Spectrum-shrinking maps and nonlinear preservers on matrix domains

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Jordan homomorphisms

Let $\mathcal{A}$ and $\mathcal{B}$ be rings (associative algebras). A **Jordan homomorphism** is an additive (linear) map $\phi : \mathcal{A} \to \mathcal{B}$ such that

$$\phi(ab + ba) = \phi(a)\phi(b) + \phi(b)\phi(a), \quad \text{for all } a, b \in \mathcal{A}.$$

When the rings (algebras) are 2-torsion-free, this is equivalent to

$$\phi(a^2) = \phi(a)^2, \quad \text{for all } a \in \mathcal{A}.$$

- Jordan homomorphisms are morphisms in the category of Jordan algebras, a class of nonassociative algebras introduced by Pascual Jordan in 1933 in the context of quantum mechanics.
- Most practically relevant Jordan algebras arise as subalgebras of associative algebras equipped with the *symmetric product* $x \circ y := xy + yx$.
- Unlike Lie algebras, which always embed into associative algebras via the *antisymmetric product* $[x, y] := xy - yx$, some Jordan algebras (known as **exceptional Jordan algebras**) do not arise this way.

Typical examples of Jordan homomorphisms include additive (linear) multiplicative and antimultiplicative maps.

Let $\mathcal{A}$ be a unital algebra, and let $p \in \mathcal{A}$ be a central idempotent. Suppose that $\phi, \psi : \mathcal{A} \to \mathcal{A}$ are an algebra endomorphism and an antiendomorphism, respectively. Then the map

$$x \mapsto p\phi(x) + (1 - p)\psi(x)$$

defines a Jordan endomorphism of $\mathcal{A}$, which, in general, is neither multiplicative nor antimultiplicative.

One of the central problems, with a long and extensive history: is to identify conditions on rings (algebras) $\mathcal{A}$ and $\mathcal{B}$ under which any (typically surjective) Jordan homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is either multiplicative or antimultiplicative, or more generally, can be expressed as a suitable combination of these types.

Let us briefly mention some state-of-the-art results:

- **(Jacobson–Rickart, 1950)** Every Jordan homomorphism from an arbitrary ring into an integral domain is either a multiplicative or antimultiplicative.
- **(Jacobson–Rickart, 1950)** Let R be a unital ring, S any ring, and $n \geq 2$. Then every Jordan homomorphism $\mathcal{M}_n(R) \to S$ is the sum of a ring homomorphism and an antihomomorphism.
- **(Herstein, 1956; Smiley, 1957)** Every Jordan epimorphism from an arbitrary ring into a prime ring (i.e., $aRb = 0$ implies $a = 0$ or $b = 0$) is either a multiplicative or antimultiplicative.

In particular, for the matrix algebra $\mathcal{M}_n(\mathbb{F})$ over a field $\mathbb{F}$, a direct consequence of Herstein's theorem and the Skolem–Noether theorem is that every nonzero Jordan endomorphism ϕ of $\mathcal{M}_n(\mathbb{F})$ is of the form

$$\phi(X) = TXT^{-1} \quad \text{or} \quad \phi(X) = TX^{\mathrm{t}}T^{-1},$$

for some invertible matrix $T \in \mathcal{M}_n(\mathbb{F})$.

The study of Jordan homomorphisms is of particular importance in the theory of Banach algebras.

Let $\mathcal{A}$ be a unital complex algebra, and let $a \in \mathcal{A}$. We denote the *spectrum* of $a \in \mathcal{A}$ by $\mathrm{sp}(a)$, i.e.

$$\mathrm{sp}(a) := \{\lambda \in \mathbb{C} : \lambda 1_{\mathcal{A}} - a \text{ is not invertible in } \mathcal{A}\}.$$

Let $\phi : \mathcal{A} \to \mathcal{B}$ be a map between unital algebras. We say that ϕ:

- **preserves invertibility** if: $\forall a \in A$, a invertible $\implies \phi(a)$ invertible,
- **preserves invertibility in both directions** if: $\forall a \in A$, a is invertible $\iff \phi(a)$ is invertible,
- **shrinks the spectrum** if: $\forall a \in A$, $\mathrm{sp}(\phi(a)) \subseteq \mathrm{sp}(a)$,
- **preserves the spectrum** if: $\forall a \in A$, $\mathrm{sp}(\phi(a)) = \mathrm{sp}(a)$.

