Derivations, elementary operators and local multipliers of $C^*$-algebras

Ilja Gogić

Department of Mathematics, University of Zagreb

Functional Analysis and Algebra Seminar
Faculty of Mathematics and Physics
Ljubljana, February 27, 2014
In quantum mechanics a physical system is typically described via a unital $C^*$-algebra $A$. The self-adjoint elements of $A$ are thought of as the observables; they are the measurable quantities of the system. A state of the system is defined as a positive functional on $A$ (i.e. a linear map $\omega: A \to \mathbb{C}$ such that $\omega(a^*a) \geq 0$ for all $a \in A$) with $\omega(1_A) = 1$. If the system is in the state $\omega$, then $\omega(a)$ is the expected value of the observable $a$. Automorphisms correspond to the symmetries, while one-parameter automorphism groups $\{\Phi_t\}_{t \in \mathbb{R}}$ describe the reversible time evolution of the system (in the Heisenberg picture). Their infinitesimal generators $\delta(x) := \lim_{t \to 0} \frac{1}{t} (\Phi_t(x) - x)$ are the $*$-derivations.
In quantum mechanics a physical system is typically described via a unital $C^*$-algebra $A$.

- The self-adjoint elements of $A$ are thought of as the observables; they are the measurable quantities of the system.
In quantum mechanics a physical system is typically described via a unital $C^*$-algebra $A$.

- The self-adjoint elements of $A$ are thought of as the observables; they are the measurable quantities of the system.
- A state of the system is defined as a positive functional on $A$ (i.e. a linear map $\omega : A \to \mathbb{C}$ such that $\omega(a^*a) \geq 0$ for all $a \in A$) with $\omega(1_A) = 1$. If the system is in the state $\omega$, then $\omega(a)$ is the expected value of the observable $a$. 

In quantum mechanics a physical system is typically described via a unital \( C^* \)-algebra \( A \).

- The self-adjoint elements of \( A \) are thought of as the observables; they are the measurable quantities of the system.

- A state of the system is defined as a positive functional on \( A \) (i.e. a linear map \( \omega : A \to \mathbb{C} \) such that \( \omega(a^*a) \geq 0 \) for all \( a \in A \)) with \( \omega(1_A) = 1 \). If the system is in the state \( \omega \), then \( \omega(a) \) is the expected value of the observable \( a \).

- Automorphisms correspond to the symmetries, while one-parameter automorphism groups \( \{\Phi_t\}_{t \in \mathbb{R}} \) describe the reversible time evolution of the system (in the Heisenberg picture). Their infinitesimal generators

\[
\delta(x) := \lim_{t \to 0} \frac{1}{t}(\Phi_t(x) - x)
\]

are the \( \ast \)-derivations.
Throughout, $A$ will be a $C^*$-algebra.
Throughout, $A$ will be a $C^*$-algebra.

By an ideal of $A$ we always mean a closed two-sided ideal.
Throughout, $A$ will be a $C^*$-algebra.

By an ideal of $A$ we always mean a closed two-sided ideal.

An ideal $I$ of $A$ is said to be **essential** if $I$ has a non-zero intersection with every other non-zero ideal of $A$. This is equivalent to say that its annihilator $I^\perp := \{ a \in A : aI = \{0\} \}$ is zero.
Throughout, $A$ will be a $C^*$-algebra.

By an ideal of $A$ we always mean a closed two-sided ideal.

An ideal $I$ of $A$ is said to be **essential** if $I$ has a non-zero intersection with every other non-zero ideal of $A$. This is equivalent to say that its annihilator $I^\perp := \{ a \in A : al = \{0\} \}$ is zero.

**Definition**

The multiplier algebra of $A$ is the $C^*$-subalgebra $M(A)$ of the enveloping von Neumann algebra $A^{**}$ that consists of all $x \in A^{**}$ such that $ax \in A$ and $xa \in A$ for all $a \in A$. 
Throughout, $A$ will be a $C^*$-algebra.

By an ideal of $A$ we always mean a closed two-sided ideal.

An ideal $I$ of $A$ is said to be **essential** if $I$ has a non-zero intersection with every other non-zero ideal of $A$. This is equivalent to say that its annihilator $I^\perp := \{a \in A : al = \{0\}\}$ is zero.

**Definition**

The multiplier algebra of $A$ is the $C^*$-subalgebra $M(A)$ of the enveloping von Neumann algebra $A^{**}$ that consists of all $x \in A^{**}$ such that $ax \in A$ and $xa \in A$ for all $a \in A$.

$M(A)$ is a unital extension of $A$ in which $A$ sits as an essential ideal. Moreover, $M(A)$ satisfies the following universal property: Whenever $A$ sits as an ideal in a $C^*$-algebra $B$, then the identity map on $A$ extends uniquely to a $*$-homomorphism from $B$ to $M(A)$ with kernel $A^\perp$. Hence, $M(A)$ is the largest unitization of $A$. 
Definition

Derivation of $A$ is a linear map $\delta : A \rightarrow A$ satisfying the Leibniz rule

$$\delta(xy) = \delta(x)y + x\delta(y)$$

for all $x, y \in A$. 
Definition

Derivation of $A$ is a linear map $\delta : A \to A$ satisfying the Leibniz rule

\[ \delta(xy) = \delta(x)y + x\delta(y) \]

for all $x, y \in A$.

Example

If $A$ is a $C^*$-subalgebra of a $C^*$-algebra $B$, then each element $a \in B$ which derives $A$ (i.e. $ax - xa \in A$, for all $x \in A$) implements a derivation $\delta_a : A \to A$ given by

\[ \delta_a(x) := ax - xa. \]
**Definition**

A **derivation** of $A$ is a linear map $\delta : A \to A$ satisfying the **Leibniz rule**

$$\delta(xy) = \delta(x)y + x\delta(y)$$

for all $x, y \in A$.

**Example**

If $A$ is a $C^*$-subalgebra of a $C^*$-algebra $B$, then each element $a \in B$ which derives $A$ (i.e. $ax - xa \in A$, for all $x \in A$) implements a derivation $\delta_a : A \to A$ given by

$$\delta_a(x) := ax - xa.$$ 

**Definition**

A derivation $\delta$ of $A$ is said to be **inner** if there exists a multiplier $a \in M(A)$ such that $\delta = \delta_a$. 
In the application to physics, innerness of a derivation corresponds to the question whether the Hamiltonian of the system under consideration belongs to the algebraic model.
In the application to physics, innerness of a derivation corresponds to the question whether the Hamiltonian of the system under consideration belongs to the algebraic model.

**Notation**

We denote by $\text{Der}(A)$ (resp. $\text{Inn}(A)$) the set of all derivations (resp. inner derivations) of $A$. 

**Main problem**

Which $C^*$-algebras admit only inner derivations?

Some classes of $C^*$-algebras which admit only inner derivations:

- Von Neumann algebras (Kadison-Sakai, 1966).
- Simple $C^*$-algebras (Sakai, 1968).
- $AW^*$-algebras (Olesen, 1974).
In the application to physics, innerness of a derivation corresponds to the question whether the Hamiltonian of the system under consideration belongs to the algebraic model.

**Notation**

We denote by $\text{Der}(A)$ (resp. $\text{Inn}(A)$) the set of all derivations (resp. inner derivations) of $A$.

**Main problem**

Which $C^*$-algebras admit only inner derivations?
In the application to physics, innerness of a derivation corresponds to the question whether the Hamiltonian of the system under consideration belongs to the algebraic model.

**Notation**

We denote by $\text{Der}(A)$ (resp. $\text{Inn}(A)$) the set of all derivations (resp. inner derivations) of $A$.

**Main problem**

Which $C^*$-algebras admit only inner derivations?

**Some classes of $C^*$-algebras which admit only inner derivations:**

- Von Neumann algebras (Kadison-Sakai, 1966).
- Simple $C^*$-algebras (Sakai, 1968).
- AW*-algebras (Olesen, 1974).
In the application to physics, innerness of a derivation corresponds to the question whether the Hamiltonian of the system under consideration belongs to the algebraic model.

