The cb-norm approximation of derivations and automorphisms by elementary operators

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- The self-adjoint elements of A are thought of as the observables; they are the measurable quantities of the system.
- A state of the system is defined as a positive functional on A (i.e. a linear map $\omega : A \to \mathbb{C}$ such that $\omega(a^*a) \geq 0$ for all $a \in A$) with $\omega(1_A) = 1$. If the system is in the state ω, then $\omega(a)$ is the expected value of the observable a.

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- Automorphisms correspond to the symmetries, while one-parameter automorphism groups $\{\Phi_t\}_{t\in\mathbb{R}}$ describe the reversible time evolution of the system (in the Heisenberg picture). Their infinitesimal generators

$$\delta(x) := \lim_{t\to 0} \frac{1}{t}(\Phi_t(x) - x)$$

are the $*$-derivations.

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Definition

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Derivation *of* A *is a linear map* $\delta : A \to A$ *satisfying the* **Leibniz rule**

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for all $x, y \in A$.
If there exists a multiplier $a \in M(A)$ such that $\delta(x) = ax - xa$ for all $x \in A$, δ is said to be an **inner derivation**.

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- simple C^*-algebras (Sakai 1968);
- AW^*-algebras (Olesen 1974);
- homogeneous C^*-algebras (Sproston 1976 - unital case; G. 2013 - extension to the non-unital case).

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- All type I AW^*-algebras are monotone complete (Hamana 1981), but it is unknown whether all AW^*-algebras are monotone complete; this is a long standing open problem dating back to the work of Kaplansky.

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- More generally, if E is an algebraic $\mathbb{M}_n$-bundle over a locally compact Hausdorff space X, i.e. E is a locally trivial fibre bundle with fibre $\mathbb{M}_n$ and structure group $\mathrm{Aut}(\mathbb{M}_n) \cong PU(n)$ (the projective unitary group), then the set $\Gamma_0(E)$ of all continuous sections of E vanishing at infinity is an n-homogeneous C^*-algebra, with respect to the fiberwise operations and sup-norm.

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- By a famous theorem due to Fell and Tomiyama-Takesaki from 1961, every n-homogeneous C^*-algebra A can be realized as $A = \Gamma_0(E)$ for some algebraic $\mathbb{M}_n$-bundle E over $\mathrm{Prim}(A)$.

Back to the main problem, the separable case was completely solved in 1979:

Theorem (Akemann, Elliott, Pedersen and Tomiyama 1979)

Let A be a separable C^*-algebra, Then the following conditions are equivalent:

(i) *A admits only inner derivations.*

(ii) *$A = A_1 \oplus A_2$, where A_1 is a continuous-trace C^*-algebra, and A_2 is a direct sum of simple C^*-algebras.*

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On the other hand, for inseparable C^*-algebras the problem of innerness of derivations remains widely open, even for the simplest cases such as subhomogeneous C^*-algebras (i.e. C^*-algebras which have finite-dimensional irreducible representations of bounded degree).

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Theorem (Pedersen 1978)

Every derivation of a C^*-algebra A extends uniquely and under preservation of the norm to a derivation of $M_{\mathrm{loc}}(A)$. Moreover, if A is separable (or more generally, if every essential closed ideal of A is σ-unital), this extension becomes inner in $M_{\mathrm{loc}}(A)$.

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In particular, Pedersen's result entails Sakai's theorem that every derivation of a simple unital C^*-algebra is inner.

Since $M_{\mathrm{loc}}(A) = M(A)$ if A is simple, and $M_{\mathrm{loc}}(A) = A$ if A is an AW^*-algebra, an affirmative answer in the inseparable case would cover, extend and unify the results that all derivations of simple C^*-algebras and AW^*-algebras are inner.

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- C^*-algebras with finite-dimensional irreducible representations; in this case $M_{\text{loc}}(A)$ coincides with the injective envelope of A (G. 2013).

The cb-norm approximation by elementary operators

Let A be a C^*-algebra. An attractive and fairly large class of bounded linear maps $\phi : A \to A$ that preserve all ideals of A is the class of **elementary operators**, that is, those that can be expressed as a finite sum

$$\phi = \sum_i M_{a_i,b_i}$$

of **two-sided multiplications** $M_{a_i,b_i} : x \mapsto a_i x b_i$, where $a_i, b_i \in M(A)$.

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$$\|\phi\|_{cb} := \sup_{n\in\mathbb{N}} \|\phi_n\| < \infty,$$

where for each n, ϕ_n is an induced map on $M_n(A)$, i.e.

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Let us denote by $\mathcal{E}\ell(A)$ the set of all elementary operators on A and by $\overline{\mathcal{E}\ell(A)}_{cb}$ its cb-norm closure.

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Theorem (G. 2013)

If A is a unital C^*-algebra whose every Glimm ideal is prime, then a derivation δ of A lies in $\overline{\mathcal{E}\ell(A)}_{cb}$ if and only if δ is an inner derivation.

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Corollary

The Pederesen's problem has a positive solution if and only if for each C^*-algebra A, every derivation of $M_{\mathrm{loc}}(A)$ lies in $\overline{\mathcal{E}\ell(M_{\mathrm{loc}}(A))}_{cb}$.

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For prime C^*-algebras we also established the following result:

Theorem (G. 2019)

If A is a prime C^*-algebra then an algebra epimorphism $\sigma : A \to A$ lies in $\overline{\mathcal{E}\ell(A)}_{cb}$ if and only if σ is an inner automorphism of A.

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On the other hand:

Proposition

Let A be a separable n-homogeneous C^*-algebra whose primitive spectrum X is locally contractable. Then every $Z(M(A))$-linear automorphism of A becomes inner when extended to $M_{\mathrm{loc}}(A)$. In particular, all (outer) elementary automorphisms on $A_n = C(PU(n), \mathbb{M}_n)$ become inner in $M_{\mathrm{loc}}(A_n)$.

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Example

Let A be a C^*-subalgebra of $B = C([1, \infty], \mathbb{M}_2)$ that consists of all $a \in B$ such that If

$$a(n) = \begin{bmatrix} \lambda_n(a) & 0 \\ 0 & \lambda_{n+1}(a) \end{bmatrix} \qquad (n \in \mathbb{N})$$

for some convergent sequence $(\lambda_n(a))$ of complex numbers. Then A admits outer derivations and outer automorphisms that are also elementary operators. In fact, there are outer derivations of A of the form $\delta = M_{a,b} - M_{b,a}$ for suitable $a, b \in A$.

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Problem

Does every automorphism of a C^*-algebra A that is also an elementary operator become inner when extended to $M_{\mathrm{loc}}(A)$?