Topologically finitely generated Hilbert $C(X)$-modules

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This talk is based on my papers:

- I. Gogić, *Topologically finitely generated Hilbert $C(X)$-modules* (2011), submitted to J.M.A.A.
- I. Gogić, *On derivations and elementary operators on C^*-algebras* (2011), submitted to Proc. Edinburgh Math. Soc.
- I. Gogić, *Elementary operators and subhomogeneous C^*-algebras*, Proc. Edinburgh Math. Soc. 54 (2011), no. 1, 99–111.

- Hilbert C^*-modules form a category in between Banach spaces (they have a little extra geometrical structure) and Hilbert spaces (they are not as well behaving as these).
- A Hilbert C^*-module obeys the same axioms as an ordinary Hilbert space, except that the inner product takes values in a more general C^*-algebras than $\mathbb{C}$.
- Hilbert C^*-modules were first introduced in the work of I. Kaplansky in 1953, who developed the theory for unital commutative C^*-algebras. In the 1970s the theory was extended to non-commutative C^*-algebras independently by W. Paschke and M. Rieffel.
- Hilbert C^*-modules appear naturally in many areas of C^*-algebra theory, such as KK-theory, Morita equivalence of C^*-algebras, and completely positive operators.

Definition

Let A be a C^*-algebra. A (left) **Hilbert A-module** is a left A-module V, equipped with an A-valued inner product $\langle \cdot, \cdot \rangle$ which is A-linear in the first and conjugate linear in the second variable, such that V is a Banach space with the norm

$$\|v\| := \sqrt{\|\langle v, v\rangle\|_A}.$$

Example

Every C^*-algebra A becomes a Hilbert A-module with respect to the inner product

$$\langle a, b\rangle := ab^*.$$

Example (continued)

Similarly, the direct sum A^n of n-copies of A becomes a A-Hilbert module with respect to the pointwise operations and the inner product

$$\langle a_1 \oplus \cdots \oplus a_n, b_1 \oplus \cdots \oplus b_n \rangle := \sum_{k=1}^{n} a_k b_k^*.$$

More generally, let

$$\mathcal{H}_A := \{(a_k) \in \prod_1^\infty \ : \ \sum_{k=1}^{\infty} a_k a_k^* \text{ is norm convergent}\}.$$

Then the pointwise operations and the inner product

$$\langle (a_k), (b_k) \rangle := \sum_{k=1}^{\infty} a_k b_k^*$$

turn $\mathcal{H}_A$ into a Hilbert A-module.

$\mathcal{H}_A$ is known as a **standard Hilbert A-module**.

Definition

Let V be a Hilbert A-module. We say that V is

- **algebraically finitely generated** if there exists a finite subset of V whose A-linear span equals V;
- **topologically finitely generated** if there exists a finite subset of V whose A-linear span is dense in V;
- **countably generated** if there exists a countable subset of V whose A-linear span is dense in V.

When the C^*-algebra A is unital and commutative, there exists a categorical equivalence between Hilbert A-modules and (F) Hilbert bundles over the spectrum of the algebra. (F) Hilbert bundles provide a natural generalization of standard vector bundles from topology.

Definition

By an **(F) Hilbert bundle** ((F) stands for Fell) we mean a triple $\mathcal{E} := (p, E, X)$ where E and X are topological spaces with a continuous open surjection $p : E \to X$, together with operations and norms making each **fibre** $E_x := p^{-1}(x)$ $(x \in X)$ into a complex Hilbert space, such that the following conditions are satisfied:

(A1) The maps $\mathbb{C} \times E \to E$, $E \times_X E \to E$ and $E \to \mathbb{R}$, given in each fibre by scalar multiplication, addition, and the norm, respectively, are continuous. Here $E \times_X E$ denotes the **Whitney sum**

$$\{(e, f) \in E \times E \ : \ p(e) = p(f)\}.$$

(A2) If $x \in X$ and if (e_α) is a net in E such that $\|e_\alpha\| \to 0$ and $p(e_\alpha) \to x$ in X, then $e_\alpha \to 0_x$ in E (where 0_x is the zero-element of E_x).

As usual, we say that p is the **projection**, E is the **bundle space** and X is the **base space** of $\mathcal{E}$.