**Notation**

We denote by $\text{Der}(A)$ (resp. $\text{Inn}(A)$) the set of all derivations (resp. inner derivations) of $A$.

**Main problem**

Which $C^*$-algebras admit only inner derivations?

**Some classes of $C^*$-algebras which admit only inner derivations:**

- Von Neumann algebras (Kadison-Sakai, 1966).
In the application to physics, innerness of a derivation corresponds to the question whether the Hamiltonian of the system under consideration belongs to the algebraic model.

**Notation**

We denote by $\text{Der}(A)$ (resp. $\text{Inn}(A)$) the set of all derivations (resp. inner derivations) of $A$.

**Main problem**

Which $C^*$-algebras admit only inner derivations?

**Some classes of $C^*$-algebras which admit only inner derivations:**

- Von Neumann algebras (Kadison-Sakai, 1966).
- Simple $C^*$-algebras (Sakai, 1968).
In the application to physics, innerness of a derivation corresponds to the question whether the Hamiltonian of the system under consideration belongs to the algebraic model.

**Notation**

We denote by $\text{Der}(A)$ (resp. $\text{Inn}(A)$) the set of all derivations (resp. inner derivations) of $A$.

**Main problem**

Which $C^*$-algebras admit only inner derivations?

**Some classes of $C^*$-algebras which admit only inner derivations:**

- Von Neumann algebras (Kadison-Sakai, 1966).
- Simple $C^*$-algebras (Sakai, 1968).
- $AW^*$-algebras (Olesen, 1974).
In the application to physics, innerness of a derivation corresponds to the question whether the Hamiltonian of the system under consideration belongs to the algebraic model.

**Notation**

We denote by $\text{Der}(A)$ (resp. $\text{Inn}(A)$) the set of all derivations (resp. inner derivations) of $A$.

**Main problem**

Which $C^*$-algebras admit only inner derivations?

**Some classes of $C^*$-algebras which admit only inner derivations:**

- Von Neumann algebras (Kadison-Sakai, 1966).
- Simple $C^*$-algebras (Sakai, 1968).
- $AW^*$-algebras (Olesen, 1974).
An $AW^*$-algebra is a $C^*$-algebra $A$ such that the left annihilator of each right ideal of $A$ is of the form $Ap$ for some projection $p \in A$. 

$AW^*$-algebras were introduced by Kaplansky in 1951 in an attempt to give an abstract algebraic characterization of von Neumann algebras. It is easy to see that every von Neumann algebra is an $AW^*$-algebra, but the converse fails. Just as for von Neumann algebras, $AW^*$-algebras can be divided into type I, type II, and type III. A commutative $C^*$-algebra is an $AW^*$-algebra if and only if its maximal ideal space is Stonean (i.e. an extremely disconnected compact Hausdorff space). A $C^*$-algebra $A$ is an $AW^*$-algebra if and only if every MASA of $A$ is monotone complete. All $AW^*$-algebras of type I are themselves monotone complete (Hamana, 1981). It is unknown whether all $AW^*$-algebras are monotone complete. In fact, this is a long standing open problem dating back to the work of Kaplansky.
**AW*-algebras**

An **AW*-algebra** is a $C^*$-algebra $A$ such that the left annihilator of each right ideal of $A$ is of the form $Ap$ for some projection $p \in A$.

- **AW*-algebras were introduced by Kaplansky in 1951 in an attempt to give an abstract algebraic characterization of von Neumann algebras.**
$AW^*$-algebras

An $AW^*$-algebra is a $C^*$-algebra $A$ such that the left annihilator of each right ideal of $A$ is of the form $Ap$ for some projection $p \in A$.

- $AW^*$-algebras were introduced by Kaplansky in 1951 in an attempt to give an abstract algebraic characterization of von Neumann algebras.
- It is easy to see that every von Neumann algebra is an $AW^*$-algebra, but the converse fails. Just as for von Neumann algebras, $AW^*$-algebras can be divided into type I, type II, and type III.
**AW*-algebras**

An **AW*-algebra** is a $C^*$-algebra $A$ such that the left annihilator of each right ideal of $A$ is of the form $A p$ for some projection $p \in A$.

- **AW*-algebras** were introduced by Kaplansky in 1951 in an attempt to give an abstract algebraic characterization of von Neumann algebras.
- It is easy to see that every von Neumann algebra is an AW*-algebra, but the converse fails. Just as for von Neumann algebras, AW*-algebras can be divided into type I, type II, and type III.
- A commutative $C^*$-algebra is an AW*-algebra if and only if its maximal ideal space is Stonean (i.e. an extremely disconnected compact Hausdorff space).
An $AW^*$-algebra is a $C^*$-algebra $A$ such that the left annihilator of each right ideal of $A$ is of the form $Ap$ for some projection $p \in A$.

- $AW^*$-algebras were introduced by Kaplansky in 1951 in an attempt to give an abstract algebraic characterization of von Neumann algebras.
- It is easy to see that every von Neumann algebra is an $AW^*$-algebra, but the converse fails. Just as for von Neumann algebras, $AW^*$-algebras can be divided into type $I$, type $II$, and type $III$.
- A commutative $C^*$-algebra is an $AW^*$-algebra if and only if its maximal ideal space is Stonean (i.e. an extremely disconnected compact Hausdorff space).
- A $C^*$-algebra $A$ is an $AW^*$-algebra if and only if every MASA of $A$ is monotone complete. All $AW^*$-algebras of type $I$ are themselves monotone complete (Hamana, 1981).
**$AW^*$-algebras**

An **$AW^*$-algebra** is a $C^*$-algebra $A$ such that the left annihilator of each right ideal of $A$ is of the form $Ap$ for some projection $p \in A$.

- $AW^*$-algebras were introduced by Kaplansky in 1951 in an attempt to give an abstract algebraic characterization of von Neumann algebras.

- It is easy to see that every von Neumann algebra is an $AW^*$-algebra, but the converse fails. Just as for von Neumann algebras, $AW^*$-algebras can be divided into type $I$, type $II$, and type $III$.

- A commutative $C^*$-algebra is an $AW^*$-algebra if and only if its maximal ideal space is Stonean (i.e. an extremely disconnected compact Hausdorff space).

- A $C^*$-algebra $A$ is an $AW^*$-algebra if and only if every MASA of $A$ is monotone complete. All $AW^*$-algebras of type $I$ are themselves monotone complete (Hamana, 1981).

- It is unknown whether all $AW^*$-algebras are monotone complete. In fact, this is a long standing open problem dating back to the work of Kaplansky.
Homogeneous $C^*$-algebras

A $C^*$-algebra $A$ is said to be $(n)$-homogeneous if all irreducible representations of $A$ have the same finite dimension $n$. 
**Homogeneous $C^*$-algebras**

A $C^*$-algebra $A$ is said to be $(n)$-**homogeneous** if all irreducible representations of $A$ have the same finite dimension $n$.

- The 1-homogeneous $C^*$-algebras are precisely the commutative ones, hence of the form $A = C_0(X)$ for some locally compact Hausdorff space $X$. 

---

Ilja Gogić (University of Zagreb)
Homogeneous $C^*$-algebras

A $C^*$-algebra $A$ is said to be $(n)$-homogeneous if all irreducible representations of $A$ have the same finite dimension $n$.

- The 1-homogeneous $C^*$-algebras are precisely the commutative ones, hence of the form $A = C_0(X)$ for some locally compact Hausdorff space $X$.
- For each locally compact Hausdorff space $X$, the $C^*$-algebra $C_0(X) \otimes \mathbb{M}_n$ is $n$-homogeneous.
Homogeneous $C^*$-algebras

A $C^*$-algebra $A$ is said to be $(n)$-homogeneous if all irreducible representations of $A$ have the same finite dimension $n$.

- The 1-homogeneous $C^*$-algebras are precisely the commutative ones, hence of the form $A = C_0(X)$ for some locally compact Hausdorff space $X$.

- For each locally compact Hausdorff space $X$, the $C^*$-algebra $C_0(X) \otimes \mathbb{M}_n$ is $n$-homogeneous.

