On derivations and elementary operators on $C^*$-algebras

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Infinite Dimensional Function Theory: Present Progress and Future Problems

University College Dublin
January 8-9, 2015

(joint work in progress with Richard Timoney)
Definition

A $C^*$-algebra is a (complex) Banach $*$-algebra $A$ whose norm $\| \cdot \|$ satisfies the $C^*$-identity. More precisely:

- $A$ is a Banach algebra over the field $\mathbb{C}$.
- $A$ is equipped with an involution, i.e. a map $\ast : A \to A$, $a \mapsto a^\ast$ satisfying the properties:
  - $(\alpha a + \beta b)^\ast = \alpha a^\ast + \beta b^\ast$,
  - $(ab)^\ast = b^\ast a^\ast$,
  - $(a^\ast)^\ast = a$,
  for all $a, b \in A$ and $\alpha, \beta \in \mathbb{C}$.
- Norm $\| \cdot \|$ satisfies the $C^*$-identity, i.e. $\| a^\ast a \| = \| a \|^2$ for all $a \in A$.

Throughout this talk, we assume that all $C^*$-algebras have identity elements.
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  $$(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*, \quad (ab)^* = b^* a^*, \quad \text{and} \quad (a^*)^* = a,$$
  
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Throughout this talk, we assume that all \(C^*\)-algebras have identity elements.
Fundamental examples

- Let \( X \) be a compact Hausdorff space and let \( C(X) \) be the set of all continuous complex-valued functions on \( X \). Then \( C(X) \) becomes a \( C^* \)-algebra with respect to the pointwise operations, involution \( f^*(x) := \overline{f(x)} \), and max-norm \( \|f\|_\infty := \sup\{|f(x)| : x \in X\} \). Obviously, \( C(X) \) is commutative \( C^* \)-algebra. Moreover, every commutative \( C^* \)-algebra arises in this fashion (Gelfand-Naimark theorem).

- The set \( B(H) \) of bounded linear operators on a Hilbert space \( H \) becomes a \( C^* \)-algebra with respect to the standard operations, usual adjoint and operator norm. In fact, every \( C^* \)-algebra can be isometrically embedded as a norm-closed self-adjoint subalgebra of \( B(H) \) for some Hilbert space \( H \) (Gelfand-Naimark theorem).

- The category of \( C^* \)-algebras is closed under the formation of direct products, direct sums, extensions, direct limits, pullbacks, pushouts, (some) tensor products, etc.
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- Let $X$ be a compact Hausdorff space and let $C(X)$ be the set of all continuous complex-valued functions on $X$. Then $C(X)$ becomes a $C^*$-algebra with respect to the pointwise operations, involution $f^*(x) := \overline{f(x)}$, and max-norm $\|f\|_\infty := \sup\{|f(x)| : x \in X\}$. Obviously, $C(X)$ is commutative $C^*$-algebra. Moreover, every commutative $C^*$-algebra arises in this fashion (Gelfand-Naimark theorem).

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A **derivation** on a C*-algebra $A$ is a linear map $\delta : A \to A$ satisfying the Leibniz rule

$$\delta(xy) = \delta(x)y + x\delta(y) \quad \text{for all } x, y \in A.$$
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- $\delta$ vanishes on the centre of $A$ (i.e. $\delta(z) = 0$ for all $z \in Z(A)$). In particular, commutative $C^*$-algebras don’t admit non-zero derivations.
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By $\text{Der}(A)$ and $\text{Inn}(A)$ we denote, respectively, the set of all derivations on $A$ and the set of all inner derivations on $A$. 

Some classes of $C^*$-algebras which admit only inner derivations:

- simple $C^*$-algebras (Sakai, 1968).
- $AW^*$-algebras (Olesen, 1974).
- homogeneous $C^*$-algebras (Sproston, 1976).
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*For a separable $C^*$-algebra $A$ the following conditions are equivalent:*

- $A$ admits only inner derivations.

- $A$ is a direct sum of a finite number of $C^*$-subalgebras which are either homogeneous or simple.

- $\text{Der}(A)$ is separable in the operator norm.

On the other hand, for inseparable $C^*$-algebras the main problem remains widely open, even for the simplest cases such as subhomogeneous $C^*$-algebras (i.e. $C^*$-algebras which have finite-dimensional irreducible representations of bounded degree).
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Motivation

We often try to understand the structure of operators and spaces on which they act in terms of approximation by finite rank maps. On $C^*$-algebras $A$, however, it is natural to regard two-sided multiplication maps $M_{a,b}: x \mapsto axb$ ($a, b \in A$) as basic building blocks (instead of rank one operators). We can therefore try to approximate a more general map on $A$, one that preserves ideals, by finite sums of two-sided multiplication maps, that is, by elementary operators.

