim}(A)$.

$C(X)$-algebras

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The natural candidate for the base space X is $\mathrm{Prim}(A)$, the primitive spectrum of A. However, since the topology on $\mathrm{Prim}(A)$ can be awkward to deal with, a natural alternative is to find a compact Hausdorff space X (which will turn out to be a continuous image of $\mathrm{Prim}(A)$) over which A fibres in a nice way.

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Such algebras are known as $C(X)$-algebras and were introduced by G. Kasparov in 1988:

Definition

*Suppose that X is a compact Hausdorff space. A unital C^*-algebra A is said to be a $C(X)$-**algebra** if A is endowed with a unital $*$-homomorphism ψ_A from $C(X)$ to the centre of A.*

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An **upper semicontinuous** C^***-bundle** *is a triple* $\mathfrak{A} = (p, \mathcal{A}, X)$ *where* $\mathcal{A}$ *is a topological space with a continuous open surjection* $p : \mathcal{A} \to X$*, together with operations and norms making each* **fibre** $\mathcal{A}_x := p^{-1}(x)$ *into a* C^**-algebra, such that the following conditions are satisfied:*

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(A1) *The maps* $\mathbb{C} \times \mathcal{A} \to \mathcal{A}$, $\mathcal{A} \times_X \mathcal{A} \to \mathcal{A}$, $\mathcal{A} \times_X \mathcal{A} \to \mathcal{A}$ *and* $\mathcal{A} \to \mathcal{A}$ *given in each fibre by scalar multiplication, addition, multiplication and involution, respectively, are continuous (*$\mathcal{A} \times_X \mathcal{A}$ *denotes the Whitney sum over* X*).*

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(A3) *If $x \in X$ and if (a_i) is a net in $\mathcal{A}$ such that $\|a_i\| \to 0$ and $p(a_i) \to x$ in X, then $a_i \to 0_x$ in $\mathcal{A}$ (0_x denotes the zero-element of $\mathcal{A}_x$).*

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Example

If A is a C^*-algebra, then the simplest example of a continuous C^*-bundle is the **product bundle** over X with fibre A,

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By a **section** of an upper semicontinuous C^*-bundle $\mathfrak{A}$ we mean a map $s : X \to \mathcal{A}$ such that $p(s(x)) = x$ for all $x \in X$. We denote by $\Gamma(\mathfrak{A})$ the set of all continuous sections of $\mathfrak{A}$. Then $\Gamma(\mathfrak{A})$ becomes a $C(X)$-algebra with respect to the natural pointwise operations and sup-norm.

On the other hand, given a $C(X)$-algebra A, one can always associate an upper semicontinuous C^*-bundle $\mathfrak{A}$ over X such that $A \cong \Gamma(\mathfrak{A})$, as follows:

- Set $J_x := C_0(X \setminus \{x\}) \cdot A$ and note that J_x is a closed two-sided ideal in A (by Cohen factorization theorem). The quotient $A_x := A/J_x$ is called the **fibre** at the point x.

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- If $a \in A$, let a_x be the image of a in A_x. We define the map $\hat{a} : X \to \mathcal{A}$ by $\hat{a}(x) := a_x$. Let $\Omega := \{\hat{a} \ : \ a \in A\}$.

On the other hand, given a $C(X)$-algebra A, one can always associate an upper semicontinuous C^*-bundle $\mathfrak{A}$ over X such that $A \cong \Gamma(\mathfrak{A})$, as follows:

- Set $J_x := C_0(X \setminus \{x\}) \cdot A$ and note that J_x is a closed two-sided ideal in A (by Cohen factorization theorem). The quotient $A_x := A/J_x$ is called the **fibre** at the point x.
- Let
$$\mathcal{A} := \bigsqcup_{x \in X} A_x,$$
and let $p : \mathcal{A} \to X$ be the canonical associated projection.
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- For each $a \in A$ we have
$$\|a_x\| = \inf\{\| [1 - f + f(x)] \cdot a\| \ : \ f \in C(X)\}.$$
In particular, all norm functions $x \mapsto \|a_x\|$ $(a \in A)$ are upper semicontinuous on X.