- More generally, if $E$ is an algebraic $\mathbb{M}_n$-bundle over a locally compact Hausdorff space $X$, i.e. $E$ is a locally trivial fibre bundle with fibre $\mathbb{M}_n$ and structure group $\text{Aut}(\mathbb{M}_n) \cong PU(n)$ (the projective unitary group), then the set $\Gamma_0(E)$ of all continuous sections of $E$ vanishing at infinity is an $n$-homogeneous $C^*$-algebra, with respect to the fiberwise operations and sup-norm.
Homogeneous $C^*$-algebras

A $C^*$-algebra $A$ is said to be ($n$-)homogeneous if all irreducible representations of $A$ have the same finite dimension $n$.

- The 1-homogeneous $C^*$-algebras are precisely the commutative ones, hence of the form $A = C_0(X)$ for some locally compact Hausdorff space $X$.

- For each locally compact Hausdorff space $X$, the $C^*$-algebra $C_0(X) \otimes \mathbb{M}_n$ is $n$-homogeneous.

- More generally, if $E$ is an algebraic $\mathbb{M}_n$-bundle over a locally compact Hausdorff space $X$, i.e. $E$ is a locally trivial fibre bundle with fibre $\mathbb{M}_n$ and structure group $\text{Aut}(\mathbb{M}_n) \cong PU(n)$ (the projective unitary group), then the set $\Gamma_0(E)$ of all continuous sections of $E$ vanishing at infinity is an $n$-homogeneous $C^*$-algebra, with respect to the fiberwise operations and sup-norm.

- By a wonderful theorem due to Fell and Tomiyama-Takesaki (1961), every $n$-homogeneous $C^*$-algebra $A$ can be realized as $A = \Gamma_0(E)$ for some algebraic $\mathbb{M}_n$-bundle $E$ over the spectrum $\hat{A}$.
Back to our main problem, the separable case was completely solved in 1979:

**Theorem (Akemann, Elliott, Pedersen and Tomiyama, 1979)**

*Let $A$ be a separable $C^*$-algebra, Then the following conditions are equivalent:*

(a) $A$ admits only inner derivations.

(b) $A = A_1 \oplus A_2$, where $A_1$ is a continuous-trace $C^*$-algebra, and $A_2$ is a direct sum of simple $C^*$-algebras.
Back to our main problem, the separable case was completely solved in 1979:

**Theorem (Akemann, Elliott, Pedersen and Tomiyama, 1979)**

Let $A$ be a separable $C^*$-algebra, Then the following conditions are equivalent:

(a) $A$ admits only inner derivations.

(b) $A = A_1 \oplus A_2$, where $A_1$ is a continuous-trace $C^*$-algebra, and $A_2$ is a direct sum of simple $C^*$-algebras.

On the other hand, for inseparable $C^*$-algebras the problem of innerness of derivations remains widely open, even for the simplest cases such as subhomogeneous $C^*$-algebras (i.e. $C^*$-algebras which have finite-dimensional irreducible representations of bounded degree).
If $I$ and $J$ are two essential ideals of $A$ such that $J \subseteq I$, then there is an embedding $M(I) \hookrightarrow M(J)$.
If $I$ and $J$ are two essential ideals of $A$ such that $J \subseteq I$, then there is an embedding $M(I) \hookrightarrow M(J)$.

In this way, we obtain a directed system of $C^*$-algebras with isometric connecting morphisms, where $I$ runs through the directed set $\text{Id}_{\text{ess}}(A)$ of all essential ideals of $A$. 
If \( I \) and \( J \) are two essential ideals of \( A \) such that \( J \subseteq I \), then there is an embedding \( M(I) \hookrightarrow M(J) \).

In this way, we obtain a directed system of \( C^* \)-algebras with isometric connecting morphisms, where \( I \) runs through the directed set \( \text{Id}_{\text{ess}}(A) \) of all essential ideals of \( A \).

**Definition**

The local multiplier algebra of \( A \) is the direct limit \( C^* \)-algebra

\[
M_{\text{loc}}(A) := (C^*)\lim \{ M(I) : I \in \text{Id}_{\text{ess}}(A) \}.
\]
If $I$ and $J$ are two essential ideals of $A$ such that $J \subseteq I$, then there is an embedding $M(I) \hookrightarrow M(J)$.

In this way, we obtain a directed system of $C^*$-algebras with isometric connecting morphisms, where $I$ runs through the directed set $\text{Id}_{\text{ess}}(A)$ of all essential ideals of $A$.

**Definition**

The local multiplier algebra of $A$ is the direct limit $C^*$-algebra

$$M_{\text{loc}}(A) := (C^*-\lim \{ M(I) : I \in \text{Id}_{\text{ess}}(A) \}).$$

Iterating the construction of $M_{\text{loc}}(A)$, one obtains the following tower of $C^*$-algebras which, a priori, does not have the largest element:

$$A \subseteq M_{\text{loc}}(A) \subseteq M_{\text{loc}}^{(2)}(A) \subseteq \cdots \subseteq M_{\text{loc}}^{(n)}(A) \subseteq \cdots,$$

where $M_{\text{loc}}^{(2)}(A) = M_{\text{loc}}(M_{\text{loc}}(A))$, $M_{\text{loc}}^{(3)}(A) = M_{\text{loc}}(M_{\text{loc}}^{(2)}(A))$, etc.
Remark

An easy, but invaluable fact is that $M_{loc}(I) = M_{loc}(A)$ for every essential ideal $I$ of $A$. This is because $\{ J \in \text{Id}_{ess}(A) : J \subseteq I \}$ is cofinal in $\text{Id}_{ess}(A)$. 

Example

If $A$ is simple, then $M_{loc}(A) = A$.

Example

If $A$ is an AW*-algebra, then $M_{loc}(A) = A$.

Example

If $A = C_0(X)$ is a commutative C*-algebra, then $M_{loc}(A)$ is a commutative AW*-algebra whose maximal ideal space can be identified with the inverse limit $\lim_{\leftarrow} \beta U$ of Stone–Čech compactifications $\beta U$ of dense open subsets $U$ of $X$. 
Remark

An easy, but invaluable fact is that $M_{loc}(I) = M_{loc}(A)$ for every essential ideal $I$ of $A$. This is because $\{J \in \text{Id}_{ess}(A) : J \subseteq I\}$ is cofinal in $\text{Id}_{ess}(A)$.

Example

If $A$ is simple, then $M_{loc}(A) = M(A)$. 
**Remark**

An easy, but invaluable fact is that $M_{\text{loc}}(I) = M_{\text{loc}}(A)$ for every essential ideal $I$ of $A$. This is because $\{J \in \text{Id}_{\text{ess}}(A) : J \subseteq I\}$ is cofinal in $\text{Id}_{\text{ess}}(A)$.

**Example**

If $A$ is simple, then $M_{\text{loc}}(A) = M(A)$.

**Example**

If $A$ is an $AW^*$-algebra, then $M_{\text{loc}}(A) = A$. 
Remark
An easy, but invaluable fact is that $M_{\text{loc}}(I) = M_{\text{loc}}(A)$ for every essential ideal $I$ of $A$. This is because $\{J \in \text{Id}_{\text{ess}}(A) : J \subseteq I\}$ is cofinal in $\text{Id}_{\text{ess}}(A)$.

Example
If $A$ is simple, then $M_{\text{loc}}(A) = M(A)$.

Example
If $A$ is an $AW^*$-algebra, then $M_{\text{loc}}(A) = A$.

Example
If $A = C_0(X)$ is a commutative $C^*$-algebra, then $M_{\text{loc}}(A)$ is a commutative $AW^*$-algebra whose maximal ideal space can be identified with the inverse limit $\lim_{\leftarrow} \beta U$ of Stone-Čech compactifications $\beta U$ of dense open subsets $U$ of $X$. 
The concept of the local multiplier algebra was introduced by G. Pedersen in 1978 (he called it the "\( C^* \)-algebra of essential multipliers").
The concept of the local multiplier algebra was introduced by G. Pedersen in 1978 (he called it the "$C^*$-algebra of essential multipliers").

