# On derivations and elementary operators on $C^*$-algebras

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Present Progress and Future Problems

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- $A$ is a Banach algebra over the field $\mathbb{C}$.
- $A$ is equipped with an involution, i.e. a map $* : A \to A$, $a \mapsto a^*$ satisfying the properties:

$$(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*, \quad (ab)^* = b^* a^*, \quad \text{and} \quad (a^*)^* = a,$$

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Throughout this talk, we assume that all $C^*$-algebras have identity elements.

## Fundamental examples

- Let $X$ be a compact Hausdorff space and let $C(X)$ be the set of all continuous complex-valued functions on $X$. Then $C(X)$ becomes a $C^*$-algebra with respect to the pointwise operations, involution $f^*(x) := \overline{f(x)}$, and max-norm $\|f\|_\infty := \sup\{|f(x)| \ : \ x \in X\}$. Obviously, $C(X)$ is commutative $C^*$-algebra. Moreover, every commutative $C^*$-algebra arises in this fashion (Gelfand-Naimark theorem).

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- The set $\mathbb{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ becomes a $C^*$-algebra with respect to the standard operations, usual adjoint and operator norm. In fact, every $C^*$-algebra can be isometrically embedded as a norm-closed self-adjoint subalgebra of $\mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (Gelfand-Naimark theorem).

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The category of $C^*$-algebras is closed under the formation of direct products, direct sums, extensions, direct limits, pullbacks, pushouts, (some) tensor products, etc.

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*A* **derivation** *on a $C^*$-algebra $A$ is a linear map $\delta : A \to A$ satisfying the* **Leibniz rule**

$$\delta(xy) = \delta(x)y + x\delta(y) \qquad \textit{for all } x, y \in A.$$

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- $\delta$ preserves the (closed two-sided) ideals of $A$ (i.e. $\delta(I) \subseteq I$ for every ideal $I$ of $A$).
- $\delta$ vanishes on the centre of $A$ (i.e. $\delta(z) = 0$ for all $z \in Z(A)$). In particular, commutative $C^*$-algebras don't admit non-zero derivations.

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- $AW^*$-algebras (Olesen, 1974).
- homogeneous $C^*$-algebras (Sproston, 1976).

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On the other hand, for inseparable $C^*$-algebras the main problem remains widely open, even for the simplest cases such as subhomogeneous $C^*$-algebras (i.e. $C^*$-algebras which have finite-dimensional irreducible representations of bounded degree).

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- We can therefore try to approximate a more general map on $A$, one that preserves ideals, by finite sums of two-sided multiplication maps, that is, by **elementary operators**.

By $\mathcal{E}\ell(A)$ we denote the set of all elementary operators on $A$. It is easy to see that every elementary operator on $A$ is completely bounded, with

$$\left\| \sum_i M_{a_i,b_i} \right\|_{cb} \leq \left\| \sum_i a_i \otimes b_i \right\|_h ,$$

where $\| \cdot \|_h$ is the Haagerup tensor norm on $A \otimes A$.

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Which derivations of a $C^*$-algebra $A$ admit a completely bounded approximation by elementary operators? That is, which derivations of $A$ lie in the cb-norm closure $\overline{\mathcal{E}\ell(A)}^{cb}$ of $\mathcal{E}\ell(A)$?

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**Remark**

- Since each inner derivation is an elementary operator (of length 2) on $A$, $\overline{\mathcal{E}\ell(A)}^{cb}$ includes the cb-corm closure of $\mathrm{Inn}(A)$.

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- Since the cb-norm of an inner derivation of a $C^*$-algebra coincides with its operator norm (easy to verify), the cb-norm closure of $\mathrm{Inn}(A)$ coincides with the operator norm closure of $\mathrm{Inn}(A)$. We denote this closure by $\overline{\mathrm{Inn}(A)}$.

**Problem (G., 2013)**

Does every $C^*$-algebra satisfy the condition

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In fact, we have the following beautiful characterization:

**Theorem (Somerset, 1993)**

*The set $\mathrm{Inn}(A)$ is closed in the operator norm, as a subset of $\mathrm{Der}(A)$, if and only if $A$ has a finite connecting order.*

## Connecting order of a $C^*$-algebra

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- A **path** of length $n$ from $P$ to $Q$ is a sequence of points $P = P_0, P_1, \ldots, P_n = Q$ such that $P_{i-1}$ is adjacent to $P_i$ for all $1 \leq i \leq n$.

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- The **distance** $d(P, Q)$ from $P$ to $Q$ is defined as follows:
  - ▷ $d(P, P) := 1$.
  - ▷ If $P \neq Q$ and there exists a path from $P$ to $Q$, then $d(P, Q)$ is equal to the minimal length of a path from $P$ to $Q$.
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  - If there is no path from $P$ to $Q$, $d(P, Q) := \infty$.
- The **connecting order** $\mathrm{Orc}(A)$ of $A$ is then defined by

$$\mathrm{Orc}(A) := \sup\{d(P, Q) : \ P, Q \in \mathrm{Prim}(A) \text{ such that } d(P, Q) < \infty\}.$$

**Theorem (G., 2013)**

*The equality* $\mathrm{Der}(A) \cap \overline{\overline{\mathcal{E}\ell(A)}}^{cb} = \overline{\overline{\mathrm{Inn}(A)}}$ *holds true for all* $C^*$*-algebras* $A$ *in which every Glimm ideal is prime.*

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Recall that the **Glimm ideals** of a $C^*$-algebra $A$ are the ideals generated by the maximal ideals of the centre of $A$.

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Recall that the **Glimm ideals** of a $C^*$-algebra $A$ are the ideals generated by the maximal ideals of the centre of $A$.

If a $C^*$-algebra $A$ has only prime Glimm ideals, then $\mathrm{Orc}(A) = 1$, so Somerset's theorem yields that $\mathrm{Inn}(A)$ is closed in the operator norm. Hence:

**Corollary**

*If every Glimm ideal of a* $C^*$*-algebra* $A$ *is prime, then every derivation of* $A$ *which lies in* $\overline{\overline{\mathcal{E}\ell(A)}}^{cb}$ *is inner.*

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Motivated by our previous discussion, it is natural to start looking for possible examples in the class of $C^*$-algebras with $\mathrm{Orc}(A) = \infty$.

**Example (G., 2010)**

Let $A$ be a $C^*$-algebra consisting of all elements $a \in C([0,\infty]) \otimes \mathbb{M}_2$ such that

$$a(n) = \begin{bmatrix} \lambda_n(a) & 0 \\ 0 & \lambda_{n+1}(a) \end{bmatrix} \quad (n \in \mathbb{N}),$$

for some convergent sequence $(\lambda_n(a))$ of complex numbers. Then:

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More recently, Richard Timoney showed that the above $C^*$-algebra admits outer derivations $\delta$ of the form $\delta = M_{a,b} - M_{b,a}$ for some $a, b \in A$. In particular $A$ has outer elementary derivations of length 2.

I end this lecture with some problems of current interest:

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What can be said about the lengths of outer elementary derivations? In particular, can one for each $n \geq 2$ find a $C^*$-algebra $A$ which admits an (outer) elementary derivation of length $n$?

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What can be said about $\mathrm{Der}(A) \cap \overline{\overline{\mathcal{E}\ell(A)}}$?