When are the lengths of elementary operators uniformly bounded?

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Preliminaries

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- The **primitive spectrum** of A, which we denote by $\mathrm{Prim}(A)$, is the set of all primitive ideals of A equipped with the Jacobson topology. Hence, if S is some set of primitive ideals, its closure is

$$\overline{S} = \left\{ P \in \mathrm{Prim}(A) \; : \; P \supseteq \bigcap_{Q \in S} Q \right\}.$$

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  $\mathrm{Prim}(A)$ is a compact space which in general satisfies only T_0-separation axiom.
- A linear map $\phi : A \to A$ is said to be **completely bounded** if

$$\|\phi\|_{cb} := \sup_{n \in \mathbb{N}} \|\phi \otimes \mathrm{id}_{\mathbb{M}_n}\| < \infty.$$

  As usual, by $\mathrm{CB}(A)$ we denote the set of all completely bounded maps on A.

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- It is not difficult to see that an ideal Q of A is 2-primal if and only if for all $P_1, P_2 \in \mathrm{Prim}(A/Q)$ there exists a net in $\mathrm{Prim}(A)$ which converges simultaneously to P_1 and P_2.

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- In particular, $\mathrm{Prim}(A)$ is Hausdorff if and only if

$$\mathrm{Glimm}(A) = \mathrm{Primal}_2(A) \setminus \{A\} = \mathrm{Prim}(A).$$

Basic example

Let A be a C^*-algebra consisting of all elements $a \in C([0,1], \mathbb{M}_3)$ s.t.

$$a(1) = \begin{bmatrix} \lambda_1(a) & 0 & 0 \\ 0 & \lambda_2(a) & 0 \\ 0 & 0 & \lambda_3(a) \end{bmatrix}$$

for some $\lambda_i(a) \in \mathbb{C}$. Let P_t ($t \in [0,1)$) and R_i ($i = 1,2,3$) be, respectively, the kernels of irreducible representations $A \to \mathbb{M}_3$, $a \mapsto a(t)$ and $A \to \mathbb{C}$, $a \mapsto \lambda_i(a)$. Then:

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$$\begin{aligned} \mathrm{Primal}_2(A) &= \{A\} \cup \{P_t \ : \ t \in [0,1)\} \cup \{R_1, R_2, R_3\} \\ &\quad \cup \{R_1 \cap R_2, R_1 \cap R_3, R_2 \cap R_3\} \cup \{R_1 \cap R_2 \cap R_3\}. \end{aligned}$$

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- A famous theorem of Fell and Tomiyama-Takesaki asserts that for any n-homogeneous C^*-algebra B with primitive spectrum X there is a locally trivial bundle $\mathcal{E}$ over X with fibre $\mathbb{M}_n$ and structure group $PU(n) = \mathrm{Aut}(\mathbb{M}_n)$ such that A is isomorphic to the algebra $\Gamma_0(\mathcal{E})$ of sections of $\mathcal{E}$ which vanish at infinity.

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- If each homogeneous sub-quotient of B has the finite type property, we say that B has the finite type property.

Elementary operators and canonical contraction θ_A

Definition

An **elementary operator** *on A is a map $\phi : A \to A$ which can be written as a finite sum of two-sided multiplication maps $M_{a,b} : x \mapsto axb$ ($a, b \in A$). By $\mathcal{E}\ell(A)$ we denote the set of all elementary operators on A*

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Hence, if we endow the algebraic tensor product $A \otimes A$ with the Haagerup norm

$$\|t\|_h := \inf \left\{ \left\| \sum_i a_i a_i^* \right\|^{\frac{1}{2}} \left\| \sum_i b_i^* b_i \right\|^{\frac{1}{2}} \ : \ t = \sum_i a_i \otimes b_i \right\},$$

we obtain a well-defined contraction

$$(A \otimes A, \|\cdot\|_h) \to (\mathcal{E}\ell(A), \|\cdot\|_{cb}), \quad \text{given by} \quad \sum_i a_i \otimes b_i \mapsto \sum_i M_{a_i,b_i}.$$

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Remark

If A contains a pair of non-zero orthogonal ideals, then θ_A cannot be injective. Hence, the necessary condition for the injectivity of θ_A is that A must be a prime C^*-algebra.

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Theorem (Haagerup 1980, Chatterjee-Sinclair 1992, Mathieu 2003)

$$A \text{ is prime} \iff \theta_A \text{ is injective} \iff \theta_A \text{ is isometric.}$$

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Theorem (Somerset 1998)

$$\ker \theta_A = \bigcap \{Q \otimes_h A + A \otimes_h Q \ : \ Q \in \mathrm{Primal}_2(A)\}.$$

Length of elementary operators

The **length** of an elementary operator $\phi \neq 0$ is the smallest $\ell = \ell(\phi) \in \mathbb{N}$ such that $\phi = \sum_{i=1}^{\ell} M_{a_i,b_i}$ for some $a_i, b_i \in A$. We also define $\ell(0) = 0$.

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In particular, if $A = \mathbb{M}_n$, then $\ell(\phi) \leq n^2$ for all $\phi \in \mathcal{E}\ell(A)$. Further, if ϕ is the transpose map $X \mapsto X^\tau$, then $\phi = \sum_{i,j=1}^{n} M_{e_{ij},e_{ij}}$ (where (e_{ij}) are standard matrix units in $\mathbb{M}_n$), so $\ell(\phi) = n^2$. Hence,

$$\sup\{\ell(\phi) \ : \ \phi \in \mathcal{E}\ell(\mathbb{M}_n)\} = n^2.$$

Let us consider the following conditions of a C^*-algebra A:

(P1) A as a Z-module is finitely generated, i.e. there exists finitely many elements $a_1, \ldots, a_n \in A$ such that every $a \in A$ can be written as $a = \sum_{i=1}^{n} z_i a_i$ for some $z_i \in Z$.

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Theorem (G. 2011)

A C^*-algebra A satisfies (P1) if and only if A is a finite direct sum of unital homogeneous C^*-algebras.

Results

Theorem (G. 2011, 2012)

If A satisfies (P3) then

$$\sup\{\dim(A/Q)\ :\ Q \in \mathrm{Primal}_2(A)\} < \infty.$$

In particular, A is subhomogeneous. Further, if A is separable, A must have a finite type property.

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Theorem (G. 2012; Hausdorffness of $\mathrm{Prim}(A)$ is crucial)

There exists a compact subset X of $\mathbb{R}$ and a unital C^*-subalgebra A of $C(X, \mathbb{M}_2)$ with trivial homogeneous sub-quotients such that $\sup\{\dim(A/Q) : Q \in \mathrm{Primal}_2(A)\} = \infty$. Hence, A doesn't satisfy (P3).

Corollary

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$(P3) \not\Rightarrow (P2)$ when $\mathrm{Prim}(A)$ is not Hausdorff in general:

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$$a(1/n) = \begin{bmatrix} \lambda_n(a) & 0 \\ 0 & \lambda_{n+1}(a) \end{bmatrix} \quad (n \in \mathbb{N}),$$

  for some convergent sequence $(\lambda_n(a))$ of complex numbers. Then A satisfies (P3) but has a Glimm ideal of infinite codimension (namely $G = \bigcap_{i=1}^{\infty} \ker \lambda_i$). In particular, A doesn't satisfy (P2).