Every derivation of a $C^*$-algebra $A$ extends uniquely and under preservation of the norm to a derivation of $M_{loc}(A)$. 
The concept of the local multiplier algebra was introduced by G. Pedersen in 1978 (he called it the ”$C^*$-algebra of essential multipliers”).

Every derivation of a $C^*$-algebra $A$ extends uniquely and under preservation of the norm to a derivation of $M_{\text{loc}}(A)$.

Pedersen proved that every derivation of a separable $C^*$-algebra $A$ becomes inner when extended to a derivation of $M_{\text{loc}}(A)$. Moreover, it suffices to assume that every essential closed ideal of $A$ is $\sigma$-unital.
The concept of the local multiplier algebra was introduced by G. Pedersen in 1978 (he called it the ”$C^*$-algebra of essential multipliers”).

Every derivation of a $C^*$-algebra $A$ extends uniquely and under preservation of the norm to a derivation of $M_{loc}(A)$.

Pedersen proved that every derivation of a separable $C^*$-algebra $A$ becomes inner when extended to a derivation of $M_{loc}(A)$. Moreover, it suffices to assume that every essential closed ideal of $A$ is $\sigma$-unital.

In particular, Pedersen’s result entails Sakai’s theorem that every derivation of a simple unital $C^*$-algebra is inner.
Since $M_{\text{loc}}(A) = M(A)$ if $A$ is simple, and $M_{\text{loc}}(A) = A$ if $A$ is an $\text{AW}^*$-algebra, only an affirmative answer in the inseparable case would cover, extend and unify the results that every derivation of a simple $C^*$-algebra is inner in its multiplier algebra and that all derivations of $\text{AW}^*$-algebras are inner.
Since $M_{\text{loc}}(A) = M(A)$ if $A$ is simple, and $M_{\text{loc}}(A) = A$ if $A$ is an $AW^*$-algebra, only an affirmative answer in the inseparable case would cover, extend and unify the results that every derivation of a simple $C^*$-algebra is inner in its multiplier algebra and that all derivations of $AW^*$-algebras are inner.

This led Pedersen to ask:
Since $M_{\text{loc}}(A) = M(A)$ if $A$ is simple, and $M_{\text{loc}}(A) = A$ if $A$ is an $AW^*$-algebra, only an affirmative answer in the inseparable case would cover, extend and unify the results that every derivation of a simple $C^*$-algebra is inner in its multiplier algebra and that all derivations of $AW^*$-algebras are inner.

This led Pedersen to ask:

**Problem of innerness of derivations of $M_{\text{loc}}(A)$**

If $A$ is an arbitrary $C^*$-algebra, is every derivation of $M_{\text{loc}}(A)$ inner?
Since $M_{\text{loc}}(A) = M(A)$ if $A$ is simple, and $M_{\text{loc}}(A) = A$ if $A$ is an $AW^*$-algebra, only an affirmative answer in the inseparable case would cover, extend and unify the results that every derivation of a simple $C^*$-algebra is inner in its multiplier algebra and that all derivations of $AW^*$-algebras are inner.

This led Pedersen to ask:

**Problem of innerness of derivations of $M_{\text{loc}}(A)$**

If $A$ is an arbitrary $C^*$-algebra, is every derivation of $M_{\text{loc}}(A)$ inner?

**Stability problem of $M_{\text{loc}}(A)$**

Is $M_{\text{loc}}^{(2)}(A) = M_{\text{loc}}(A)$ for every $C^*$-algebra $A$?
There is another important characterisation of $M_{\text{loc}}(A)$, which was first obtained by Frank and Paulsen in 2003.
There is another important characterisation of $M_{\text{loc}}(A)$, which was first obtained by Frank and Paulsen in 2003.

For a $C^*$-algebra $A$, let us denote by $I(A)$ its **injective envelope** as introduced by Hamana in 1979.
There is another important characterisation of $M_{\text{loc}}(A)$, which was first obtained by Frank and Paulsen in 2003.

For a $C^*$-algebra $A$, let us denote by $I(A)$ its injective envelope as introduced by Hamana in 1979.

$I(A)$ is not an injective object in the category of $C^*$-algebras and $\ast$-homomorphisms, but in the category of operator spaces and complete positive maps, i.e. for every inclusion $E \subseteq F$ of operator systems, each completely positive map $\phi : E \to I(A)$ has a completely positive extension $\tilde{\phi} : F \to I(A)$. 
There is another important characterisation of $M_{loc}(A)$, which was first obtained by Frank and Paulsen in 2003.

For a $C^*$-algebra $A$, let us denote by $I(A)$ its **injective envelope** as introduced by Hamana in 1979.

$I(A)$ is not an injective object in the category of $C^*$-algebras and $\ast$-homomorphisms, but in the category of operator spaces and complete positive maps, i.e. for every inclusion $E \subseteq F$ of operator systems, each completely positive map $\phi : E \to I(A)$ has a completely positive extension $\tilde{\phi} : F \to I(A)$.

However, it turns out that (nevertheless) $I(A)$ is a $C^*$-algebra canonically containing $A$ as a $C^*$-subalgebra. Moreover, $I(A)$ is monotone complete, so in particular, $I(A)$ is an $AW^*$-algebra.
Theorem (Frank and Paulsen, 2003)

Under this embedding of $A$ into $I(A)$, $M_{\text{loc}}(A)$ is the norm closure of the set of all $x \in I(A)$ which act as a multiplier on some $l \in \text{Id}_{\text{ess}}(A)$, i.e.

$$M_{\text{loc}}(A) = \left( \bigcup_{l \in \text{Id}_{\text{ess}}(A)} \{ x \in I(A) : xl + lx \subseteq l \} \right)^{=}$$

Thus, we have the following inclusion of $C^*$-algebras:

$$A \subseteq M_{\text{loc}}(A) \subseteq A \subseteq I(A),$$

where $A$ is the regular monotone completion of $A$. Moreover, it can be seen that $I(A) = I(M_{\text{loc}}(A))$, so we have an additional sequence of inclusions of $C^*$-algebras:

$$A \subseteq M_{\text{loc}}(A) \subseteq M_{\text{loc}}(2) \subseteq \cdots \subseteq A \subseteq I(A).$$
Theorem (Frank and Paulsen, 2003)

Under this embedding of $A$ into $I(A)$, $M_{\text{loc}}(A)$ is the norm closure of the set of all $x \in I(A)$ which act as a multiplier on some $l \in \text{Id}_{\text{ess}}(A)$, i.e.

$$M_{\text{loc}}(A) = \left( \bigcup_{l \in \text{Id}_{\text{ess}}(A)} \{ x \in I(A) : xl + lx \subseteq l \} \right)^{\text{c}}$$

Thus, we have the following inclusion of $C^*$-algebras:

$$A \subseteq M_{\text{loc}}(A) \subseteq \overline{A} \subseteq I(A),$$

where $\overline{A}$ is the regular monotone completion of $A$. 
Theorem (Frank and Paulsen, 2003)

Under this embedding of $A$ into $I(A)$, $M_{\text{loc}}(A)$ is the norm closure of the set of all $x \in I(A)$ which act as a multiplier on some $l \in \text{Id}_{\text{ess}}(A)$, i.e.

$$M_{\text{loc}}(A) = \left( \bigcup_{l \in \text{Id}_{\text{ess}}(A)} \{ x \in I(A) : xl + lx \subseteq l \} \right) =$$

Thus, we have the following inclusion of $C^*$-algebras:

$$A \subseteq M_{\text{loc}}(A) \subseteq \overline{A} \subseteq I(A),$$

where $\overline{A}$ is the regular monotone completion of $A$.

Moreover, it can be seen that $I(M_{\text{loc}}(A)) = I(A)$, so we have an additional sequence of inclusions of $C^*$-algebras:

$$A \subseteq M_{\text{loc}}(A) \subseteq M_{\text{loc}}^{(2)}(A) \subseteq \cdots \subseteq \overline{A} \subseteq I(A).$$
Very difficult problem

When is $M_{\text{loc}}(A) = I(A)$, or at least $M_{\text{loc}}(A) = \overline{A}$?
Very difficult problem

When is $M_{loc}(A) = I(A)$, or at least $M_{loc}(A) = \bar{A}$?

