Derivations, elementary operators and local multipliers of C^*-algebras

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C^*-algebraic formulation of Quantum Mechanics

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- The self-adjoint elements of A are thought of as the observables; they are the measurable quantities of the system.
- A state of the system is defined as a positive functional on A (i.e. a linear map $\omega : A \to \mathbb{C}$ such that $\omega(a^*a) \geq 0$ for all $a \in A$) with $\omega(1_A) = 1$. If the system is in the state ω, then $\omega(a)$ is the expected value of the observable a.

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- Automorphisms correspond to the symmetries, while one-parameter automorphism groups $\{\Phi_t\}_{t\in\mathbb{R}}$ describe the reversible time evolution of the system (in the Heisenberg picture). Their infinitesimal generators

$$\delta(x) := \lim_{t\to 0} \frac{1}{t}(\Phi_t(x) - x)$$

are the $*$-derivations.

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Definition

The multiplier algebra *of A is the C^*-subalgebra $M(A)$ of the enveloping von Neumann algebra A^{**} that consists of all $x \in A^{**}$ such that $ax \in A$ and $xa \in A$ for all $a \in A$.*

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$M(A)$ is a unital extension of A in which A sits as an essential ideal. Moreover, $M(A)$ satisfies the following universal property: Whenever A sits as an ideal in a C^*-algebra B, then the identity map on A extends uniquely to a $*$-homomorphism from B to $M(A)$ with kernel $A^\perp$. Hence, $M(A)$ is the largest unitization of A.

Definition

Derivation *of* A *is a linear map* $\delta : A \to A$ *satisfying the* **Leibniz rule**

$$\delta(xy) = \delta(x)y + x\delta(y)$$

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Example

If A is a C^*-subalgebra of a C^*-algebra B, then each element $a \in B$ which derives A (i.e. $ax - xa \in A$, for all $x \in A$) implements a derivation $\delta_a : A \to A$ given by

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A derivation δ of A is said to be **inner** *if there exists a multiplier $a \in M(A)$ such that $\delta = \delta_a$.*

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- AW^*-algebras (Olesen, 1974).
- Homogeneous C^*-algebras (Sproston, 1976 - unital case; G., 2013 - extension to the non-unital case).

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- It is unknown whether all AW^*-algebras are monotone complete. In fact, this is a long standing open problem dating back to the work of Kaplansky.

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- More generally, if E is an algebraic $\mathbb{M}_n$-bundle over a locally compact Hausdorff space X, i.e. E is a locally trivial fibre bundle with fibre $\mathbb{M}_n$ and structure group $\mathrm{Aut}(\mathbb{M}_n) \cong PU(n)$ (the projective unitary group), then the set $\Gamma_0(E)$ of all continuous sections of E vanishing at infinity is an n-homogeneous C^*-algebra, with respect to the fiberwise operations and sup-norm.

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- By a wonderful theorem due to Fell and Tomiyama-Takesaki (1961), every n-homogeneous C^*-algebra A can be realized as $A = \Gamma_0(E)$ for some algebraic $\mathbb{M}_n$-bundle E over the spectrum $\widehat{A}$.

Back to our main problem, the separable case was completely solved in 1979:

Theorem (Akemann, Elliott, Pedersen and Tomiyama, 1979)

Let A be a separable C^*-algebra, Then the following conditions are equivalent:

(a) *A admits only inner derivations.*

(b) *$A = A_1 \oplus A_2$, where A_1 is a continuous-trace C^*-algebra, and A_2 is a direct sum of simple C^*-algebras.*

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On the other hand, for inseparable C^*-algebras the problem of innerness of derivations remains widely open, even for the simplest cases such as subhomogeneous C^*-algebras (i.e. C^*-algebras which have finite-dimensional irreducible representations of bounded degree).

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In this way, we obtain a directed system of C^*-algebras with isometric connecting morphisms, where I runs through the directed set $\mathrm{Id}_{ess}(A)$ of all essential ideals of A.

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Iterating the construction of $M_{\mathrm{loc}}(A)$, one obtains the following tower of C^*-algebras which, a priori, does not have the largest element:

$$A \subseteq M_{\mathrm{loc}}(A) \subseteq M_{\mathrm{loc}}^{(2)}(A) \subseteq \cdots \subseteq M_{\mathrm{loc}}^{(n)}(A) \subseteq \cdots,$$

where $M_{\mathrm{loc}}^{(2)}(A) = M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$, $M_{\mathrm{loc}}^{(3)}(A) = M_{\mathrm{loc}}(M_{\mathrm{loc}}^{(2)}(A))$, etc.