Note that for *linear unital* maps $\phi : \mathcal{A} \to \mathcal{B}$ we have:

$$\phi \text{ preserves invertibility} \iff \phi \text{ is spectrum-shrinking},$$

$$\phi \text{ preserves invertibility in both directions} \iff \phi \text{ is spectrum-preserving}$$

It is well-known (and easy to verify) that any unital Jordan homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ between unital algebras $\mathcal{A}$ and $\mathcal{B}$ preserves invertibility.

Recall the following classical results:

Theorem (Marcus-Purves, 1959)

Every linear unital invertibility-preserving map on $\mathcal{M}_n = \mathcal{M}_n(\mathbb{C})$ is either multiplicative or antimultiplicative, and consequently a Jordan automorphism of $\mathcal{M}_n$.

Theorem (Gleason–Kahane–Żelazko, 1967-1968)

Let $\mathcal{A}$ be a unital Banach algebra and $\phi : \mathcal{A} \to \mathbb{C}$ a linear unital map which preserves invertibility. Then ϕ is multiplicative. The same is true if $\mathbb{C}$ is replaced by any unital commutative semisimple Banach algebra $\mathcal{B}$.

Kaplansky-Aupetit Conjecture, 1970, 2000

Let $\mathcal{A}$ and $\mathcal{B}$ be unital semisimple Banach algebras, and let $\phi : \mathcal{A} \to \mathcal{B}$ be a surjective unital invertibility-preserving linear map. Then ϕ is a Jordan homomorphism.

When $\mathcal{B}$ is semisimple, it is well-known that any surjective unital linear map $\phi : \mathcal{A} \to \mathcal{B}$ for which there exists a constant $M \geq 0$ such that

$$r(\phi(a)) \leq M\, r(a), \quad \text{for all } a \in \mathcal{A},$$

($r(\cdot)$ denotes the spectral radius) is necessarily continuous (Aupetit, 1991).

The following example illustrates that surjectivity is indispensable when $\mathcal{A} \neq \mathcal{B}$.

Example (Russo, 1966)

Define a linear map $\phi : \mathcal{M}_2 \to \mathcal{M}_4$ by

$$\phi(X) := \begin{bmatrix} X & X - X^t \\ 0 & X \end{bmatrix}.$$

This map is unital, injective, and spectrum-preserving. However, ϕ is not a Jordan homomorphism, since for any non-symmetric matrix X, we have

$$\phi(X^2) - \phi(X)^2 = \begin{bmatrix} 0 & (X - X^t)^2 \\ 0 & 0 \end{bmatrix} \neq 0.$$

Semisimplicity is essential, even when ϕ is a spectrum-presering bijection.

Example (Aupetit, 1979)

Let $\mathcal{A} \subseteq \mathcal{M}_4$ consist of all block upper-triangular matrices of the form

$$X = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}, \quad \text{with } A, B, C \in \mathcal{M}_2.$$

Define $\phi : \mathcal{A} \to \mathcal{A}$ by

$$\phi\left(\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}\right) := \begin{bmatrix} A & B \\ 0 & C^t \end{bmatrix}.$$

Then ϕ is a unital spectrum-preserving linear bijection. However, it fails to be a Jordan homomorphism, since in general:

$$\phi(X^2) - \phi(X)^2 = \begin{bmatrix} 0 & B(C - C^t) \\ 0 & 0 \end{bmatrix} \neq 0.$$

The Kaplansky–Aupetit conjecture has attracted considerable interest, and several special cases have been resolved. In particular, it has been verified under the following assumptions:

- **(Aupetit, 1979)**: $\mathcal{B}$ admits a separating family of finite-dimensional irreducible representations.
- **(Jafarian–Sorour, 1986)**: $\mathcal{A}$ and $\mathcal{B}$ are full algebras of bounded linear operators on Banach spaces, and ϕ is spectrum-preserving. (A more concise proof was provided by Šerml in 2002.)
- **(Aupetit–Mouton, 1994)**: The socle of $\mathcal{B}$ (i.e., the sum of all minimal left ideals) is an essential ideal in $\mathcal{B}$.
- **(Sorour, 1996)**: $\mathcal{A}$ and $\mathcal{B}$ are full algebras of bounded linear operators on Banach spaces, and ϕ is bijective.
- **(Aupetit, 2000; Cui–Hou, 2004)**: $\mathcal{A}$ is a von Neumann algebra.

Despite this progress, the conjecture remains unresolved in full generality, even in the setting of C^*-algebras.

