Centrally Stable Algebras

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joint work with Matej Brešar

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Introduction

Let A be an algebra with centre $Z(A)$. If I is an ideal of A and $q_I : A \to A/I$ the canonical map, then $q_I(Z(A)) = (Z(A) + I)/I$ is obviously contained in, but is not necessarily equal to $Z(A/I)$.

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Definition (Archbold 1972)

A C^*-algebra A is said to have the **centre-quotient property** *(shortly, the CQ-property) if for any closed ideal I of A, $Z(A/I) = (Z(A) + I)/I$.*

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Theorem (Vesterstrøm 1971, Archbold-G. 2020)

For a C^*-algebra A the following conditions are equivalent:

(i) *A has the CQ-property.*

(ii) *A is* **weakly central**, *that is no modular maximal ideal of A contains $Z(A)$ and for any pair of modular maximal ideals M and N of A, $M \cap Z(A) = N \cap Z(A)$ implies $M = N$.*

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- The most prominent examples of weakly central C^*-algebras A are those satisfying the **Dixmier property**, that is for each $x \in A$ the closure of the convex hull of the unitary orbit of x intersects $Z(A)$ (Archbold 1972). In particular, von Neumann algebras are weakly central (Dixmier 1949, Misonou 1952).

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- A unital simple C^*-algebra satisfies the Dixmier property if and only if it admits at most one tracial state (Haagerup-Zsidó 1984). In particular, weak centrality does not imply the Dixmier property.

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- In 2008 Magajna gave a characterisation of weak centrality in terms of more general averaging that involves unital completely positive elementary operators (i.e. $x \mapsto \sum_i a_i^* x a_i$, where $\sum_i a_i^* a_i = 1$).

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- Finally, in 2017 Archbold, Robert and Tikuisis found the exact gap between weak centrality and the Dixmier property for unital C^*-algebras and showed that a postliminal C^*-algebra has the (singleton) Dixmier property if and only if it has the CQ-property.

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Definition (Brešar-G. 2019)

An algebra A over a filed $\mathbb{F}$ is said to be **centrally stable** *(shortly, CS) if for any ideal I of A, $Z(A/I) = (Z(A) + I)/I$.*

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The next proposition presents some alternative definitions of centrally stable algebras.

Proposition

Let A be an algebra. The following conditions are equivalent:

(i) *A is CS.*

(ii) *For every algebra epimorhism ϕ from A to another algebra B, $Z(B) = \phi(Z(A))$.*

(iii) *For every $a \in A$, $a \in Z(A) + \mathrm{Id}([a, A])$.*

Remarks and Examples

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Recall that an algebra A is said to be **central** if it is unital $Z(A) = \mathbb{F}1$. Obviously, a central algebra A is CS if and only if A/I is a central algebra for every ideal I of A.

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Caution

There is also a notion of centrality for C^*-algebras, which differs from the one given above (i.e. a C^*-algebra A is central if it is quasi-central and for any par of primitive ideals P and Q of A, $P \cap Z(A) = Q \cap Z(A)$ implies $P = Q$). In this talk, by a central algebra we always mean that $Z(A) = \mathbb{F}1$.

Example

Let V be an infinite-dimensional vector space over $\mathbb{F}$. Recall that every proper ideal of the algebra $\mathrm{End}_{\mathbb{F}}(V)$ of all linear operators on V is of the form

$$\mathrm{F}_\kappa(V) = \{T \in \mathrm{End}_{\mathbb{F}}(V) : \dim_{\mathbb{F}} T(V) < \kappa\}$$

for some cardinal number $\aleph_0 \le \kappa \le \dim_{\mathbb{F}} V$. By using Zorn's Lemma one can show that for each such cardinal number κ, the algebra $\mathrm{End}_{\mathbb{F}}(V)/\mathrm{F}_\kappa(V)$ is central, so that $\mathrm{End}_{\mathbb{F}}(V)$ is a central CS algebra.

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Example

Let V be an infinite-dimensional vector space over $\mathbb{F}$ and let B be any central simple subalgebra of $\mathrm{End}_{\mathbb{F}}(V)$ that contains the identity operator (e.g. one of Weyl algebras). Then the algebra $A := B + \mathrm{F}_{\aleph_0}(V)$ is CS.

Remark

In contrast to the algebra $\mathrm{End}_{\mathbb{F}}(V)$, if V is a (real or complex) infinite-dimensional Banach space, then the algebra $\mathrm{B}(V)$ of all bounded linear operators on V does not need to be CS. This is due to the fact the centre of the Calkin algebra $\mathrm{B}(V)/\mathrm{K}(V)$ can be quite large, even though $\mathrm{B}(V)$ is central. In fact, Motakis, Puglisi and Zisimopoulou shown in 2016 that for each countably infinite compact metric space X, there is a Banach space V such that the Calkin algebra $\mathrm{B}(V)/\mathrm{K}(V)$ is isomorphic to the algebra $C(X)$ of scalar-valued continuous functions on X.

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If $\mathcal{H}$ is a separable infinite-dimensional Hilbert space, it is easy to check directly that the C^*-algebra $\mathrm{B}(\mathcal{H})$ has the CQ-property. Namely, it has a unique non-trivial closed ideal, namely the ideal $\mathrm{K}(\mathcal{H})$ and the Calkin algebra $\mathrm{B}(\mathcal{H})/\mathrm{K}(\mathcal{H})$ is central, as a unital simple C^*-algebra. However, $\mathrm{B}(\mathcal{H})$ has many non-closed ideals (e.g. the Schatten p-ideals).

