Applications of algebraic topology to operator algebras: Homogeneous C*-algebras

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In memory to my dear friend Prof. Stana Nikčević

C^*-algebras as noncommutative topology

Definition

A (unital) C^*-**algebra** is a complex Banach $*$-algebra A whose norm $\|\cdot\|$ satisfies the C^*-identity. More precisely:

- A is a Banach algebra with identity over $\mathbb{C}$.
- A is equipped with an involution, i.e. a map $* : A \to A$, $a \mapsto a^*$ satisfying the properties:

$$(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*, \quad (ab)^* = b^* a^*, \quad \text{and} \quad (a^*)^* = a,$$

  for all $a, b \in A$ and $\alpha, \beta \in \mathbb{C}$.
- Norm $\|\cdot\|$ satisfies the C^*-**identity**, i.e.

$$\|a^* a\| = \|a\|^2$$

  for all $a \in A$.

Example

Let X be a CH (compact Hausdorff) space and let $C(X)$ be the set of all continuous complex-valued functions on X. Then $C(X)$ becomes a commutative C^*-algebra with respect to the pointwise operations, involution $f^*(x) := \overline{f(x)}$, and sup-norm $\|f\|_\infty := \sup\{|f(x)| \ : \ x \in X\}$.

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In fact, all unital commutative C^*-algebras arise in this fashion:

Theorem (Gelfand-Naimark)

The (contravariant) functor $X \rightsquigarrow C(X)$ defines an equivalence of categories of CH spaces (with continuous maps as morphisms) and commutative C^*-algebras (with $*$-homomorphisms as morphisms).

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In other words: By passing from the space X the function algebra $C(X)$, no information is lost. In fact, X can be recovered from $C(X)$. Thus, topological properties of X can be translated into algebraic properties of $C(X)$, and vice versa. Therefore, the theory of C^*-algebras is often thought of as **noncommutative topology**.

Example

- The set $\mathbb{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ becomes a C^*-algebra with respect to the standard operations, usual adjoint and operator norm. In particular, the complex matrix algebras $\mathbb{M}_n = M_n(\mathbb{C})$ are C^*-algebras.
- In fact, every C^*-algebra can be isometrically embedded as a norm-closed self-adjoint subalgebra of $\mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (the noncommutative Gelfand-Naimark theorem).
- To every locally compact group G, one can associate a C^*-algebra $C^*(G)$. Everything about the representation theory of G is encoded in $C^*(G)$.
- The category of C^*-algebras is closed under the formation of direct products, direct sums, extensions, direct limits, pullbacks, pushouts, (some) tensor products, etc.

Homogeneous C^*-algebras

In the light of noncommutative topology it is natural to try to view a given unital C^*-algebra A as a set of sections of some sort of the bundle. For example, $C(X)$ is the family of sections of a trivial bundle over X.

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This idea in particularly works well for the following class of C^*-algebras:

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Example

For any CH space X, the C^*-algebra $C(X) \otimes \mathbb{M}_n$ is n-homogeneous.

Example

More generally, if $\mathcal{E}$ is a locally trivial fibre bundle over a CH space X with fibre $\mathbb{M}_n$ and structure group $\mathrm{Aut}(\mathbb{M}_n) \cong PU(n) = U(n)/\mathbb{S}^1$ (the projective unitary group), then the set $\Gamma(\mathcal{E})$ of all continuous sections of $\mathcal{E}$ is an n-homogeneous C^*-algebra, with respect to the fiberwise operations and sup-norm.

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Theorem (Fell & Tomiyama-Takesaki)

If A is an n-homogeneous C^*-algebra, then its spectrum X is a CH space and there is a locally trivial bundle $\mathcal{E}$ over X with fibre $\mathbb{M}_n$ and structure group $PU(n)$ such that A is isomorphic to the section algebra $\Gamma(\mathcal{E})$. Moreover, any two such algebras $A_i = \Gamma(\mathcal{E}_i)$ with spectra X_i are isomorphic if and only if there is a homeomorphism $f : X_1 \to X_2$ such that $\mathcal{E}_1 \cong f^*(\mathcal{E}_2)$ as bundles over X_1.

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In particular, the classification problem of n-homogeneous C^*-algebras over X is equivalent to the classification problem of $PU(n)$-bundles over X.