Theorem (Fell & Lee)

There exists a unique topology on $\mathcal{A}$ *for which* $\mathfrak{A} := (p, \mathcal{A}, X)$ *becomes an upper semicontinuous* C^**-bundle such that* $\Omega = \Gamma(\mathfrak{A})$*. Moreover, the* **generalized Gelfand transform** $\mathcal{G} : a \mapsto \hat{a}$, $\mathcal{G} : A \to \Gamma(\mathfrak{A})$, *defines an isomorphism of* $C(X)$*-algebras.*

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Definition

If all norm functions $x \mapsto \|a_x\|$ $(a \in A)$ are continuous on X, we say that A is a **continuous** $C(X)$**-algebra**. *This is equivalent to say that the associated bundle $\mathfrak{A}$ is continuous.*

Example

Let D be any unital C^*-algebra. Then $A := C(X, D)$ becomes a continuous $C(X)$-algebra in a natural way:

$$\psi_A(f)(x) := f(x) \cdot 1_A \qquad (f \in C(X)).$$

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Example (Degenerate example)

Let A be any unital C^*-algebra and let us fix a point $x_0 \in X$. Then A becomes a $C(X)$-algebra via the map

$$\psi_A(f) := f(x_0) \cdot 1_A \qquad (f \in C(X)).$$

In this example, every fibre A_x is zero, except for $x = x_0$, where $A_{x_0} = A$.

Remark

To avoid such pathological examples, we shall always assume that the $*$-homomorphism ψ_A is injective. Then we may identify $C(X)$ with the C^*-subalgebra $\psi_A(C(X))$ of $Z(A)$.

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- For each $x \in X$, every fibre $C(Y)_x$ is $*$-isomorphic to $C(F^{-1}(x))$.
- $C(Y)$ is a continuous $C(X)$-algebra if and only if F is an open map.

In fact, the previous example is not nearly as specialized as it might seem at first:

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- *If there exists a continuous map $F_A : \mathrm{Prim}(A) \to X$, then A becomes a $C(X)$-algebra with*

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- *Moreover, every unital $C(X)$-algebra arises is this way.*
- *A $C(X)$-algebra A is continuous if end only if the associated map $F_A : \mathrm{Prim}(A) \to X$ is open.*

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Example

- $C(X, \mathbb{M}_n)$ is a (continuous) homogeneous $C(X)$-algebra with fibre $\mathbb{M}_n$.
- Let

$$A := \{f \in C([0,1], \mathbb{M}_n) \ : \ f(0) \text{ is a diagonal matrix}\}.$$

  Then A is a (continuous) $C([0,1])$-algebra with $A_0 = \mathbb{C}^n$ and $A_x = \mathbb{M}_n$ for $0 < x \leq 1$.

If D is a finite-dimensional C^*-algebra, recall that A is isomorphic to the finite direct sums of matrix algebras $\mathbb{M}_{n_i}$. We define the **rank** of D as

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  As in the finite-dimensional case, we call this number as **rank** of A.
- If A is continuous and homogeneous with fibre D, then by an important result of J. Fell from 1961, A is automatically locally trivial. This intuitively means that for every point $x \in X$ there exists a compact neighborhood U of x such that the restriction of A on U looks like $C(U, D)$.

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Theorem (Y. Tomiyama, 1957)

A map $E : A \to B$ is a C.E. if and only if E is a projection of norm one.

Definition

A C.E. $E : A \to B$ *is said to be of* **finite index** *(abbreviated C.E.F.I.) if there exists a constant* $K \geq 1$ *such that the map* $(K \cdot E - \mathrm{id}_A) : A \to A$ *is positive.*

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However, attempts to describe the more general situation of conditional expectations on C^*-algebras with arbitrary centers to be "of finite index" in some sense(s) went into difficulties. In fact, M. Baillet, Y. Denizeau and J.-F. Havet showed that even in the case of normal faithful conditional expectations E on W^*-algebras M with non-trivial centres, the index value can be calculated only in situations when there exists a number $L \geq 1$ such that the mapping $(L \cdot E - \mathrm{id}_A)$ is completely positive.