Automatic spectrum preservation for spectrum-shrinking maps

In many situations it is more convenient to deal with spectrum-preserving maps. Hence, given a result for spectrum-preserving maps, a natural question is whether it extends to spectrum-shrinking counterpart.

However, the literature indicates that such extensions are generally *highly nontrivial*.

For example, Sorour's 1996 extension of the Jafarian–Sorour (1986) characterization of spectrum-preserving linear bijections between algebras of bounded operators on Banach spaces to the spectrum-shrinking case involves significantly more intricate techniques and a considerably longer proof based on complex analysis.

This leads us to the following general problem:

Problem

For which matrix or operator domains $\mathcal{X}$ and $\mathcal{Y}$ are all continuous spectrum-shrinking maps $\phi : \mathcal{X} \to \mathcal{Y}$ (when they exist) automatically spectrum-preserving?

To state our results, we begin by fixing some notation.

- Given a subset $L \subseteq \mathbb{C}^n$, by Δ_L we denote the subset of L that consists of elements with at least two equal coordinates:

$$\Delta_L := \{(x_1, \dots, x_n) \in L : x_j = x_k \text{ for some } j \neq k\}.$$

- We naturally identify the symmetric group S_n with the $n \times n$ permutation matrices, so that S_n forms a subgroup of the general linear group $\mathrm{GL}(n)$.
- Assuming that L is invariant under the action of S_n in $\mathbb{C}^n$ (by conjugation), S_n also naturally acts on the set of connected components of $L \setminus \Delta_L$.
- If V is a subspace of the algebra $\mathcal{T}_n^+$ of $n \times n$ strictly upper-triangular matrices, denote by $\mathcal{T}_{L,V}$ the space of upper-triangular matrices with diagonal in L and strictly upper-triangular component in V, i.e.

$$\mathcal{T}_{L,V} := \{\mathrm{diag}(\lambda_1, \dots, \lambda_n) + v : (\lambda_1, \dots, \lambda_n) \in L,\ v \in V\}.$$

Theorem (Chirvasitu-G.-Tomašević, preprint, 2025)

Given $n \in \mathbb{Z}_{\geq 1}$, a closed connected subgroup G of $\mathsf{GL}(n)$, a linear subspace V of $\mathcal{T}_n^+$, and a subset $L \subseteq \mathbb{C}^n$, denote

$$\mathcal{X}_n := \mathrm{Ad}_{\mathsf{G}}\, \mathcal{T}_{L,V} = \{SXS^{-1} \,:\, X \in \mathcal{T}_{L,V},\, S \in \mathsf{G}\}.$$

Assume that:

- *$L \setminus \Delta_L$ is dense in L;*
- *L is invariant under the action of S_n in $\mathbb{C}^n$;*
- *and the isotropy groups of the connected components of $L \setminus \Delta_L$ in $\mathsf{G} \cap S_n$ are transitive on $[n] = \{1, 2, \ldots, n\}$.*

Then for an arbitrary $m \in \mathbb{Z}_{\geq 1}$ there exists a continuous spectrum-shrinking map $\phi : \mathcal{X}_n \to \mathcal{M}_m$ if and only n divides m and in that case we have the equality of characteristic polynomials

$$k_{\phi(X)} = (k_X)^{\frac{m}{n}}, \qquad \text{for all } X \in \mathcal{X}_n.$$

The theorem applies to a wide array of matrix domains $\mathcal{X}_n$, including:

(a) the matrix algebra $\mathcal{M}_n$ itself,

(b) the general linear group $\mathrm{GL}(n)$,

(c) the special linear group $\mathrm{SL}(n)$,

(d) the unitary group $\mathrm{U}(n)$,

(e) the set N_n of $n \times n$ normal matrices.

It also holds for the sets of diagonalizable matrices in $\mathcal{M}_n$, $\mathrm{GL}(n)$ & $\mathrm{SL}(n)$.

Example: $\mathcal{X}_n = \mathrm{SL}(n)$

We have $\mathrm{SL}(n) = \mathrm{Ad}_{\mathrm{G}}\, \mathcal{T}_{L,V}$ for $\mathrm{G} = \mathrm{GL}(n)$, $V = \mathcal{T}_n^+$ and

$$L = \left\{ (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n \ : \ \prod_{j=1}^{n} \lambda_j = 1 \right\}.$$

Then $L \setminus \Delta_L$ is connected (L is a connected complex algebraic variety, and Δ_L is its closed algebraic subset).