Stability problem has a negative solution
Very difficult problem

When is $M_{loc}(A) = I(A)$, or at least $M_{loc}(A) = \overline{A}$?

Stability problem has a negative solution

- The first class of examples of $C^*$-algebras for which the stability problem of local multiplier algebras has a negative answer was given by Ara and Mathieu (2006): There exist unital separable approximately finite-dimensional primitive $C^*$-algebras $A$ such that $M_{loc}^{(2)}(A) \neq M_{loc}(A)$. 
Very difficult problem

When is $M_{loc}(A) = I(A)$, or at least $M_{loc}(A) = \overline{A}$?

Stability problem has a negative solution

- The first class of examples of $C^*$-algebras for which the stability problem of local multiplier algebras has a negative answer was given by Ara and Mathieu (2006): There exist unital separable approximately finite-dimensional primitive $C^*$-algebras $A$ such that $M_{loc}^{(2)}(A) \neq M_{loc}(A)$.

- After that, Argerami, Farenick and Massey (2009) showed that a relatively well-behaved $C^*$-algebra $C([0,1]) \otimes \mathbb{K}$ also fails to satisfy $M_{loc}^{(2)}(A) = M_{loc}(A)$. 
Stability problem has a negative solution

- The first class of examples of $C^*$-algebras for which the stability problem of local multiplier algebras has a negative answer was given by Ara and Mathieu (2006): There exist unital separable approximately finite-dimensional primitive $C^*$-algebras $A$ such that $M_{loc}(A) \neq M_{loc}(A)$.

- After that, Argerami, Farenick and Massey (2009) showed that a relatively well-behaved $C^*$-algebra $C([0,1]) \otimes K$ also fails to satisfy $M_{loc}^{(2)}(A) = M_{loc}(A)$.

- This example was further developed by Ara and Mathieu (2011), who showed that if $X$ is a perfect, second countable locally compact Hausdorff space, and $A = C_0(X) \otimes B$ for some non-unital separable simple $C^*$-algebra $B$, then $M_{loc}^{(2)}(A) \neq M_{loc}(A)$. 
This leads to the following restatement of the stability problem of $M_{\text{loc}}(A)$:

**Problem**

When is $M_{\text{loc}}^{(2)}(A) = M_{\text{loc}}(A)$?
This leads to the following restatement of the stability problem of $M_{\text{loc}}(A)$:

**Problem**

When is $M_{\text{loc}}^{(2)}(A) = M_{\text{loc}}(A)$?

We have the following partial answer:

**Theorem (Somerset, 2000; Ara and Mathieu, 2011)**

If $A$ is a unital (or more generally quasi-central), separable C*-algebra such that $\text{Prim}(A) (= \text{the primitive ideal space of } A)$ contains a dense $G_\delta$ subset of closed points, then $M_{\text{loc}}^{(2)}(A) = M_{\text{loc}}(A)$. Moreover, in this case $M_{\text{loc}}(A)$ has only inner derivations.
This leads to the following restatement of the stability problem of $M_{\text{loc}}(A)$:

**Problem**

When is $M_{\text{loc}}^{(2)}(A) = M_{\text{loc}}(A)$?

We have the following partial answer:

**Theorem (Somerset, 2000; Ara and Mathieu, 2011)**

If $A$ is a unital (or more generally quasi-central), separable C*-algebra such that $\text{Prim}(A) (= \text{the primitive ideal space of} A)$ contains a dense $G_\delta$ subset of closed points, then $M_{\text{loc}}^{(2)}(A) = M_{\text{loc}}(A)$. Moreover, in this case $M_{\text{loc}}(A)$ has only inner derivations.

Recall that a C*-algebra is said to be **quasicentral** if no primitive ideal of $A$ contains the centre $Z(A)$. This is equivalent to say that $A$ admits an approximate unit $(e_i)$ such that $e_i \in Z(A)$ for all $i$. 
On the other hand, $M_{10c}^{(2)}(A)$ is always a type I $AW^*$-algebra, whenever $A$ is separable and liminal. More generally:

**Theorem (Somerset, 2000; Argerami and Farenick, 2005)**

If the injective envelope of a $C^*$-algebra $A$ is an $AW^*$-algebra of type I, then $A$ has a liminal essential ideal. The converse is also true if $A$ is separable. Moreover, in this case $M_{10c}^{(2)}(A)$ is an $AW^*$-algebra of type I.

There is also a partial converse in a non-separable direction:

**Theorem (Argerami, Farenick and Massey, 2010)**

If $A$ is a spatial Fell algebra, then $M_{10c}^{(2)}(A)$ is an $AW^*$-algebra of type I.

The above result in particular applies to the algebras of the form $A = C_0(X) \otimes K$, for any locally compact Hausdorff space $X$.
On the other hand, $M_{\text{loc}}^{(2)}(A)$ is always a type I $\mathcal{A}\mathcal{W}^*$-algebra, whenever $A$ is separable and liminal. More generally:

**Theorem (Somerset, 2000; Argerami and Farenick, 2005)**

*If the injective envelope of a $C^*$-algebra $A$ is an $\mathcal{A}\mathcal{W}^*$-algebra of type I, then $A$ has a liminal essential ideal. The converse is also true if $A$ is separable. Moreover, in this case $M_{\text{loc}}^{(2)}(A)$ is an $\mathcal{A}\mathcal{W}^*$-algebra of type I.*
On the other hand, $M^{(2)}_{\text{loc}}(A)$ is always a type $I$ $\text{AW}^*$-algebra, whenever $A$ is separable and liminal. More generally:

**Theorem (Somerset, 2000; Argerami and Farenick, 2005)**

*If the injective envelope of a $C^*$-algebra $A$ is an $\text{AW}^*$-algebra of type $I$, then $A$ has a liminal essential ideal. The converse is also true if $A$ is separable. Moreover, in this case $M^{(2)}_{\text{loc}}(A)$ is an $\text{AW}^*$-algebra of type $I$.***

There is also a partial converse in a non-separable direction:

**Theorem (Argerami, Farenick and Massey, 2010)**

*If $A$ is a spatial Fell algebra, then $M^{(2)}_{\text{loc}}(A)$ is an $\text{AW}^*$-algebra of type $I$.***
On the other hand, $M_{loc}^{(2)}(A)$ is always a type $I$ $AW^*$-algebra, whenever $A$ is separable and liminal. More generally:

**Theorem (Somerset, 2000; Argerami and Farenick, 2005)**

*If the injective envelope of a $C^*$-algebra $A$ is an $AW^*$-algebra of type $I$, then $A$ has a liminal essential ideal. The converse is also true if $A$ is separable. Moreover, in this case $M_{loc}^{(2)}(A)$ is an $AW^*$-algebra of type $I$.***

There is also a partial converse in a non-separable direction:

**Theorem (Argerami, Farenick and Massey, 2010)**

*If $A$ is a spatial Fell algebra, then $M_{loc}^{(2)}(A)$ is an $AW^*$-algebra of type $I$.***

The above result in particular applies to the algebras of the form $A = C_0(X) \otimes \mathbb{K}$, for any locally compact Hausdorff space $X$. 
On the other hand, $M_{\text{loc}}^{(2)}(A)$ is always a type $I$ $AW^*$-algebra, whenever $A$ is separable and liminal. More generally:

**Theorem (Somerset, 2000; Argerami and Farenick, 2005)**

*If the injective envelope of a $C^*$-algebra $A$ is an $AW^*$-algebra of type $I$, then $A$ has a liminal essential ideal. The converse is also true if $A$ is separable. Moreover, in this case $M_{\text{loc}}^{(2)}(A)$ is an $AW^*$-algebra of type $I$."

There is also a partial converse in a non-separable direction:

**Theorem (Argerami, Farenick and Massey, 2010)**

*If $A$ is a spatial Fell algebra, then $M_{\text{loc}}^{(2)}(A)$ is an $AW^*$-algebra of type $I$."

The above result in particular applies to the algebras of the form $A = C_0(X) \otimes \mathbb{K}$, for any locally compact Hausdorff space $X$.