If in (A2) one only requires that the norm function is upper semicontinuous, one gets the notion of an **(H) Hilbert bundle** ((H) stands for Hofmann).

If $\mathcal{E}$ is an (F) Hilbert bundle, then using a polarization identity together with the continuity of the norm and operations, it is an immediate consequence that the map $E \times_X E \to \mathbb{C}$ given by the inner product in each fibre is continuous.

For the (F) Hilbert bundles $\mathcal{E} = (p, E, X)$ and $\mathcal{E}' = (p', E', X')$ we say that $\Phi : \mathcal{E} \to \mathcal{E}'$ is a **Hilbert bundle map** if Φ is a pair $\Phi = (\phi, f)$ of maps, where $\phi : E \to E'$ and $f : X \to X'$ are continuous maps such that

(i) the following diagram

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E' \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p'} \\ X & \xrightarrow{f} & X' \end{array}$$

is commutative,

(ii) for each $x \in X$, ϕ defines a linear map from E_x into $E'_{f(x)}$.

It is usually said that Φ **covers** f. If in addition ϕ defines an isometric isomorphism of each fibre E_x onto $E'_{f(x)}$, then we say that Φ is a **strong Hilbert bundle map**. If $X' = X$, we write $\Phi : \mathcal{E} \cong \mathcal{E}'$ to say that Φ is an **isomorphism of Hilbert bundles**, that is, Φ is a strong Hilbert bundle map covering the identity map $\mathrm{id}_X : X \to X$.

Example

The simplest example of an (F) Hilbert bundle is the **product bundle** over X with fibre H,

$$\epsilon(X, H) := (\mathrm{proj}_1, X \times H, H),$$

where H is a Hilbert space.

Example

Suppose that $\mathcal{E}$ is an n-dimensional (locally trivial) complex vector bundle over a compact Hausdorff space. Then $\mathcal{E}$ becomes an (F) Hilbert bundle if one chooses a Riemannian metric on $\mathcal{E}$. Furthermore, if $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are two Riemannian metrics on $\mathcal{E}$, then the formal identity map $\mathrm{id} : (\mathcal{E}, \langle\cdot,\cdot\rangle_1) \to (\mathcal{E}, \langle\cdot,\cdot\rangle_2)$ defines an isomorphism of (F) Hilbert bundles. If we make a polar decomposition $\mathrm{id} = UP$, where P is positive and U is unitary, then U provides a strong Hilbert bundle map between these two bundles. Hence, an (F) Hilbert bundle structure on a vector bundle is essentially unique.

If $\mathcal{E} = (p, E, X)$ is an (F) Hilbert bundle and $Y \subseteq X$ then we denote by

$$\mathcal{E}|_Y := (p|_{p^{-1}(Y)}, p^{-1}(Y), Y)$$

the **restriction** of $\mathcal{E}$ to Y.

We say that $\mathcal{E} = (p, E, X)$ is

- **trivial** if $\mathcal{E} \cong \epsilon(X, H)$ for some Hilbert space H;
- **locally trivial** if there exists a Hilbert space H and an open cover $\mathcal{U}$ of X such that for each $U \in \mathcal{U}$ we have $\mathcal{E}|_U \cong \epsilon(U, H)$.
- If in addition X admits a finite open cover over which $\mathcal{E}$ is locally trivial, we say that $\mathcal{E}$ is of **finite type**.

If all fibres of an (F) Hilbert bundle $\mathcal{E}$ have the same finite dimension n, then we say that $\mathcal{E}$ is n-**homogeneous**.

The next fact is an easy consequence of the continuity of the operations, and a pointwise application of the Gram-Schmidt orthonormalization process.

Proposition

If $\mathcal{E}$ is an n-homogeneous (F) Hilbert bundle, then $\mathcal{E}$ is locally trivial. In particular, if the base space X of $\mathcal{E}$ is compact, then $\mathcal{E}$ is of finite type.

Remark

Hence, the category of n-homogeneous (F) Hilbert bundles over compact Hausdorff spaces is equivalent to the category of n-dimensional (locally trivial) complex vector bundles.