From the general theory we know that any topological group G admits the **universal** G**-bundle** $\mathrm{E}G$ over $\mathrm{B}G$ (where $\mathrm{B}G$ is the **classifying space** of G), which has the property that any G-bundle E over a CW-complex X is isomorphic to the induced G-bundle $f^*(\mathrm{E}G)$ for some continuous map $f : X \to \mathrm{B}G$.

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Since any two homotopic maps induce isomorphic bundles, the map $[f] \mapsto [f^*(\mathrm{E}G)]$ defines a bijection between the homotopy classes $[X, \mathrm{B}G]$ onto the isomorphism classes $\mathrm{Bun}(X, G)$ of G-bundles over X .

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We know that the classifying space of $U(n)$ is $G_n(\mathbb{C}^\infty)$, i.e. the inductive limits of complex Grassmanians. What is the classifying space of $PU(n)$?

As an illustration, we present a result which can be used in order to classify the $PU(n)$-bundles over spaces of the form $\Sigma(Y)$ (the suspension of Y).

Theorem

If the group G is path-connected, then there exists a bijection between the equivalence classes of G-bundles over $X = \Sigma(Y)$ and the homotopy classes $[Y, G]$.

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In particular, since $\Sigma(\mathbb{S}^{k-1}) = \mathbb{S}^k$, we have:

Corollary

If the group G is path-connected, then there is a bijection between the equivalence classes of G-bundles over $\mathbb{S}^k$ and the elements of $(k-1)$th-homotopy group $\pi_{k-1}(G)$.

The lower homotopy groups of $G = PU(n)$ are known. In particular, for $X = \mathbb{S}^k$, we get:

No. of isomorphism classes of n-homogeneous C^*-algebras over $\mathbb{S}^k$

	$\mathbb{M}_1$	$\mathbb{M}_2$	$\mathbb{M}_3$	$\mathbb{M}_4$	$\mathbb{M}_5$	$\mathbb{M}_6$	$\mathbb{M}_7$	$\mathbb{M}_8$	$\mathbb{M}_9$	$\mathbb{M}_{10}$
$\mathbb{S}^1$	1	1	1	1	1	1	1	1	1	1
$\mathbb{S}^2$	1	2	3	4	5	6	7	8	9	10
$\mathbb{S}^3$	1	1	1	1	1	1	1	1	1	1
$\mathbb{S}^4$	1	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$
$\mathbb{S}^5$	1	2	1	1	1	1	1	1	1	1
$\mathbb{S}^6$	1	2	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$
$\mathbb{S}^7$	1	12	6	1	1	1	1	1	1	1

We end this part of the talk with the following interesting result:

Theorem (Antonevič-Krupnik)

If $X = \mathbb{S}^k$, then:

(a) *Any $PU(n)$-bundle over X is trivial as a vector bundle.*

(b) *Any $PU(n)$-bundle $\mathcal{E}$ over X is of the form $\mathcal{E} = \mathrm{End}(\mathcal{V})$ for some n-dimensional vector bundle $\mathcal{V}$ over X.*

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Problem

Which manifolds/CW-complexes X satisfy the property (a) or (b) of the preceeding theorem?

Algebraic characterization of homogeneous C^*-algebras

Standard polynomial of degree k is a polynomial in k non-commuting variables $x_1, \dots, x_k$ defined by

$$s_k(x_1, \dots, x_k) := \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(k)},$$

where S_k is a symmetric group of order k.

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Definition

We say that a ring R satisfies the **standard identity** s_k *if for each k-tuple $(r_1, \ldots, r_k)$ of elements in R we have $s_k(r_1, \ldots, r_k) = 0$.*

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Theorem (Amitsur-Levitzki)

If R is a unital commutative ring, then the ring $M_n(R)$ of $n \times n$ matrices over R satisfies the standard identity s_{2n}.

Definition

*A (unital) ring R is said to be an A_n-**ring** if:*

(a) *R satisfies the standard identity s_{2n}; and*

(b) *No non-zero homomorphic image of R satisfies the standard identity $s_{2(n-1)}$.*

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Definition

*A (unital) ring R with centre Z is said to be **Azumaya** over Z if:*

(a) *R is a finitely generated projective Z-module; and*

(b) *The canonical homomorphism*

$$\theta : A \otimes_Z A^\circ \to \mathrm{End}_Z(R), \quad \theta(a \otimes b)(x) = axb$$

is an isomorphism.