**Problem**

Is $M_{\text{loc}}^{(3)}(A) = M_{\text{loc}}^{(2)}(A)$ for every $C^*$-algebra $A$?
On the other hand, a fairly interesting class of type I C*-algebras is the class \( \text{FIN} \), which consists of all C*-algebras with finite-dimensional irreducible representations.
On the other hand, a fairly interesting class of type I $C^*$-algebras is the class **FIN**, which consists of all $C^*$-algebras with finite-dimensional irreducible representations.

**Problem**
What can be said about $M_{loc}(A)$ and $I(A)$ if $A$ belongs to **FIN**?

Theorem (G., 2013)
If a $C^*$-algebra $A$ belongs to **FIN**, then $M_{loc}(A)$ is a finite or countable direct product of $C^*$-algebras of the form $C(X_n) \otimes M_n$, where each space $X_n$ is Stonean. In particular, $M_{loc}(A) = M(2)_{loc}(A) = I(A)$, and $M_{loc}(A)$ admits only inner derivations.
On the other hand, a fairly interesting class of type I $C^*$-algebras is the class $\text{FIN}$, which consists of all $C^*$-algebras with finite-dimensional irreducible representations.

**Problem**

What can be said about $M_{\text{loc}}(A)$ and $I(A)$ if $A$ belongs to $\text{FIN}$?

**Theorem (G., 2013)**

If a $C^*$-algebra $A$ belongs to $\text{FIN}$, then $M_{\text{loc}}(A)$ is a finite or countable direct product of $C^*$-algebras of the form $C(X_n) \otimes M_n$, where each space $X_n$ is Stonean. In particular, $M_{\text{loc}}(A) = M_{\text{loc}}^{(2)}(A) = I(A)$, and $M_{\text{loc}}(A)$ admits only inner derivations.
Proof, Step 1

We first show that every $C^*$-algebra in $\text{FIN}$ contains a quasi-central essential ideal $J$ of continuous trace. The proof essentially relies on the fact that the spectrum of a $C^*$-algebra is a Baire space.
Proof, Step 1

We first show that every $C^*$-algebra in $\text{FIN}$ contains a quasi-central essential ideal $J$ of continuous trace. The proof essentially relies on the fact that the spectrum of a $C^*$-algebra is a Baire space.

Note that all quasi-central continuous-trace $C^*$-algebras belong to $\text{FIN}$. They have a particularly nice description:
Proof, Step 1

We first show that every $C^*$-algebra in $\text{FIN}$ contains a quasi-central essential ideal $J$ of continuous trace. The proof essentially relies on the fact that the spectrum of a $C^*$-algebra is a Baire space.

Note that all quasi-central continuous-trace $C^*$-algebras belong to $\text{FIN}$. They have a particularly nice description:

**Theorem (Archbold, 1972)**

Let $J$ be a $C^*$-algebra in $\text{FIN}$. Then the following conditions are equivalent:

(a) $J$ is quasi-central and has a continuous trace.

(b) Dimension function $d : \hat{J} \to \mathbb{N}$, $d : [\pi] \mapsto \dim \pi$, is continuous.

(c) $J$ is isomorphic to the $C^*$-direct sum $\bigoplus_{n=1}^{\infty} J_n$ of a sequence $(J_n)$ of $C^*$-algebras, where each $J_n$ is either zero, or $n$-homogeneous.
Note that this reduces the proof to the homogeneous case. Indeed, if $J$ is an essential quasi-central continuous trace ideal of $A$, and if $J = \bigoplus_{n=1}^{\infty} J_n$, where $J_n$ are as in Archbold’s theorem, then

$$M_{\text{loc}}(A) = M_{\text{loc}} \left( \bigoplus_{n=1}^{\infty} J_n \right) = \prod_{n=1}^{\infty} M_{\text{loc}}(J_n).$$
Proof, Step 2

Note that this reduces the proof to the homogeneous case. Indeed, if $J$ is an essential quasi-central continuous trace ideal of $A$, and if $J = \bigoplus_{n=1}^{\infty} J_n$, where $J_n$ are as in Archbold’s theorem, then

$$M_{\text{loc}}(A) = M_{\text{loc}} \left( \bigoplus_{n=1}^{\infty} J_n \right) = \prod_{n=1}^{\infty} M_{\text{loc}}(J_n).$$

Hence, in the sequel we shall assume that $A$ is $n$-homogeneous. Then by Fell-Tomiyama-Takesaki theorem we have $A = \Gamma_0(E)$ for an algebraic $\mathbb{M}_n$-bundle $E$ over $\hat{A}$. 
Proof, Step 3

If \( A = \Gamma_0(E) \) as above, then using the Zorn’s lemma we find an open dense subset \( U \) of \( \hat{A} \) such that the restriction bundle \( E |_U \) is trivial (i.e. \( E |_U \cong U \times \mathbb{M}_n \) as \( PU(n) \)-bundles). Then \( I := \Gamma_0(E |_U) \cong C_0(U) \otimes \mathbb{M}_n \) is an essential ideal of \( A \), so

\[
M_{\text{loc}}(A) = M_{\text{loc}}(I) \cong M_{\text{loc}}(C_0(U) \otimes \mathbb{M}_n) \cong M_{\text{loc}}(C_0(U)) \otimes \mathbb{M}_n \\
\cong C(X) \otimes \mathbb{M}_n,
\]

where \( X \) is the maximal ideal space of \( M_{\text{loc}}(C_0(U)) \). Finally, since \( M_{\text{loc}}(C_0(U)) \) is a commutative \( AW^* \)-algebra, \( X \) is a Stonean space.
Proof, Step 3

If \( A = \Gamma_0(E) \) as above, then using the Zorn’s lemma we find an open dense subset \( U \) of \( \hat{A} \) such that the restriction bundle \( E|_U \) is trivial (i.e. \( E|_U \cong U \times \mathbb{M}_n \) as \( PU(n) \)-bundles). Then \( I := \Gamma_0(E|_U) \cong C_0(U) \otimes \mathbb{M}_n \) is an essential ideal of \( A \), so

\[
M_{\text{loc}}(A) = M_{\text{loc}}(I) \cong M_{\text{loc}}(C_0(U) \otimes \mathbb{M}_n) \cong M_{\text{loc}}(C_0(U)) \otimes \mathbb{M}_n \\
\cong C(X) \otimes \mathbb{M}_n,
\]

where \( X \) is the maximal ideal space of \( M_{\text{loc}}(C_0(U)) \). Finally, since \( M_{\text{loc}}(C_0(U)) \) is a commutative \( AW^* \)-algebra, \( X \) is a Stonean space.

Summary

- We have no example in which \( M_{\text{loc}}^{(2)}(A) = M_{\text{loc}}(A) \) and we do not know that every derivation of \( M_{\text{loc}}(A) \) is inner.
- We have no example in which \( M_{\text{loc}}^{(2)}(A) \neq M_{\text{loc}}(A) \) and we know every derivation of \( M_{\text{loc}}(A) \) is inner.
We often try to understand the structure of operators and spaces on which they act in terms of approximation by finite rank maps. On a $C^*$-algebra $A$, however, it is natural to regard two-sided multiplication maps $M_{a,b}: x \mapsto axb$ ($a,b \in M(A)$) as basic building blocks (instead of rank one operators). We can therefore try to approximate a more general map on $A$, one that preserves ideals, by finite sums of two-sided multiplication maps, that is, by elementary operators. This procedure in particular applies to derivations of $C^*$-algebras.
Motivation

- We often try to understand the structure of operators and spaces on which they act in terms of approximation by finite rank maps.
Motivation

- We often try to understand the structure of operators and spaces on which they act in terms of approximation by finite rank maps.
- On a $C^*$-algebra $A$, however, it is natural to regard two-sided multiplication maps $M_{a,b}: x \mapsto axb$ ($a, b \in M(A)$) as basic building blocks (instead of rank one operators).
Motivation

- We often try to understand the structure of operators and spaces on which they act in terms of approximation by finite rank maps.
- On a $C^*$-algebra $A$, however, it is natural to regard two-sided multiplication maps

$$M_{a,b}: x \mapsto axb \quad (a, b \in M(A))$$

as basic building blocks (instead of rank one operators).
- We can therefore try to approximate a more general map on $A$, one that preserves ideals, by finite sums of two-sided multiplication maps, that is, by elementary operators.
Motivation

- We often try to understand the structure of operators and spaces on which they act in terms of approximation by finite rank maps.
- On a $C^*$-algebra $A$, however, it is natural to regard two-sided multiplication maps
  \[ M_{a,b} : x \mapsto axb \quad (a, b \in M(A)) \]
as basic building blocks (instead of rank one operators).
- We can therefore try to approximate a more general map on $A$, one that preserves ideals, by finite sums of two-sided multiplication maps, that is, by **elementary operators**.