If all fibres of an (F) Hilbert bundle $\mathcal{E}$ are finite dimensional with

$$n := \sup_{x \in X} \dim E_x < \infty,$$

then we say that $\mathcal{E}$ is n-**subhomogeneous**. In this case every restriction bundle of $\mathcal{E}$ over a set where $\dim E_x$ is constant is locally trivial, by the previous Proposition. If in addition every such restriction bundle is of finite type, then we say that $\mathcal{E}$ is n-**subhomogeneous of finite type**.

By a **section** of an (F) Hilbert bundle $\mathcal{E} = (p, E, X)$ we mean a map $s : X \to E$ such that

$$p(s(x)) = x \quad (x \in X).$$

By $\Gamma(\mathcal{E})$ we denote the set of all continuous of sections of $\mathcal{E}$.

If X is compact, then $\Gamma(\mathcal{E})$ becomes a Hilbert $C(X)$-module with respect to the action

$$(\varphi s)(x) := \varphi(x)s(x)$$

and inner product

$$\langle s, u\rangle(x) := \langle s(x), u(x)\rangle_x,$$

where $\langle \cdot, \cdot \rangle_x$ denotes the inner product on fibre E_x.

At the end of this introductory, let us briefly describe how for a given Hilbert $C(X)$-module V (X is a compact Hausdorff space) one constructs a canonical Hilbert bundle $\mathcal{E}_V$.

- For $x \in X$ let I_x be the maximal ideal of $C(X)$ consisting of all functions which vanish at x, and put

$$J_x := I_x V = \{\varphi v \ : \ \varphi \in I_x, v \in V\}.$$

  Then J_x is a closed submodule of V, by the Hewitt-Cohen factorization theorem.

- Set $E_x := V/J_x$, let $\pi_x : V \to E_x$ be the quotient map, let

$$E := \bigsqcup_{x \in X} E_x,$$

  and let $p : E \to X$ be the canonical projection.

- Since for each $v \in V$ and $x \in X$ we have $\|\pi_x(v)\| = \sqrt{\langle v, v\rangle(x)}$, the function $X \to \mathbb{R}_+$, $x \mapsto \|\pi_x(v)\|$ is continuous.
- Hence, by Fell's theorem, there exists a unique topology on E for which $\mathcal{E}_V := (p, E, X)$ becomes an (F) Hilbert bundle. We say that $\mathcal{E}_V$ is the **canonical (F) Hilbert bundle associated to** V.

Now we can define the **generalized Gelfand transform** $\Gamma_V : V \to \Gamma(\mathcal{E}_V)$, which sends $v \in V$ to $\hat{v} \in \Gamma(\mathcal{E}_V)$, where

$$\hat{v}(x) := v(x) := \pi_x(v) \quad (x \in X).$$

Theorem

Γ_V is an isometric $C(X)$-linear isomorphism between Hilbert $C(X)$-modules V and $\Gamma(\mathcal{E}_V)$.

The next theorem is just a Hilbert module version of the celebrated Serre-Swan theorem.

Theorem

Let V be a Hilbert $C(X)$-module, where X is a compact Hausdorff space, and let $\mathcal{E} := \mathcal{E}_V$. Then the following conditions are equivalent:

- **(i)** *V is a.f.g.;*
- **(ii)** *V is a.f.g. and projective;*
- **(iii)** *there exists a finite clopen partition $X = X_1 \sqcup \cdots \sqcup X_k$ such that each restriction bundle $\mathcal{E}|_{X_i}$ is homogeneous.*

Proof.

(i) $\Rightarrow$ (ii). Every a.f.g. Hilbert module over a unital (not necessarily commutative) C^*-algebra A is automatically projective. This is a consequence of the Kasparov stabilization theorem, which says that if W is a c.g. Hilbert A-module, then $E \oplus \mathcal{H}_A \cong \mathcal{H}_A$, where $\mathcal{H}_A$ is a standard Hilbert A-module.

Proof (continued).