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Question

Is the algebra $\mathrm{B}(\mathcal{H})$ CS?

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Example

Let V be a real vector space of countably infinite dimension. Let $T \in \mathrm{End}_{\mathbb{R}}(V)$ be any operator such that $T^2 = -1$ (e.g. if $\{e_1, e_2, \dots\}$ is a basis of V, we may define T by $T(e_{2n-1}) = e_{2n}$ and $T(e_{2n}) = -e_{2n-1}$ for all $n \in \mathbb{N}$). Set

$$A := \{\alpha 1 + \beta T + V : \alpha, \beta \in \mathbb{R}, V \in \mathrm{F}_{\aleph_0}(V)\} \subset \mathrm{End}_{\mathbb{R}}(V).$$

Then A is a central real algebra and $\mathrm{F}_{\aleph_0}(V)$ is the unique non-trivial ideal of A. On the other hand $A/\mathrm{F}_{\aleph_0}(V) \cong \mathbb{C}$, so A is not centrally stable.

Central stability is preserved under some algebraic constructions.

Proposition

(a) *A homomorphic image of a CS algebra is CS.*

(b) *The direct sum of a family of algebras $\{A_j\}$ is CS if and only if all algebras A_j are CS.*

(c) *A non-unital algebra is CS if and only if its unitization is CS.*

(d) *If A and B are unital algebras and if the algebra $A \otimes B$ is CS, then so are A and B.*

(e) *If one of the unital algebras A or B is CS and the other one is central and simple, then $A \otimes B$ is CS.*

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Corollary

A unital algebra A is CS if and only if $\mathrm{M}_n(A) \cong \mathrm{M}_n(\mathbb{F}) \otimes A$ ($n \in \mathbb{N}$) is CS.

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Question

If A and B are two unital CS algebras, is $A \otimes B$ necessarily CS?

Centrally Stable Finite-Dimensional Algebras

Under a mild assumption that $\mathbb{F}$ is perfect, we were able to determine the structure of an arbitrary centrally stable finite-dimensional unital algebra over $\mathbb{F}$. Recall that $\mathbb{F}$ is said to be *perfect* if every irreducible polynomial in $\mathbb{F}[X]$ is separable (a polynomial $P(X) \in \mathbb{F}[X]$ is separable if its roots are distinct in an algebraic closure of $\mathbb{F}$, that is, the number of distinct roots is equal to the degree of $P(X)$). The basic examples of perfect fields are: fields of characteristic zero, finite fields and algebraically closed fields.

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Let now A be a finite-dimensional algebra over a field $\mathbb{F}$. By $\mathrm{rad}(A)$ we denote the *radical* of A; that is, $\mathrm{rad}(A)$ is a unique maximal nilpotent ideal of A. If A is *semisimple* (i.e. $\mathrm{rad}(A) = 0$), then Wedderburn's Theorem tells us that A is isomorphic to a finite direct product of simple algebras of the form $\mathrm{M}_{n_i}(\mathbb{D}_i)$ where $n_i \geq 1$ and $\mathbb{D}_i$ is a division algebra over $\mathbb{F}$. Combining this with our previous results, we see that every finite-dimensional semisimple algebra is CS.

We also record the next simple result which is useful for (counter)examples.

Proposition

Let A be a finite-dimensional algebra. If $A/\operatorname{rad}(A)$ is commutative and A is not commutative, then A is not CS. In particular, a nilpotent algebra is CS if and only if it is commutative.

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It is easy to show that any closed ideal of a C^*-algebra with the CQ-property also has the CQ-property. The analogue of this is not in general true for the ideals of CS algebras.

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Example

If C is a commutative finite-dimensional unital algebra, then $A := \mathrm{M}_n(C)$ is a CS algebra and $\operatorname{rad}(A) = \mathrm{M}_n(\operatorname{rad}(C))$. In particular, if $n > 1$ and $\operatorname{rad}(C)^2 \neq 0$, $\operatorname{rad}(A)$ is a noncommutative nilpotent algebra and therefore is not CS (by the previous proposition). This shows the following:

(a) An ideal of a CS algebra may not be CS.

(b) The algebra of $n \times n$ matrices over a commutative algebra without identity may not be CS.

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Theorem (Brešar-G. 2019)

Let A be a finite-dimensional unital algebra over a perfect field $\mathbb{F}$. The following conditions are equivalent:

(i) *A is centrally stable.*

(ii) $\mathrm{rad}(A) = \mathrm{Id}\big(Z(A) \cap \mathrm{rad}(A)\big)$.

(iii) *A is isomorphic to a finite direct product of algebras of the form $C_i \otimes_{\mathbb{F}_i} A_i$, where $\mathbb{F}_i$ is a finite field extension of $\mathbb{F}$, C_i is a commutative $\mathbb{F}_i$-algebra, and A_i is a central simple $\mathbb{F}_i$-algebra.*

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Corollary

A finite-dimensional unital algebra A over an algebraically closed field $\mathbb{F}$ is centrally stable if and only if A is isomorphic to a finite direct product of algebras of the form $\mathrm{M}_{n_i}(C_i)$, where each C_i is a commutative unital $\mathbb{F}$-algebra.

The proof of our main theorem uses the classical theory of finite-dimensional algebras, including Wedderburn's structure theory, the Skolem-Noether Theorem, and the Artin-Whaples Theorem.