Remark

An easy, but invaluable fact is that $M_{\mathrm{loc}}(I) = M_{\mathrm{loc}}(A)$ for every essential ideal I of A. This is because $\{J \in \mathrm{Id}_{ess}(A) \ : \ J \subseteq I\}$ is cofinal in $\mathrm{Id}_{ess}(A)$.

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If $A = C_0(X)$ is a commutative C^*-algebra, then $M_{\rm loc}(A)$ is a commutative AW^*-algebra whose maximal ideal space can be identified with the inverse limit $\varprojlim \beta U$ of Stone-Čech compactifications βU of dense open subsets U of X.

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Pedersen proved that every derivation of a separable C^*-algebra A becomes inner when extended to a derivation of $M_{\text{loc}}(A)$. Moreover, it suffices to assume that every essential closed ideal of A is σ-unital.

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Pedersen proved that every derivation of a separable C^*-algebra A becomes inner when extended to a derivation of $M_{\mathrm{loc}}(A)$. Moreover, it suffices to assume that every essential closed ideal of A is σ-unital.

In particular, Pedersen's result entails Sakai's theorem that every derivation of a simple unital C^*-algebra is inner.

Since $M_{\mathrm{loc}}(A) = M(A)$ if A is simple, and $M_{\mathrm{loc}}(A) = A$ if A is an AW^*-algebra, only an affirmative answer in the inseparable case would cover, extend and unify the results that every derivation of a simple C^*-algebra is inner in its multiplier algebra and that all derivations of AW^*-algebras are inner.

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Problem of innerness of derivations of $M_{\text{loc}}(A)$

If A is an arbitrary C^*-algebra, is every derivation of $M_{\text{loc}}(A)$ inner?

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Stability problem of $M_{\mathrm{loc}}(A)$

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$I(A)$ is not an injective object in the category of C^*-algebras and $*$-homomorphisms, but in the category of operator spaces and complete positive maps, i.e. for every inclusion $E \subseteq F$ of operator systems, each completely positive map $\phi : E \to I(A)$ has a completely positive extension $\widetilde{\phi} : F \to I(A)$.

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However, it turns out that (nevertheless) $I(A)$ is a C^*-algebra canonically containing A as a C^*-subalgebra. Moreover, $I(A)$ is monotone complete, so in particular, $I(A)$ is an AW^*-algebra.

Theorem (Frank and Paulsen, 2003)

Under this embedding of A into $I(A)$, $M_{\mathrm{loc}}(A)$ is the norm closure of the set of all $x \in I(A)$ which act as a multiplier on some $I \in \mathrm{Id}_{ess}(A)$, i.e.

$$M_{\mathrm{loc}}(A) = \left(\bigcup_{I \in \mathrm{Id}_{ess}(A)} \{x \in I(A) \ : xI + Ix \subseteq I\} \right)^{=}$$

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Moreover, it can be seen that $I(M_{\mathrm{loc}}(A)) = I(A)$, so we have an additional sequence of inclusions of C^*-algebras:

$$A \subseteq M_{\mathrm{loc}}(A) \subseteq M_{\mathrm{loc}}^{(2)}(A) \subseteq \cdots \subseteq \overline{A} \subseteq I(A).$$

Very difficult problem

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- After that, Argerami, Farenick and Massey (2009) showed that a relatively well-behaved C^*-algebra $C([0,1]) \otimes \mathbb{K}$ also fails to satisfy $M_{\mathrm{loc}}^{(2)}(A) = M_{\mathrm{loc}}(A)$.

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- This example was further developed by Ara and Mathieu (2011), who showed that if X is a perfect, second countable locally compact Hausdorff space, and $A = C_0(X) \otimes B$ for some non-unital separable simple C^*-algebra B, then $M_{\rm loc}^{(2)}(A) \neq M_{\rm loc}(A)$.

This leads to the following restatement of the stability problem of $M_{\mathrm{loc}}(A)$:

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We have the following partial answer:

Theorem (Somerset, 2000; Ara and Mathieu, 2011)

If A is a unital (or more generally quasi-central), separable C^*-algebra such that $\mathrm{Prim}(A)$(= the primitive ideal space of A) contains a dense G_δ subset of closed points, then $M_{\mathrm{loc}}^{(2)}(A) = M_{\mathrm{loc}}(A)$. Moreover, in this case $M_{\mathrm{loc}}(A)$ has only inner derivations.

This leads to the following restatement of the stability problem of $M_{\mathrm{loc}}(A)$:

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Recall that a C^*-algebra is said to be **quasicentral** if no primitive ideal of A contains the centre $Z(A)$. This is equivalent to say that A admits an approximate unit (e_i) such that $e_i \in Z(A)$ for all i.