(ii) $\Rightarrow$ (iii). We may assume that

$$V = PC(X)^n \quad (n \in \mathbb{N})$$

for some ($C(X)$-linear self-adjoint) projection $P \in \mathbb{B}(C(X)^n) = \mathrm{M}_n(C(X))$. Then $E_x = P(x)\ell_2^n$ for all $x \in X$. Since $\mathrm{rank}(P(x)) = \mathrm{trace}(P(x))$, the dimension function

$$\dim : X \to \{0, 1, \ldots, n\}, \quad x \mapsto \dim E_x = \mathrm{rank}(P(x))$$

is continuous. If $0 \leq n_1 < \ldots < n_k \leq n$ are its values, put

$$X_i := \{x \in X \ : \ \dim E_x = n_i\}.$$

Then $X = X_1 \sqcup \cdots \sqcup X_k$ is a desired clopen partition of X.
(iii) $\Rightarrow$ (i). This is easy. □

The main difference between a.f.g and t.f.g. Hilbert $C(X)$-modules is the fact that t.f.g. Hilbert $C(X)$-modules are not generally projective. Hence, the dimension of the fibres of the canonical (F) Hilbert bundle may vary, even if X is connected.

Example

let X be the unit interval $[0,1]$ and let $V := C_0((0,1])$. Then V becomes a Hilbert $C([0,1])$-module with respect to the standard action and inner product $\langle f, g \rangle = f^* g$. Note that V is topologically singly generated (for instance, the identity function $f(x) = x$ is such generator, by the Weierstrass approximation theorem). On the other hand, each fibre E_x of $\mathcal{E}_V$ is one-dimensional, except E_0, which is zero.

However, this phenomenon is in fact the only major difference between the classes of a.f.g. and t.f.g. Hilbert $C(X)$-modules, at least when X is metrizable.

Theorem (G. 2011)

Let X be a compact metrizable space and let V be a Hilbert $C(X)$-module with the canonical (F) Hilbert bundle $\mathcal{E}_V$. Then the following conditions are equivalent:

(i) *V is t.f.g.;*

(ii) *$\mathcal{E}_V$ is subhomogeneous of finite type.*

The proof of the theorem relays on the next two facts:

Lemma

Let $\mathcal{E}$ be an (F) Hilbert bundle over a compact metrizable space X. Then the following conditions are equivalent:

(i) *$\mathcal{E}$ is subhomogeneous of finite type;*

(ii) *There exists a finite number of sections $s_1, \ldots, s_m \in \Gamma(\mathcal{E})$ which satisfy*

$$\mathrm{span}_{\mathbb{C}}\{s_1(x), \ldots, s_m(x)\} = E_x$$

for all $x \in X$.

Proposition

Let $\mathcal{E}$ be an (F) Banach bundle over a compact space X. A $C(X)$-submodule $W \subseteq \Gamma(\mathcal{E})$ is dense in $\Gamma(\mathcal{E})$ if and only if for each $x \in X$,

$$\{s(x) \ : \ s \in W\}$$

is dense in E_x.

Proof of theorem.

Let $\mathcal{E} := \mathcal{E}_V$. We identify V with $\Gamma(\mathcal{E})$ using the generalized Gelfand transform.

(i) $\Rightarrow$ (ii). Let $s_1, \ldots, s_m \in \Gamma(\mathcal{E})$ be sections whose $C(X)$-linear span is dense in $\Gamma(\mathcal{E})$. Obviously,

$$W_x := \mathrm{span}_{\mathbb{C}}\{s_1(x), \ldots, s_m(x)\}$$

is dense in E_x for each $x \in X$. Since obviously $\dim W_x < \infty$, we conclude that $W_x = E_x$ for each $x \in X$. Now we apply our lemma.

Proof of theorem (continued).

(ii) $\Rightarrow$ (i). By lemma, there exist $s_1, \ldots, s_m \in \Gamma(\mathcal{E})$ which satisfy

$$\mathrm{span}_{\mathbb{C}}\{s_1(x), \ldots, s_m(x)\} = E_x$$

for all $x \in X$. The claim now follows from proposition. $\square$

Now we shall present another characterizations of t.f.g. Hilbert $C(X)$-modules.

First recall that if X is a locally compact Hausdroff space and if V and W are (left) Banach $C_0(X)$-modules, then the $C_0(X)$-**projective tensor product** $V \overset{\pi}{\otimes}_{C_0(X)} W$ of V and W is by definition the quotient of the (completed) projective tensor product $V \overset{\pi}{\otimes} W$ by the closure of the liner span of tensors of the form

$$\varphi v \otimes w - v \otimes \varphi w,$$

where $v \in V$, $w \in W$ and $\varphi \in C_0(X)$. For $t \in V \overset{\pi}{\otimes} W$, by t_X we denote the canonical image of t in $V \overset{\pi}{\otimes}_{C_0(X)} W$.