If R iz Azumaya over Z, then R is a finitely generated projective Z-module and hence has a rank function $\mathrm{Spec}(R) \to \mathbb{N}_0$. If this function is constant then R is said to be of **constant rank** (this number is a perfect square).

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Theorem (Artin)

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Therefore, for a C^*-algebra A we have:
A is n-homogeneous $\iff$ A is an A_n-ring $\iff$ A is Azumaya of constant rank n^2.

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Therefore, for a C^*-algebra A we have:
A is n-homogeneous $\Longleftrightarrow$ A is an A_n-ring $\Longleftrightarrow$ A is Azumaya of constant rank n^2.

Theorem (G.)

For a C^*-algebra A with centre Z the following conditions are equivalent:

(a) *A is Azumaya.*

(b) *A is finitely generated Z-module (projectivity is not assumed).*

(c) *A is a finite direct sum of homogeneous C^*-algebras.*

Hilbert C^*-modules

- Hilbert C^*-modules form a category in between Banach spaces (they have a little extra geometrical structure) and Hilbert spaces (they are not as well behaving as these).
- A Hilbert C^*-module obeys the same axioms as an ordinary Hilbert space, except that the inner product takes values in a more general C^*-algebras than $\mathbb{C}$.
- Hilbert C^*-modules were first introduced in the work of I. Kaplansky in 1953, who developed the theory for unital commutative C^*-algebras. In the 1970s the theory was extended to non-commutative C^*-algebras independently by W. Paschke and M. Rieffel.
- Hilbert C^*-modules appear naturally in many areas of C^*-algebra theory, such as KK-theory, Morita equivalence of C^*-algebras, and completely positive operators.

Definition

Let A be a C^*-algebra. A (left) **Hilbert A-module** *is a left A-module M, equipped with an A-valued inner product $\langle \cdot, \cdot \rangle$ which is A-linear in the first and conjugate linear in the second variable, such that M is a Banach space with the norm*

$$\|v\| := \sqrt{\|\langle v, v \rangle\|_A}.$$

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Example

Every C^*-algebra A becomes a Hilbert A-module with respect to the inner product

$$\langle a, b \rangle := ab^*.$$

Example

Similarly, the direct sum A^n of n-copies of A becomes an A-Hilbert module with respect to the pointwise operations and the inner product

$$\langle a_1 \oplus \cdots \oplus a_n, b_1 \oplus \cdots \oplus b_n \rangle := \sum_{k=1}^{n} a_k b_k^*.$$

More generally, let

$$\mathcal{H}_A := \left\{ (a_k) \in \prod_1^\infty A \ : \ \sum_{k=1}^\infty a_k a_k^* \text{ is norm convergent} \right\}.$$

Then the pointwise operations and the inner product

$$\langle (a_k), (b_k) \rangle := \sum_{k=1}^\infty a_k b_k^*$$

turn $\mathcal{H}_A$ into a Hilbert A-module – a **standard Hilbert A-module**.

When a C^*-algebra A is unital and commutative, $A = C(X)$, there exists a categorical equivalence between Hilbert A-modules and (F) Hilbert bundles over X. (F) Hilbert bundles provide a natural generalization of standard vector bundles from topology.

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Definition

An **(F) Hilbert bundle** is a triple $\mathcal{E} := (p, E, X)$ where E and X are topological spaces with a continuous open surjection $p : E \to X$, together with operations and norms making each **fibre** $E_x := p^{-1}(x)$ $(x \in X)$ into a complex Hilbert space, such that the following conditions are satisfied:

- The maps $\mathbb{C} \times E \to E$, $E \oplus_X E \to E$ and $E \oplus_X E \to \mathbb{C}$ given in each fibre by scalar multiplication, addition, and the inner product, respectively, are continuous. Here $E \oplus_X E$ denotes the Whitney sum

$$\{(e, f) \in E \times E \ : \ p(e) = p(f)\}.$$

- If $x \in X$ and if (e_α) is a net in E such that $\|e_\alpha\| \to 0$ and $p(e_\alpha) \to x$ in X, then $e_\alpha \to 0_x$ in E (where 0_x is the zero-element of E_x).