On the other hand, $M_{\mathrm{loc}}^{(2)}(A)$ is always a type I AW^*-algebra, whenever A is separable and liminal. More generally:

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Theorem (Somerset, 2000; Argerami and Farenick, 2005)

If the injective envelope of a C^*-algebra A is an AW^*-algebra of type I, then A has a liminal essential ideal. The converse is also true if A is separable. Moreover, in this case $M_{\mathrm{loc}}^{(2)}(A)$ is an AW^*-algebra of type I.

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There is also a partial converse in a non-separable direction:

Theorem (Argerami, Farenick and Massey, 2010)

If A is a spatial Fell algebra, then $M_{\mathrm{loc}}^{(2)}(A)$ is an AW^*-algebra of type I.

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Problem

Is $M_{\text{loc}}^{(3)}(A) = M_{\text{loc}}^{(2)}(A)$ for every C^*-algebra A?

On the other hand, a fairly interesting class of type I C^*-algebras is the class **FIN**, which consists of all C^*-algebras with finite-dimensional irreducible representations.

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What can be said about $M_{\mathrm{loc}}(A)$ and $I(A)$ if A belongs to **FIN**?

Theorem (G., 2013)

If a C^*-algebra A belongs to **FIN**, *then $M_{\mathrm{loc}}(A)$ is a finite or countable direct product of C^*-algebras of the form $C(X_n) \otimes \mathbb{M}_n$, where each space X_n is Stonean. In particular, $M_{\mathrm{loc}}(A) = M_{\mathrm{loc}}^{(2)}(A) = I(A)$, and $M_{\mathrm{loc}}(A)$ admits only inner derivations.*

Proof, Step 1

We first show that every C^*-algebra in **FIN** contains a quasi-central essential ideal J of continuous trace. The proof essentially relies on the fact that the spectrum of a C^*-algebra is a Baire space.

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Note that all quasi-central continuous-trace C^*-algebras belong to **FIN**. They have a particularly nice description:

Theorem (Archbold, 1972)

Let J be a C^*-algebra in **FIN***. Then the following conditions are equivalent:*

(a) *J is quasi-central and has a continuous trace.*

(b) *Dimension function $d : \hat{J} \to \mathbb{N}$, $d : [\pi] \mapsto \dim \pi$, is continuous.*

(c) *J is isomorphic to the C^*-direct sum $\bigoplus_{n=1}^{\infty} J_n$ of a sequence (J_n) of C^*-algebras, where each J_n is either zero, or n-homogeneous.*

Proof, Step 2

Note that this reduces the proof to the homogeneous case. Indeed, if J is an essential quasi-central continuous trace ideal of A, and if $J = \bigoplus_{n=1}^{\infty} J_n$, where J_n are as in Archbold's theorem, then

$$M_{\mathrm{loc}}(A) = M_{\mathrm{loc}}\left(\bigoplus_{n=1}^{\infty} J_n\right) = \prod_{n=1}^{\infty} M_{\mathrm{loc}}(J_n).$$

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Hence, in the sequel we shall assume that A is n-homogeneous. Then by Fell-Tomiyama-Takesaki theorem we have $A = \Gamma_0(E)$ for an algebraic $\mathbb{M}_n$-bundle E over $\widehat{A}$.

Proof, Step 3

If $A = \Gamma_0(E)$ as above, then using the Zorn's lemma we find an open dense subset U of $\widehat{A}$ such that the restriction bundle $E|_U$ is trivial (i.e. $E|_U \cong U \times \mathbb{M}_n$ as $PU(n)$-bundles). Then $I := \Gamma_0(E|_U) \cong C_0(U) \otimes \mathbb{M}_n$ is an essential ideal of A, so

$$\begin{aligned} M_{\mathrm{loc}}(A) &= M_{\mathrm{loc}}(I) \cong M_{\mathrm{loc}}(C_0(U) \otimes \mathbb{M}_n) \cong M_{\mathrm{loc}}(C_0(U)) \otimes \mathbb{M}_n \\ &\cong C(X) \otimes \mathbb{M}_n, \end{aligned}$$

where X is the maximal ideal space of $M_{\mathrm{loc}}(C_0(U))$. Finally, since $M_{\mathrm{loc}}(C_0(U))$ is a commutative AW^*-algebra, X is a Stonean space.

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Summary

- We have no example in which $M_{\mathrm{loc}}^{(2)}(A) = M_{\mathrm{loc}}(A)$ and we do not know that every derivation of $M_{\mathrm{loc}}(A)$ is inner.
- We have no example in which $M_{\mathrm{loc}}^{(2)}(A) \neq M_{\mathrm{loc}}(A)$ and we know every derivation of $M_{\mathrm{loc}}(A)$ is inner.