Definition

Let V be a non-degenerate Banach $C_0(X)$-module, where X is a locally compact Hausdorff space.

(i) If W is another Banach $C_0(X)$-module, then for $t \in V \overset{\pi}{\otimes} W$ we define a $C_0(X)$-**projective rank of** t, denoted by $\mathrm{rank}_X^\pi(t)$, as the smallest nonnegative integer k for which there exists a rank k tensor $u \in V \otimes W$ such that $t_X = u_X$ in $V \overset{\pi}{\otimes}_{C_0(X)} W$. If such k does not exist, we define $\mathrm{rank}_X^\pi(t) := \infty$.

(ii) If there exists $K \in \mathbb{N}$ such that for every Banach $C_0(X)$-module W, and every tensor $t \in V \overset{\pi}{\otimes} W$ we have $\mathrm{rank}_X^\pi(t) \leq K$, then we say that V is of **finite** $C_0(X)$-**projective rank**. The smallest number K with this property is denoted by $\mathrm{rank}_X^\pi(V)$. If such K does not exist, we define $\mathrm{rank}_X^\pi(V) := \infty$.

Here is a sufficient condition for V to be of finite $C_0(X)$-projective rank.

Proposition

Let V be a non-degenerate Banach $C_0(X)$-module, where X is a locally compact Hausdorff space. Let us say that V satisfies the **condition (P)** *if there exists $K \in \mathbb{N}$ such that for every sequence $(a_i) \in \ell^1(V)$ there exist $k \leq K$, elements $v_1, \ldots, v_k \in V$ and sequences $(\varphi_{i,1})_i, \ldots, (\varphi_{i,k})_i \in \ell^1(C_0(X))$ such that*

$$a_i = \sum_{j=1}^{k} \varphi_{i,j} v_j$$

for all $i \in \mathbb{N}$. If V satisfies (P), then $\mathrm{rank}_X^{\pi}(V) \leq K$.

Remark

Note that the condition (P) in particular implies that V is **weakly algebraically finitely generated**, in a sense that every a.f.g. submodule of V can be generated by $k \leq K$ generators.

Now we are ready to state the final result.

Theorem (G. 2011)

Let V be Hilbert $C(X)$-module, where X is a compact metrizable space. The following conditions are equivalent:

(i) *V is t.f.g.;*

(ii) *V satisfies the condition (P);*

(iii) *V is of finite $C(X)$-projective rank;*

(iv) *V is weakly a.f.g.*

Remark

Our proof of the above theorem essentially relies on sectional representation $V = \Gamma(\mathcal{E}_V)$.

- We can also try to generalize obtained results for a larger class of Banach $C(X)$-modules.
- One can similarly define a notion of an (F) and (H) Banach bundle.
- If M a Banach $C(X)$-module, one can also similarly construct the canonical Banach bundle $\mathcal{E}_M$. However, $\mathcal{E}_M$ is only an (H) bundle in general, and the generalized Gelfand transform $\Gamma_M : M \to \Gamma(\mathcal{E}_M)$ fails to be isometric.

If M is a Banach $C(X)$-module one can of course look for conditions which might guarantee that Γ_M is isometric. This was solved by K. Hofmann in 1974.

Definition

Let M be a banach $C(X)$-module. We say that M is $C(X)$-**locally convex** if for any pair $\varphi_1, \varphi_2 \in C(X)_+$ with $\varphi_1 + \varphi_2 = 1$ and $s_1, s_2 \in M$, we have

$$\|\varphi_1 s_1 + \varphi_2 s_2\| \leq \max\{\|s_1\|, \|s_2\|\}.$$

Theorem

If M is a Banach $C(X)$-module, then Γ_M is an isometric isomorphism from M onto $\Gamma(\mathcal{E}_M)$ if and only if M is $C(X)$-locally convex.

