The cb-norm approximation of derivations and automorphisms by elementary operators

Ilja Gogić

Department of Mathematics
University of Zagreb

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C*-algebraic formulation of Quantum Mechanics

In quantum mechanics a physical system is typically described via a unital C*-algebra $A$. The self-adjoint elements of $A$ are thought of as the observables; they are the measurable quantities of the system. A state of the system is defined as a positive functional on $A$ (i.e. a linear map $\omega: A \to \mathbb{C}$ such that $\omega(a^*a) \geq 0$ for all $a \in A$) with $\omega(1_A) = 1$. If the system is in the state $\omega$, then $\omega(a)$ is the expected value of the observable $a$.

Automorphisms correspond to the symmetries, while one-parameter automorphism groups $\{\Phi_t\}_{t \in \mathbb{R}}$ describe the reversible time evolution of the system (in the Heisenberg picture). Their infinitesimal generators $\delta(x) := \lim_{t \to 0} \frac{1}{t}(\Phi_t(x) - x)$ are the $\ast$-derivations.
Introduction

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**Definition**

The **multiplier algebra** $M(A)$ of $A$ is the largest unitization of $A$; it consists of all elements $x \in A^{**}$ (the enveloping von Neumann algebra) such that $ax \in A$ and $xa \in A$ for all $a \in A$. 
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Derivation of $A$ is a linear map $\delta : A \to A$ satisfying the Leibniz rule

$$\delta(xy) = \delta(x)y + x\delta(y)$$

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If there exists a multiplier $a \in M(A)$ such that $\delta(x) = ax - xa$ for all $x \in A$, $\delta$ is said to be an inner derivation.
In the application to physics, innerness of a derivation corresponds to the question whether the Hamiltonian of the system under consideration belongs to the algebraic model.
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**Main problem**

Which $C^*$-algebras admit only inner derivations?

Some $C^*$-algebras which admit only inner derivations:

- von Neumann algebras (Kadison-Sakai 1966);
- simple $C^*$-algebras (Sakai 1968);
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- All type I $\mathcal{AW}^*$-algebras are monotone complete (Hamana 1981), but it is unknown whether all $\mathcal{AW}^*$-algebras are monotone complete; this is a long standing open problem dating back to the work of Kaplansky.
Homogeneous $C^*$-algebras

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- More generally, if $E$ is an algebraic $\mathbb{M}_n$-bundle over a locally compact Hausdorff space $X$, i.e. $E$ is a locally trivial fibre bundle with fibre $\mathbb{M}_n$ and structure group $\text{Aut}(\mathbb{M}_n) \cong PU(n)$ (the projective unitary group), then the set $\Gamma_0(E)$ of all continuous sections of $E$ vanishing at infinity is an $n$-homogeneous $C^*$-algebra, with respect to the fiberwise operations and sup-norm.
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- By a famous theorem due to Fell and Tomiyama-Takesaki from 1961, every $n$-homogeneous $C^*$-algebra $A$ can be realized as $A = \Gamma_0(E)$ for some algebraic $M_n$-bundle $E$ over $\text{Prim}(A)$.
Back to the main problem, the separable case was completely solved in 1979:

**Theorem (Akemann, Elliott, Pedersen and Tomiyama 1979)**

Let $A$ be a separable $C^*$-algebra, then the following conditions are equivalent:

(i) $A$ admits only inner derivations.

(ii) $A = A_1 \oplus A_2$, where $A_1$ is a continuous-trace $C^*$-algebra, and $A_2$ is a direct sum of simple $C^*$-algebras.
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On the other hand, for inseparable $C^*$-algebras the problem of innerness of derivations remains widely open, even for the simplest cases such as subhomogeneous $C^*$-algebras (i.e. $C^*$-algebras which have finite-dimensional irreducible representations of bounded degree).
The local multiplier algebra

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The local multiplier algebra of $A$ is the direct limit $C^*$-algebra

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Ilja Gogić (University of Zagreb)  
Lisbon, 2019
Example

If $A = C_0(X)$ is a commutative $C^*$-algebra, then $M_{loc}(A)$ is a commutative $AW^*$-algebra whose maximal ideal space can be identified with the inverse limit $\lim_{\leftarrow} \beta U$ of Stone-Čech compactifications $\beta U$ of dense open subsets $U$ of $X$. 

The concept of the local multiplier algebra was introduced by Pedersen in 1978 (he called it the "$C^*$-algebra of essential multipliers").

Theorem (Pedersen 1978)

Every derivation of a $C^*$-algebra $A$ extends uniquely and under preservation of the norm to a derivation of $M_{loc}(A)$. Moreover, if $A$ is separable (or more generally, if every essential closed ideal of $A$ is $\sigma$-unital), this extension becomes inner in $M_{loc}(A)$.

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Since $M_{\text{loc}}(A) = M(A)$ if $A$ is simple, and $M_{\text{loc}}(A) = A$ if $A$ is an $AW^*$-algebra, an affirmative answer in the inseparable case would cover, extend and unify the results that all derivations of simple $C^*$-algebras and $AW^*$-algebras are inner.

This led Pedersen to ask:

Problem of innerness of derivations of $M_{\text{loc}}(A)$

If $A$ is an arbitrary $C^*$-algebra, is every derivation of $M_{\text{loc}}(A)$ inner?