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Example

The simplest example of an (F) Hilbert bundle is the **product bundle** over X with fibre H, $\epsilon(X, H) := (\mathrm{proj}_1, X \times H, H)$, where H is a Hilbert space.

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Example

Every locally trivial complex vector bundle $\mathcal{E}$ over a (para)compact Hausdorff space becomes an (F) Hilbert bundle for a choice of a Riemannian metric on $\mathcal{E}$. In fact, an (F) Hilbert bundle structure on $\mathcal{E}$ is essentially unique.

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If X is a CH space, then $\Gamma(\mathcal{E})$ becomes a Hilbert $C(X)$-module with respect to the action

$$(\varphi s)(x) := \varphi(x)s(x)$$

and inner product

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where $\langle \cdot, \cdot \rangle_x$ denotes the inner product on fibre E_x.

In fact, all Hilbert $C(X)$-modules arise in this fashion:

Theorem

To any Hilbert $C(X)$-module M one can associate a natural (F) Hilbert bundle $\mathcal{E}_M$ such that the (generalized) Gelfand transform $\Gamma_M : M \to \Gamma(\mathcal{E}_M)$ becomes an isometric $C(X)$-linear isomorphism.

Finitely generated Hilbert $C(X)$-modules

A Hilbert A-module M is said to be:

- **algebraically finitely generated** (AFG) if there exists a finite subset of M whose A-linear span equals M.
- **weakly algebraically finitely generated** (WAFG) if there exists a constant $k = k(A) \in \mathbb{N}$ such that every AFG submodule of M is contained in a submodule od M generated by $\leq k$ generators.
- **topologically finitely generated** (TFG) if there exists a finite subset of M whose A-linear span is dense M.
- **countably generated** (CG) if there exists a countable subset of M whose A-linear span is dense M.

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Theorem (Kasparov stabilization theorem)

If M is a CG Hilbert A-module, then $M \oplus \mathcal{H}_A \cong \mathcal{H}_A$, where $\mathcal{H}_A$ is a standard Hilbert A-module.

Corollary

Every AFG Hilbert module (over a unital C^*-algebra) is automatically projective.

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An (F) Hilbert bundle $\mathcal{E} = (p, E, X)$ is said to be:

- **Locally trivial** if there exists a Hilbert space H and an open cover $\mathcal{U}$ of X such that for each $U \in \mathcal{U}$ we have $\mathcal{E}|_U \cong \epsilon(U, H)$.
- n-**homogeneous**, if all fibres of $\mathcal{E}$ have the same finite dimension n.

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- n-**homogeneous**, if all fibres of $\mathcal{E}$ have the same finite dimension n.

Lemma

Any n-homogeneous (F) Hilbert bundle is automatically locally trivial.

In particular, when $A = C(X)$, we get a Hilbert module version of the celebrated Serre-Swan theorem:

Theorem

Let M be a Hilbert $C(X)$-module, where X is a CH space, and let $\mathcal{E} := \mathcal{E}_M$. Then M is AFG if and only if there exists a finite clopen partition $X = X_1 \sqcup \cdots \sqcup X_k$ such that each restriction bundle $\mathcal{E}|_{X_i}$ is homogeneous.

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Hence, the category of *n*-homogeneous (F) Hilbert bundles over (connected) CH spaces is equivalent to the category of *n*-dimensional (locally trivial) complex vector bundles.

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In particular, the dimension of the fibres of the canonical (F) Hilbert bundle may vary, even when X is connected:

Example

let X be the unit interval $[0,1]$ and let

$$M := C_0((0,1]) = \{f \in C([0,1]) : f(0) = 0\}.$$

Then M becomes a Hilbert $C([0,1])$-module with respect to the standard action and inner product $\langle f, g \rangle = fg^*$.

- M is topologically singly generated (for instance, the identity function $f(x) = x$ is such generator, by the Weierstrass approximation theorem).
- On the other hand, each fibre of $\mathcal{E}_M$ is one-dimensional, except at $x = 0$ which is zero.

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If all fibres of an (F) Hilbert bundle $\mathcal{E}$ are finite dimensional and

$$n := \sup_{x \in X} \dim E_x < \infty,$$

we say that $\mathcal{E}$ is n-**subhomogeneous**.