By $\mathcal{E}_\ell(A)$ we denote the set of all elementary operators on $A$. It is easy to see that every elementary operator on $A$ is completely bounded, with

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\|\sum_i M_{a_i,b_i}\|_{h} \leq \|\sum_i a_i \otimes b_i\|_{h},
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where $\|\cdot\|_h$ is the Haagerup tensor norm on $A \otimes A$. 
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Since every $\delta \in \text{Der}(A)$ preserve the ideals of $A$ and is completely bounded, the above approximation procedure in particular applies to the derivations of $C^*$-algebras:

Problem Which derivations of a $C^*$-algebra $A$ admit a completely bounded approximation by elementary operators? That is, which derivations of $A$ lie in the cb-norm closure $E_{cb}(A)$?

Remark Since each inner derivation is an elementary operator (of length 2) on $A$, $E_{cb}(A)$ includes the cb-closure of $\text{Inn}(A)$.

Since the cb-norm of an inner derivation of a $C^*$-algebra coincides with its operator norm (easy to verify), the cb-norm closure of $\text{Inn}(A)$ coincides with the operator norm closure of $\text{Inn}(A)$. We denote this closure by $\text{Inn}(A)$. 
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Ilja Gogić (TCD)  Derivations and elem. operators  UCD, January 9, 2015  8 / 14
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Problem (G., 2013)

Does every $C^*$-algebra satisfy the condition

$$\text{Der}(A) \cap \overline{\mathcal{E}(A)}^{cb} = \overline{\text{Inn}(A)}?$$

In many cases the set $\overline{\text{Inn}(A)}$ is closed in the operator norm. However, this is not always true. In fact, we have the following beautiful characterization:

Theorem (Somerset, 1993)
The set $\overline{\text{Inn}(A)}$ is closed in the operator norm, as a subset of $\overline{\text{Der}(A)}$, if and only if $A$ has a finite connecting order.
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Connecting order of a $C^*$-algebra

The connecting order of a $C^*$-algebra is a constant in $\mathbb{N} \cup \{\infty\}$ arising from a certain graph structure on the primitive spectrum $\text{Prim}(A)$:

Two primitive ideals $P, Q$ of $A$ are said to be adjacent, if $P$ and $Q$ cannot be separated by disjoint open subsets of $\text{Prim}(A)$.

A path of length $n$ from $P$ to $Q$ is a sequence of points $P = P_0, P_1, \ldots, P_n = Q$ such that $P_{i-1}$ is adjacent to $P_i$ for all $1 \leq i \leq n$.

The distance $d(P, Q)$ from $P$ to $Q$ is defined as follows:

$\Delta d(P, P) := 1$.

If $P \neq Q$ and there exists a path from $P$ to $Q$, then $d(P, Q)$ is equal to the minimal length of a path from $P$ to $Q$.

If there is no path from $P$ to $Q$, $d(P, Q) := \infty$.

The connecting order $Orc(A)$ of $A$ is then defined by $Orc(A) := \sup \{d(P, Q) : P, Q \in \text{Prim}(A) \text{ such that } d(P, Q) < \infty\}$. 
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Theorem (G., 2013)

The equality \( \text{Der}(A) \cap \mathcal{E}_\ell(A)^{cb} = \text{Inn}(A) \) holds true for all \( C^* \)-algebras \( A \) in which every Glimm ideal is prime.

Recall that the Glimm ideals of a \( C^* \)-algebra \( A \) are the ideals generated by the maximal ideals of the centre of \( A \).

If a \( C^* \)-algebra \( A \) has only prime Glimm ideals, then \( \text{Orc}(A) = 1 \), so Somerset's theorem yields that \( \text{Inn}(A) \) is closed in the operator norm. Hence:

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If every Glimm ideal of a \( C^* \)-algebra \( A \) is prime, then every derivation of \( A \) which lies in \( \mathcal{E}_\ell(A)^{cb} \) is inner.
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The class of \( C^* \)-algebras in which every Glimm ideal is prime is fairly large. It includes:

- Prime \( C^* \)-algebras.
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- Quotients of \( AW^* \)-algebras.
- Local multiplier algebras.

By an elementary derivation on a \( C^* \)-algebra \( A \) we mean every derivation on \( A \) which is also an elementary operator on \( A \).

Question: Does there exist a \( C^* \)-algebra \( A \) which admits an outer elementary derivation?

Motivated by our previous discussion, it is natural to start looking for possible examples in the class of \( C^* \)-algebras with \( \text{Orc}(A) = \infty \).
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*Example*

*Derivations and elem. operators*

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Example (G., 2010)

Let $A$ be a $C^*$-algebra consisting of all elements $a \in C([0, \infty]) \otimes \mathbb{M}_2$ such that

$$a(n) = \begin{bmatrix} \lambda_n(a) & 0 \\ 0 & \lambda_{n+1}(a) \end{bmatrix} \quad (n \in \mathbb{N}),$$

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More recently, Richard Timoney showed that the above $C^*$-algebra admits outer derivations $\delta$ of the form $\delta = M_{a,b} - M_{b,a}$ for some $a, b \in A$. In particular $A$ has outer elementary derivations of length 2.
I end this lecture with some problems of current interest:

**Problem**

What can be said about the lengths of outer elementary derivations? In particular, can one for each $n \geq 2$ find a $C^*$-algebra $A$ which admits an (outer) elementary derivation of length $n$?
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