However, the following important result resolved this issue, and consequently justified the given definition for C.E. on general C^*-algebras to be of finite index:

Theorem (M. Frank and E. Kirchberg, 1998)

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(c) *A becomes a (complete) Hilbert B-module when equipped with the inner product $\langle a_1, a_2 \rangle := E(a_1^* a_2)$.*

Moreover, if

$$K(E) := \inf\{K \geq 1 \ : \ K \cdot E - \mathrm{id}_A \textit{ is positive}\},$$

$$L(E) := \inf\{L \geq 1 \ : \ L \cdot E - \mathrm{id}_A \textit{ is c.p.}\},$$

with $K(E) = \infty$ or $L(E) = \infty$ if no such number K or L exists, then

$$K(E) \leq L(E) \leq K(E)^2.$$

The number $K(E)$ is sometimes called the **weak index** of E.

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For a unital inclusion $A \subseteq B$ of unital C^*-algebras we can now introduce the following constant, which plays an important role in our research:

$$K(A, B) := \inf\{K(E) \ : \ E : A \to B \text{ is C.E.F.I.}\},$$

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Example

Let A be a homogeneous $C(X)$-algebra $C(X, \mathbb{M}_n)$ and let $\mathrm{tr}(\cdot)$ be the standard trace on $\mathbb{M}_n$. Then

$$E(f)(x) := \frac{1}{n}\mathrm{tr}(f(x))$$

defines a C.E.F.I. from A onto $C(X)$. In this case we have $K(A, C(X)) = K(E) = n$.

Noncommutative branched coverings

Definition

Let X and Y be two CH spaces. A **branched coverings** *is an open continuous surjection $\sigma : Y \to X$ with uniformly bounded number of pre-images, i.e.*

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Theorem (A. Pavlov i E. Troitsky, 2011)

A pair (X, Y) admits a branched covering $\sigma : Y \to X$ if and only if there exists a C.E.F.I. $E : C(Y) \to C(X)$.

In light of noncommutative topology, A. Pavlov and E. Troitsky introduced the following definition:

Definition

A **noncommutative branched covering** *is a pair* (A, B) *consisting of a* C^**-algebra* A *and its* C^**-subalgebra* B *with common identity element, such that there exists a C.E.F.I. from* A *onto* B*.*

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Reinterpretation in terms of $C(X)$-algebras

If $\sigma : Y \to X$ is a continuous surjection, then (as already described) $C(Y)$ becomes a $C(X)$-algebra via

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- $\sup_{x \in X} |\sigma^{-1}(x)| < \infty$ if and only if $C(Y)$ is a subhomogeneous $C(X)$-algebra.

Therefore, if A is a unital commutative $C(X)$-algebra, then a pair $(A, C(X))$ defines a noncommutative branched covering if and only if A is a continuous subhomogeneous $C(X)$-algebra.

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We managed to prove one direction:

Theorem (E. Blanchard & I.G., 2013)

Let A be a unital $C(X)$-algebra. If a pair $(A, C(X))$ defines a noncommutative branched covering, then A is necessarily a continuous subhomogeneous $C(X)$-algebra. Moreover, in this case we have $K(A, C(X)) \geq r(A)$.

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Moreover, in both this cases the equality $K(A, C(X)) = r(A)$ is achieved.

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As a direct consequence of part (A), we get:

Corollary

If a unital $C(X)$-algebra A admits a $C(X)$-linear embedding into some unital continuous homogeneous unital $C(X)$-algebra A', then $(A, C(X))$ defines a noncommutative branched covering with $K(A, C(X)) \leq K(A', C(X))$.

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- We exhibited an example of a continuous $C(X)$-algebra A with fibres $M_2(\mathbb{C})$ and $\mathbb{C}$, where X is the Alexandroff compactification of the disjoint union $\bigsqcup_{n=1}^{\infty} \mathbb{C}P^n$ of complex projective n-dimensional spaces, which does not admit a $C(X)$-linear embedding into any unital continuous homogeneous $C(X)$-algebra.

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- On the other hand, since A is of rank 2, the part (B) implies that the pair $(A, C(X))$ defines a noncommutative branched covering, with $K(A, C(X)) = 2$.