This procedure in particular applies to derivations of $C^*$-algebras
Since derivations of $C^*$-algebras are completely bounded, we may consider the following question:

**Problem**

Which derivations of a $C^*$-algebra $A$ admit a completely bounded approximation by elementary operators? That is, which derivations of $A$ lie in the cb-norm closure $\overline{\mathcal{E}_\ell(A)}^{cb}$ of the set $\mathcal{E}_\ell(A)$ of all elementary operators on $A$?
Since derivations of $C^*$-algebras are completely bounded, we may consider the following question:

**Problem**

Which derivations of a $C^*$-algebra $A$ admit a completely bounded approximation by elementary operators? That is, which derivations of $A$ lie in the cb-norm closure $\overline{\mathcal{E}\ell(A)}^{cb}$ of the set $\mathcal{E}\ell(A)$ of all elementary operators on $A$?

**Remark**

- Since each inner derivation $\delta_a$ ($a \in M(A)$) is an elementary operator on $A$, $\overline{\mathcal{E}\ell(A)}^{cb}$ includes the cb-corm closure of $\text{Inn}(A)$.
- Since the cb-norm of an inner derivation of a $C^*$-algebra coincides with its operator norm, the cb-norm closure of $\text{Inn}(A)$ coincides with the operator norm closure of $\text{Inn}(A)$. We denote this closure by $\overline{\text{Inn}(A)}$. 
Problem (G., 2010)

Does every $C^*$-algebra satisfy the condition

$$\text{Der}(A) \cap \overline{E\ell}(A)^{cb} = \overline{\text{Inn}(A)}?$$
Problem (G., 2010)

Does every $C^*$-algebra satisfy the condition

$$\text{Der}(A) \cap \overline{E_\ell(A)}^{cb} = \overline{\text{Inn}(A)}?$$

In many cases the set $\text{Inn}(A)$ is closed in the operator norm. However, this is not always true.
Problem (G., 2010)

Does every $C^*$-algebra satisfy the condition

$$\text{Der}(A) \cap \mathcal{E}\ell(A)^{cb} = \overline{\text{Inn}(A)}?$$

In many cases the set $\text{Inn}(A)$ is closed in the operator norm. However, this is not always true.

In fact, we have the following beautiful characterization:

Theorem (Somerset, 1993)

*The set $\text{Inn}(A)$ is closed in the operator norm, as a subset of $\text{Der}(A)$, if and only if $A$ has a finite connecting order.*
Connecting order of a $C^*$-algebra

The connecting order of a $C^*$-algebra is a constant in $\mathbb{N} \cup \{\infty\}$ arising from a certain graph structure on the primitive spectrum $\text{Prim}(A)$:

- Two primitive ideals $P$, $Q$ of $A$ are said to be adjacent, if $P$ and $Q$ cannot be separated by disjoint open subsets of $\text{Prim}(A)$.
- A path of length $n$ from $P$ to $Q$ is a sequence of points $P = P_0, P_1, \ldots, P_n = Q$ such that $P_{i-1}$ is adjacent to $P_i$ for all $1 \leq i \leq n$.
- The distance $d(P, Q)$ from $P$ to $Q$ is defined as follows:
  - $\Delta d(P, P) := 1$.
  - If $P \neq Q$ and there exists a path from $P$ to $Q$, then $d(P, Q)$ is equal to the minimal length of a path from $P$ to $Q$.
  - If there is no path from $P$ to $Q$, $d(P, Q) := \infty$.
- The connecting order of $A$ is then defined by $\text{Orc}(A) := \sup \{d(P, Q) : P, Q \in \text{Prim}(A) \text{ such that } d(P, Q) < \infty\}$.
Connecting order of a \( C^* \)-algebra

The connecting order of a \( C^* \)-algebra is a constant in \( \mathbb{N} \cup \{ \infty \} \) arising from a certain graph structure on the primitive spectrum \( \text{Prim}(A) \):

- Two primitive ideals \( P, Q \) of \( A \) are said to be **adjacent**, if \( P \) and \( Q \) cannot be separated by disjoint open subsets of \( \text{Prim}(A) \).

\[
\text{Orc}(A) := \sup \left\{ d(P, Q) : P, Q \in \text{Prim}(A) \text{ such that } d(P, Q) < \infty \right\}
\]
Connecting order of a $C^*$-algebra

The connecting order of a $C^*$-algebra is a constant in $\mathbb{N} \cup \{\infty\}$ arising from a certain graph structure on the primitive spectrum $\text{Prim}(A)$:

- Two primitive ideals $P, Q$ of $A$ are said to be **adjacent**, if $P$ and $Q$ cannot be separated by disjoint open subsets of $\text{Prim}(A)$.

- A **path** of length $n$ from $P$ to $Q$ is a sequence of points $P = P_0, P_1, \ldots, P_n = Q$ such that $P_{i-1}$ is adjacent to $P_i$ for all $1 \leq i \leq n$.

**Distance** $d(P, Q)$ from $P$ to $Q$ is defined as follows:

\[ \frac{\mathcal{D}}{d(P, P)} := 1. \]

\[ \frac{\mathcal{D}}{d(P, Q)} = \begin{cases} \text{equal to the minimal length of a path from } P \text{ to } Q, & \text{if } P \neq Q \text{ and there exists a path from } P \text{ to } Q, \\ \infty, & \text{if there is no path from } P \text{ to } Q. \end{cases} \]

The **connecting order** of $A$ is then defined by

\[ \text{Orc}(A) := \sup \{ d(P, Q) : P, Q \in \text{Prim}(A) \text{ such that } d(P, Q) < \infty \} \]
Connecting order of a $C^*$-algebra

The connecting order of a $C^*$-algebra is a constant in $\mathbb{N} \cup \{\infty\}$ arising from a certain graph structure on the primitive spectrum $\text{Prim}(A)$:

- Two primitive ideals $P, Q$ of $A$ are said to be **adjacent**, if $P$ and $Q$ cannot be separated by disjoint open subsets of $\text{Prim}(A)$.

- A **path** of length $n$ from $P$ to $Q$ is a sequence of points $P = P_0, P_1, \ldots, P_n = Q$ such that $P_{i-1}$ is adjacent to $P_i$ for all $1 \leq i \leq n$.

- The **distance** $d(P, Q)$ from $P$ to $Q$ is defined as follows:
  - $d(P, P) := 1$.
  - If $P \neq Q$ and there exists a path from $P$ to $Q$, then $d(P, Q)$ is equal to the minimal length of a path from $P$ to $Q$.
  - If there is no path from $P$ to $Q$, $d(P, Q) := \infty$.
Connecting order of a $C^\ast$-algebra

The connecting order of a $C^\ast$-algebra is a constant in $\mathbb{N} \cup \{\infty\}$ arising from a certain graph structure on the primitive spectrum $\text{Prim}(A)$:

- Two primitive ideals $P, Q$ of $A$ are said to be **adjacent**, if $P$ and $Q$ cannot be separated by disjoint open subsets of $\text{Prim}(A)$.

- A **path** of length $n$ from $P$ to $Q$ is a sequence of points $P = P_0, P_1, \ldots, P_n = Q$ such that $P_{i-1}$ is adjacent to $P_i$ for all $1 \leq i \leq n$.

- The **distance** $d(P, Q)$ from $P$ to $Q$ is defined as follows:
  - $d(P, P) := 1$.
  - If $P \neq Q$ and there exists a path from $P$ to $Q$, then $d(P, Q)$ is equal to the minimal length of a path from $P$ to $Q$.
  - If there is no path from $P$ to $Q$, $d(P, Q) := \infty$.