Example

Suppose that A is a unital Banach algebra with the center Z. If C is a C^*-subalgebra of Z, then A can be viewed as a Banach C-module, under the natural action. If in addition A is a C^*-algebra, then using the Dauns-Hofmann it is easy to see that A is C-locally convex. In particular, $A = \Gamma(\mathcal{E}_A)$, where $\mathcal{E}_A$ is the canonical bundle of A over $\mathrm{Max}(Z)$.

We have a similar characterization of a.f.g. C^*-algebras over Z:

Theorem (G. 2011)

Let A be a unital C^*-algebra and let X be the spectrum of Z.

- *A as a Banach Z-module if a.f.g.;*
- *A is necessarily unital, $\mathcal{E}_A$ is an (F) bundle, and there exists a finite clopen partition $X = X_1 \sqcup \cdots \sqcup X_k$ such that every fibre of each restriction bundle $\mathcal{E}|_{X_i}$ is $*$-isomorphic to some fix matrix algebra $\mathrm{M}_{n_i}(\mathbb{C})$.*

In particular, every a.f.g. C^*-algebra over Z is projective over Z.

Remark

One can briefly say that A is a.f.g. over Z if and only if A is a finite direct sum of unital homogeneous C^*-algebras.

As we saw, the canonical bundle $\mathcal{E}_A$ of an a.f.g. C^*-algebra over Z is automatically an (F) bundle. However, this is not true in general for t.f.g. C^*-algebras over Z.

Example

Let $B := C([0,1], \mathrm{M}_2(\mathbb{C})) = \mathrm{M}_2(C([0,1]))$ and let A be a C^*-subalgebra of B consisting of all functions $f \in B$ such that

$$f(0) = \begin{bmatrix} \lambda(f) & 0 \\ 0 & \lambda(f) \end{bmatrix} \text{ and } f(1) = \begin{bmatrix} \lambda(f) & 0 \\ 0 & \mu(f) \end{bmatrix},$$

for some complex numbers $\lambda(a)$ and $\mu(a)$. One easily checks that matrices

$$\begin{bmatrix} \sin(\pi x) & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \sin(\pi x) \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \sin(\pi x) & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & \sin(\pi x) \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} \cos(2\pi x) & 0 \\ 0 & \cos(\pi x) \end{bmatrix}$$

Example (continued)

generate a dense Z-submodule of A. On the other hand, note that the spectrum of Z can be identified with $\mathbb{T}$. Hence, the canonical bundle of A over $\mathbb{T}$ is not and (F) bundle, since the quotient map $[0,1] \to \mathbb{T}$ is not open.

Hoewever, under the assumption that $\mathcal{E}_A$ is an (F) bundle we have the following result for t.f.g. C^*-algebras over Z, at least for separable ones:

Theorem (G. 2011)

Assume that A is a unital separable C^*-algebra. If $\mathcal{E}_A$ is an (F) bundle, then the following conditions are equivalent:

- **(i)** *A is t.f.g. over Z,*
- **(ii)** *Fibres E_x of $\mathcal{E}_A$ have uniformly finite dimensions, and each restriction bundle of $\mathcal{E}_A$ over a set where $\dim E_x$ is constant is of finite type (as a vector bundle).*
- **(iii)** *V satisfies the condition (P) over Z;*
- **(iv)** *A is of finite Z-projective rank;*
- **(v)** *A is weakly a.f.g over Z.*

Question

- Are the conditions (i), (iii), (iv) and (v) also equivalent without the assumption that $\mathcal{E}_A$ is an (F) bundle?
- More generally, are these conditions also equivalent for all $C(X)$-locally convex Banach modules?

Remark

Unlike a.f.g. Hilbert $C(X)$-modules, a.f.g. $C(X)$-locally convex Banach modules are not generally projective. For example, a C^*-algebra $A := C([0,1])$ is a.f.g. as a module over $C(\mathbb{T})$, with respect to the action

$$(\varphi f)(x) := \varphi(e^{2\pi i x}) f(x).$$

On the other hance, A is clearly not projective over $C(\mathbb{T})$.

However, if we assume the continuity of $\mathcal{E}_M$ from the start, we can state the following question:

Question

Suppose that M is an a.f.g. $C(X)$-locally convex Banach module such that $\mathcal{E}_M$ is an (F) bundle. Is M necessarily projective? In particular, is a dimension function $x \mapsto \dim E_x$ necessarily continuous?