- In that case every restriction bundle of $\mathcal{E}$ over a set where $\dim E_x$ is constant is locally trivial.
- If in addition every base space of such restriction bundle admits a finite trivializing open cover, then we say that $\mathcal{E}$ is n-**subhomogeneous of finite type**.

Theorem (G.)

Let X be a compact metrizable space and let M be a Hilbert $C(X)$-module with the canonical (F) Hilbert bundle $\mathcal{E}_M$. The following conditions are equivalent:

(a) *M is TFG.*

(b) *$\mathcal{E}_M$ is subhomogeneous of finite type.*

(c) *M is WAFG.*

(d) *There exists a constant $k = k(M) \in \mathbb{N}$ such that for any Banach $C(X)$-module V, each tensor in the $C(X)$-projective tensor product $M \overset{\pi}{\otimes}_{C(X)} V$ is of (finite) rank at most k.*

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Remark

Recently, A. Chirvasitu removed the metrizabilty requirement.

$C(X)$-convex Banach modules

Definition

Let A be a unital Banach algebra. A **left (unital) Banach A-module** *is Banach space M, which is also a left A-module such that the action $A \times M \to M$, $(a, x) \mapsto ax$ is continuous (i.e. $\|ax\| \leq \|a\|\|x\|$ for all $a \in A$ and $x \in M$) and $1x = x$ for all $x \in M$.*

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Example

If X is a CH space, one can similarly define a notion of an (F) or (H) bundle $\mathcal{E} = (p, E, X)$ (in the (H) case the norm $E \to X$ is only required to be upper semicontinuous). Then $\Gamma(\mathcal{E})$ is a Banach $C(X)$-module.

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For a Banach $C(X)$-module M, one can also construct a canonical Banach bundle $\mathcal{E}_M$ and the generalized Gelfand transform $\Gamma_M : M \to \Gamma(\mathcal{E}_M)$. However, in general:

- $\mathcal{E}_M$ is only an (H) bundle.
- Γ_M fails to be isometric.

If M is a Banach $C(X)$-module one can look for conditions which might guarantee that Γ_M is isometric. This was solved by K. Hofmann in 1974.

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Definition

Let M be a Banach $C(X)$-module. We say that M is $C(X)$-**convex** if for any pair $\varphi_1, \varphi_2 \in C(X)_+$ with $\varphi_1 + \varphi_2 = 1$ and $s_1, s_2 \in M$, we have

$$\|\varphi_1 s_1 + \varphi_2 s_2\| \le \max\{\|s_1\|, \|s_2\|\}.$$

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Example

If $\mathcal{E}$ is an (H) Banach bundle over a CH space X, then the Banach $C(X)$-module $\Gamma(E)$ is $C(X)$-convex.

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Example

If $\mathcal{E}$ is an (H) Banach bundle over a CH space X, then the Banach $C(X)$-module $\Gamma(E)$ is $C(X)$-convex.

In that way we essentially get all $C(X)$-convex modules:

Theorem (Hofmann)

If M is a Banach $C(X)$-module, then Γ_M defines an isometric $C(X)$-isomorphism from M onto $\Gamma(\mathcal{E}_M)$ if and only if M is $C(X)$-convex.

The next example shows that we cannot extend our results from Hilbert $C(X)$-modules to general $C(X)$-convex modules, unless the canonical bundles are (F) bundles:

Example

We consider $M := C([0,1])$ as Banach module over $C(\mathbb{S}^1)$, with respect to the action

$$(\varphi f)(x) := \varphi(e^{2\pi i x}) f(x).$$

- All fibres of $\mathcal{E}_M$ (which is an (H) Banach bundle over $\mathbb{S}^1$) are 1-dimensional, except at 1, where $\dim = 2$.
- On the other hand, it is easy to see that M is AFG (2 generators suffice).

Problem (G.)

Let $\mathcal{E}$ be an (F) Banach bundle over a CH space X and let $M := \Gamma(\mathcal{E})$.

(a) If M is AFG, is M automatically projective ($\iff$ homogeneity of $\mathcal{E}$, when X is connected)?

(b) Are all TFG conditions for M, from the Hilbert $C(X)$-module case, always equivalent?

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A several days ago A. Chirvasitu informed me that:

Theorem (Chirvasitu - arxiv.org/pdf/2405.14518)

The answer to (b) is positive.