It is known that $M_{\text{loc}}(A)$ has only inner derivations for:

- Simple $C^*$-algebras and $AW^*$-algebras (Kadison, Sakai, Olesen);
- Quasi-central separable $C^*$-algebras such that $\text{Prim}(A)$ contains a dense $G_\delta$ subset consisting of closed points (Somerset 2000, Ara-Mathieu 2011);
- $C^*$-algebras with finite-dimensional irreducible representations; in this case $M_{\text{loc}}(A)$ coincides with the injective envelope of $A$ (G. 2013).
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The cb-norm approximation by elementary operators

Let $A$ be a $C^*$-algebra. An attractive and fairly large class of bounded linear maps $\phi : A \rightarrow A$ that preserve all ideals of $A$ is the class of elementary operators, that is, those that can be expressed as a finite sum

$$\phi = \sum_i M_{a_i, b_i}$$

of two-sided multiplications $M_{a_i, b_i} : x \mapsto a_i x b_i$, where $a_i, b_i \in M(A)$. 

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$$\|\phi\|_{cb} := \sup_{n \in \mathbb{N}} \|\phi_n\| < \infty,$$

where for each $n$, $\phi_n$ is an induced map on $M_n(A)$, i.e.

$$\phi_n([a_{ij}]) = [\phi(a_{ij})].$$
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Let us denote by $\mathcal{E} \ell(A)$ the set of all elementary operators on $A$ and by $\mathcal{E} \ell(A)_{cb}$ its cb-norm closure.
We have the following general question:

Which completely bounded operators \( \phi : A \rightarrow A \) admit a cb-norm approximation by elementary operators, i.e. when do we have \( \phi \in E_\ell^c(A) \)?

Since all derivations and \( * \)-automorphisms of \( \mathbb{C}^* \)-algebras \( A \) are completely bounded, the above question in particular applies to those class of maps.

Theorem (G. 2013)

If \( A \) is a unital \( \mathbb{C}^* \)-algebra whose every Glimm ideal is prime, then a derivation \( \delta \) of \( A \) lies in \( E_\ell^c(A) \) cb if and only if \( \delta \) is an inner derivation.

The Glimm ideals of \( A \) are the ideals of \( A \) generated by the maximal ideals of \( Z(A) \).
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**Theorem (G. 2013)**

If \( A \) is a unital \( C^* \)-algebra whose every Glimm ideal is prime, then a derivation \( \delta \) of \( A \) lies in \( \mathcal{E}\ell(A)_{cb} \) if and only if \( \delta \) is an inner derivation.
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The **Glimm ideals** of \( A \) are the ideals of \( A \) generated by the maximal ideals of \( Z(A) \).
Example

The class of $\mathcal{C}_\ast$-algebras whose every Glimm ideal is prime includes:

- prime $\mathcal{C}_\ast$-algebras;
- $\mathcal{C}_\ast$-algebras with Hausdorff primitive spectrum;
- quotients of $\mathcal{AW}_\ast$-algebras;
- local multiplier algebras.

Corollary

The Pederesen's problem has a positive solution if and only if for each $\mathcal{C}_\ast$-algebra $\mathcal{A}$, every derivation of $M_{\text{loc}}(\mathcal{A})$ lies in $E_{\mathcal{L}}(M_{\text{loc}}(\mathcal{A}))_{\text{cb}}$.

For prime $\mathcal{C}_\ast$-algebras we also established the following result:

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If $\mathcal{A}$ is a prime $\mathcal{C}_\ast$-algebra then an algebra epimorphism $\sigma: \mathcal{A} \rightarrow \mathcal{A}$ lies in $E_{\mathcal{L}}(\mathcal{A})_{\text{cb}}$ if and only if $\sigma$ is an inner automorphism of $\mathcal{A}$.
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Ilja Gogić (University of Zagreb)

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In a contrast to the similar result for derivations, the above result cannot be extended even to homogeneous $C^*$-algebras, which admit only inner derivations (by Sproston’s Theorem):

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For $n \geq 2$ let $A_n = C(\mathbb{P}U(n), M_n)$. Then $A_n$ admits outer automorphisms that are simultaneously elementary operators. On the other hand:

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Let $A$ be a separable $n$-homogeneous $C^*$-algebra whose primitive spectrum $X$ is locally contractable. Then every $Z(M(A))$-linear automorphism of $A$ becomes inner when extended to $M_{loc}(A)$. In particular, all (outer) elementary automorphisms on $A_n = C(\mathbb{P}U(n), M_n)$ become inner in $M_{loc}(A_n)$.
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Moreover, if the primitive spectrum of a $C^*$-algebra $A$ is rather pathological, it can happen that $A$ admits both outer derivations and outer automorphisms that are simultaneously elementary operators:

**Example**

Let $A$ be a $C^*$-subalgebra of $B = C([1, \infty), M_2)$ that consists of all $a \in B$ such that

$$a(n) = \begin{bmatrix} \lambda_n(a) & 0 \\ 0 & \lambda_{n+1}(a) \end{bmatrix} \quad (n \in \mathbb{N})$$

for some convergent sequence $(\lambda_n(a))$ of complex numbers. Then $A$ admits outer derivations and outer automorphisms that are also elementary operators. In fact, there are outer derivations of $A$ of the form $\delta = M_a - M_b$ for suitable $a, b \in A$.

**Problem**

Does every automorphism of a $C^*$-algebra $A$ that is also an elementary operator become inner when extended to $M_{loc}(A)$?
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