- The **connecting order** of $A$ is then defined by

\[
\text{Orc}(A) := \sup\{d(P, Q) : P, Q \in \text{Prim}(A) \text{ such that } d(P, Q) < \infty\}.
\]
Theorem (G., 2013)

The equality \( \text{Der}(A) \cap \overline{\mathcal{E}\ell}(A)^{cb} = \overline{\text{Inn}(A)} \) holds true for all unital \( C^*\)-algebras \( A \) in which every Glimm ideal is prime.
Theorem (G., 2013)

The equality $\text{Der}(A) \cap \mathcal{E}_\ell(A)^{cb} = \text{Inn}(A)$ holds true for all unital $C^*$-algebras $A$ in which every Glimm ideal is prime.

Glimm ideals

Recall that the **Glimm ideals** of a unital $C^*$-algebra $A$ are the ideals generated by the maximal ideals of the centre of $A$. 
Theorem (G., 2013)

The equality $\operatorname{Der}(A) \cap \mathcal{E}\ell(A)^{cb} = \operatorname{Inn}(A)$ holds true for all unital $C^*$-algebras $A$ in which every Glimm ideal is prime.

Glimm ideals

Recall that the **Glimm ideals** of a unital $C^*$-algebra $A$ are the ideals generated by the maximal ideals of the centre of $A$.

If a unital $C^*$-algebra $A$ has only prime Glimm ideals, then $\operatorname{Orc}(A) = 1$, so Somerset's theorem yields that $\operatorname{Inn}(A)$ is closed in the operator norm. Hence:

Corollary

*If every Glimm ideal of a unital $C^*$-algebra $A$ is prime, then every derivation of $A$ which lies in $\mathcal{E}\ell(A)^{cb}$ is inner.*
**Example**

The class of $C^*$-algebras in which every Glimm ideal is prime is fairly large. It includes:

- Prime $C^*$-algebras.
- $C^*$-algebras with Hausdorff primitive spectrum.
- Quotients of $AW^*$-algebras.
- Local multiplier algebras.
The class of $C^*$-algebras in which every Glimm ideal is prime is fairly large. It includes:

- Prime $C^*$-algebras.
The class of $C^*$-algebras in which every Glimm ideal is prime is fairly large. It includes:

- Prime $C^*$-algebras.
- $C^*$-algebras with Hausdorff primitive spectrum.
The class of $C^*$-algebras in which every Glimm ideal is prime is fairly large. It includes:

- Prime $C^*$-algebras.
- $C^*$-algebras with Hausdorff primitive spectrum.
- Quotients of $AW^*$-algebras.
Example

The class of $C^*$-algebras in which every Glimm ideal is prime is fairly large. It includes:

- Prime $C^*$-algebras.
- $C^*$-algebras with Hausdorff primitive spectrum.
- Quotients of $AW^*$-algebras.
- Local multiplier algebras.
**Example**

The class of $C^*$-algebras in which every Glimm ideal is prime is fairly large. It includes:

- Prime $C^*$-algebras.
- $C^*$-algebras with Hausdorff primitive spectrum.
- Quotients of $AW^*$-algebras.
- Local multiplier algebras.

**Corollary**

*For each $C^*$-algebra $A$ the following conditions are equivalent:*

(a) $M_{\text{loc}}(A)$ admits only inner derivations.

(b) Every derivation of $M_{\text{loc}}(A)$ admits a cb-norm approximation by elementary operators.
Question

Does there exist a $C^*$-algebra $A$ which admits an outer derivation that is also an elementary operator on $A$?

Motivated by our previous discussion, it is natural to start looking for potential examples in the class of $C^*$-algebras with $\text{Orc}(A) = \infty$.

Example (G., 2010)

Let $A$ be a unital $C^*$-algebra consisting of all elements $a \in C([0, \infty]) \otimes M_2$ such that $a(n) = \begin{bmatrix} \lambda_n(a) & 0 \\ 0 & \lambda_{n+1}(a) \end{bmatrix}$ for all $n \in \mathbb{N}$, for some convergent sequence $(\lambda_n(a))$ of complex numbers. Then:

(a) $d(\ker \lambda_1, \ker \lambda_n) = n$ for all $n \in \mathbb{N}$. In particular, $\text{Orc}(A) = \infty$.

(b) $E_{\ell}(A)$ is closed in the cb-norm. In particular, $A$ admits an outer derivation that is also an elementary operator on $A$. 

Ilja Gogić (University of Zagreb)
Question

Does there exist a $C^*$-algebra $A$ which admits an outer derivation that is also an elementary operator on $A$?

Motivated by our previous discussion, it is natural to start looking for potential examples in the class of $C^*$-algebras with $\text{Orc}(A) = \infty$. 

Example (G., 2010)

Let $A$ be a unital $C^*$-algebra consisting of all elements $a \in C([0, \infty]) \otimes M_2$ such that $a(n) = [\lambda_n(a) 0 0 \lambda_{n+1}(a)]$ for all $n \in \mathbb{N}$, for some convergent sequence $(\lambda_n(a))$ of complex numbers. Then:

(a) $d(\ker \lambda_1, \ker \lambda_n) = n$ for all $n \in \mathbb{N}$. In particular, $\text{Orc}(A) = \infty$.

(b) $E_\ell(A)$ is closed in the cb-norm. In particular, $A$ admits an outer derivation that is also an elementary operator on $A$. 

Ilja Gogić (University of Zagreb)
**Question**

Does there exist a $C^*$-algebra $A$ which admits an outer derivation that is also an elementary operator on $A$?

Motivated by our previous discussion, it is natural to start looking for potential examples in the class of $C^*$-algebras with $Orc(A) = \infty$.

**Example (G., 2010)**

Let $A$ be a unital $C^*$-algebra consisting of all elements $a \in C([0, \infty)) \otimes \mathbb{M}_2$ such that

$$a(n) = \begin{bmatrix} \lambda_n(a) & 0 \\ 0 & \lambda_{n+1}(a) \end{bmatrix} \quad (n \in \mathbb{N}),$$

for some convergent sequence $(\lambda_n(a))$ of complex numbers. Then:

(a) $d(\ker \lambda_1, \ker \lambda_n) = n$ for all $n \in \mathbb{N}$. In particular, $Orc(A) = \infty$.

(b) $E\ell(A)$ is closed in the cb-norm.

In particular, $A$ admits an outer derivation that is also an elementary operator on $A$. 
I end this talk with some connected questions which I find to be interesting:

**Problem**

Do we always have $\overline{\text{Inn}(A)} \subseteq \mathcal{E}\ell(A)$? In particular, does every unital $C^*$-algebra $A$ with $\text{Orc}(A) = \infty$ admit an outer derivation that is also an elementary operator on $A$?
I end this talk with some connected questions which I find to be interesting:

**Problem**

Do we always have $\overline{\text{Inn}(A)} \subseteq \mathcal{E}_\ell(A)$? In particular, does every unital $C^*$-algebra $A$ with $\text{Orc}(A) = \infty$ admit an outer derivation that is also an elementary operator on $A$?

**Problem**

Does there exist a unital $C^*$-algebra $A$ which admits an outer derivation $\delta$ that is also an elementary operator of length 2, i.e. $\delta = M_{a_1,b_1} + M_{a_2,b_2}$ for some $a_i, b_i \in A$?
I end this talk with some connected questions which I find to be interesting:

**Problem**

Do we always have $\text{Inn}(A) \subseteq \mathcal{E}\ell(A)$? In particular, does every unital $C^*$-algebra $A$ with $\text{Orc}(A) = \infty$ admit an outer derivation that is also an elementary operator on $A$?

**Problem**

Does there exist a unital $C^*$-algebra $A$ which admits an outer derivation $\delta$ that is also an elementary operator of length 2, i.e. $\delta = M_{a_1,b_1} + M_{a_2,b_2}$ for some $a_i, b_i \in A$?

**Problem**

What can be said about $\text{Der}(A) \cap \overline{\mathcal{E}\ell(A)}$?