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This procedure in particular applies to derivations of C^*-algebras

Since derivations of C^*-algebras are completely bounded, we may consider the following question:

Problem

Which derivations of a C^*-algebra A admit a completely bounded approximation by elementary operators? That is, which derivations of A lie in the cb-norm closure $\overline{\overline{\mathcal{E}\ell(A)}}^{cb}$ of the set $\mathcal{E}\ell(A)$ of all elementary operators on A?

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Remark

- Since each inner derivation δ_a $(a \in M(A))$ is an elementary operator on A, $\overline{\overline{\mathcal{E}\ell(A)}}^{cb}$ includes the cb-corm closure of $\mathrm{Inn}(A)$.
- Since the cb-norm of an inner derivation of a C^*-algebra coincides with its operator norm, the cb-norm closure of $\mathrm{Inn}(A)$ coincides with the operator norm closure of $\mathrm{Inn}(A)$. We denote this closure by $\overline{\overline{\mathrm{Inn}(A)}}$.

Problem (G., 2010)

Does every C^*-algebra satisfy the condition

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In fact, we have the following beautiful characterization:

Theorem (Somerset, 1993)

The set $\mathrm{Inn}(A)$ is closed in the operator norm, as a subset of $\mathrm{Der}(A)$, if and only if A has a finite connecting order.

Connecting order of a C^*-algebra

The connecting order of a C^*-algebra is a constant in $\mathbb{N} \cup \{\infty\}$ arising from a certain graph structure on the primitive spectrum $\mathrm{Prim}(A)$:

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- The **distance** $d(P, Q)$ from P to Q is defined as follows:
  - ▷ $d(P, P) := 1$.
  - ▷ If $P \neq Q$ and there exists a path from P to Q, then $d(P, Q)$ is equal to the minimal length of a path from P to Q.
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  - ▷ If there is no path from P to Q, $d(P, Q) := \infty$.
- The **connecting order** of A is then defined by

$$\mathrm{Orc}(A) := \sup\{d(P, Q) : \ P, Q \in \mathrm{Prim}(A) \text{ such that } d(P, Q) < \infty\}.$$

Theorem (G., 2013)

The equality $\mathrm{Der}(A) \cap \overline{\overline{\mathcal{E}\ell(A)}}^{cb} = \overline{\overline{\mathrm{Inn}(A)}}$ *holds true for all unital C^*-algebras A in which every Glimm ideal is prime.*

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Glimm ideals

Recall that the **Glimm ideals** of a unital C^*-algebra A are the ideals generated by the maximal ideals of the centre of A.

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Recall that the **Glimm ideals** of a unital C^*-algebra A are the ideals generated by the maximal ideals of the centre of A.

If a unital C^*-algebra A has only prime Glimm ideals, then $\mathrm{Orc}(A) = 1$, so Somerset's theorem yields that $\mathrm{Inn}(A)$ is closed in the operator norm. Hence:

Corollary

If every Glimm ideal of a unital C^**-algebra* A *is prime, then every derivation of* A *which lies in* $\overline{\overline{\mathcal{E}\ell(A)}}^{cb}$ *is inner.*

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Corollary

For each C^*-algebra A the following conditions are equivalent:

(a) $M_{\mathrm{loc}}(A)$ *admits only inner derivations.*

(b) *Every derivation of $M_{\mathrm{loc}}(A)$ admits a cb-norm approximation by elementary operators.*

Question

Does there exist a C^*-algebra A which admits an outer derivation that is also an elementary operator on A?

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Example (G., 2010)

Let A be a unital C^*-algebra consisting of all elements $a \in C([0,\infty]) \otimes \mathbb{M}_2$ such that

$$a(n) = \begin{bmatrix} \lambda_n(a) & 0 \\ 0 & \lambda_{n+1}(a) \end{bmatrix} \quad (n \in \mathbb{N}),$$

for some convergent sequence $(\lambda_n(a))$ of complex numbers. Then:

(a) $d(\ker \lambda_1, \ker \lambda_n) = n$ for all $n \in \mathbb{N}$. In particular, $\mathrm{Orc}(A) = \infty$.

(b) $\mathcal{E}\ell(A)$ is closed in the cb-norm.

In particular, A admits an outer derivation that is also an elementary operator on A.

I end this talk with some connected questions which I find to be interesting:

Problem

Do we always have $\overline{\overline{\mathrm{Inn}(A)}} \subseteq \mathcal{E}\ell(A)$? In particular, does every unital C^*-algebra A with $\mathrm{Orc}(A) = \infty$ admit an outer derivation that is also an elementary operator on A?

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Problem

Does there exist a unital C^*-algebra A which admits an outer derivation δ that is also an elementary operator of length 2, i.e. $\delta = M_{a_1,b_1} + M_{a_2,b_2}$ for some $a_i, b_i \in A$?

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What can be said about $\mathrm{Der}(A) \cap \overline{\overline{\mathcal{E}\ell(A)}}$?