Example: $\mathcal{X}_n = \mathrm{U}(n)$

We have $\mathrm{U}(n) = \mathrm{Ad}_\mathsf{G}\,\mathcal{T}_{L,V}$ for $G = U(n)$, $V = \{0\}$ and $L = (\mathbb{S}^1)^n$, so that $L \setminus \Delta_L = \mathcal{C}^n(\mathbb{S}^1)$ (the n^{th} *configuration space* of $\mathbb{S}^1$). One can show that:

- $\mathcal{C}^n(\mathbb{S}^1)$ is disconnected as soon as $n \geq 3$.
- The symmetric group S_n acts transitively on the space connected components of $\mathcal{C}^n(\mathbb{S}^1)$, and the isotropy groups are the conjugates of the subgroup generated by the cycle $(1\ 2\ \cdots\ n)$.

Example: continuous spectrum-preserving maps $\mathcal{X}_n \to \mathcal{M}_{rn}$

If $m = rn$ for some $r \in \mathbb{Z}_{\geq 1}$, then an apparent class of continuous spectrum-preserving maps $\phi : \mathcal{X}_n \to \mathcal{M}_m = \mathcal{M}_r(\mathcal{M}_n)$ is given by

$$\phi(X) = S(X) \begin{bmatrix} X \otimes I_p & 0 \\ 0 & X^t \otimes I_q \end{bmatrix} S(X)^{-1},$$

where $S : \mathcal{X}_n \to \mathrm{GL}(m)$ is a continuous function and $p, q \in \mathbb{Z}_{\geq 0}$ are such that $p + q = r$.

Non-example: $\mathcal{X}_n = H_n$

The theorem does not hold for the real subspace H_n of $n \times n$ self-adjoint matrices. Indeed, the assignment

$$H_n \ni X \mapsto \lambda_{\max}(X) \in \mathrm{sp}(X),$$

where $\lambda_{\max}(X)$ denotes the largest eigenvalue of $X \in H_n$, is continuous. Then for any $m \in \mathbb{Z}_{\geq 2}$ the assignment

$$X \mapsto \lambda_{\max}(X) I_m$$

defines a continuous spectrum-shrinking map $H_n \to \mathcal{M}_m$, which is not spectrum preserving.

Non-example: $\mathcal{X}_n = \mathrm{SU}(n)$

The special unitary group $\mathrm{SU}(n)$ behaves similarly as H_n. Specifically, for any $n \in \mathbb{Z}_{\geq 1}$ there exists a continuous eigenvalue selection $\varphi : \mathrm{SU}(n) \to \mathbb{S}^1$. To see this, first identify the space $\mathrm{SU}(n)/\operatorname{Ad}_{\mathrm{SU}(n)}$ of conjugacy classes with the quotient $\mathbb{T}/S_n$ of the maximal torus

Non-example: $\mathcal{X}_n = \mathrm{SU}(n)$ (continuation)

$$\mathbb{T} := \left\{ (\lambda_1, \ldots, \lambda_n) \in \left(\mathbb{S}^1\right)^n \ : \ \prod_{j=1}^{n} \lambda_j = 1 \right\}$$

by the *Weyl group* S_n of $\mathrm{SU}(n)$. It is well-known (Morton, 1966) that

$$(x_1, \ldots, x_n) \mapsto (\exp(2\pi i x_1), \ldots, \exp(2\pi i x_n))$$

implements a homeomorphism between

$$F := \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ \sum_{j=1}^{n} x_j = 0, \ x_1 \le x_2 \le \cdots \le x_n \le x_1 + 1 \right\}$$

and $\mathbb{T}/S_n$. We can now take our continuous eigenvalue selection to be

$$\mathrm{SU}(n) \to \mathrm{SU}(n)/\operatorname{Ad}_{\mathrm{SU}(n)} \cong \mathbb{T}/S_n \cong F \ni (x_1, \ldots, x_n) \mapsto \exp(2\pi i x_1) \in \mathbb{S}^1.$$

Singular matrices

Let us now focus on subsets of singular matrices. Specifically, for $n \in \mathbb{Z}_{\geq 2}$ and $1 \leq k < n$, define

$$\mathcal{M}_n^{\leq k} := \{\text{all } n \times n \text{ matrices of rank } \leq k\}$$
$$\mathrm{Sing}(n) := \{\text{all } n \times n \text{ singular matrices}\} = \mathcal{M}_n^{\leq n-1}.$$

Our theorem does not apply to any of the sets $\mathcal{M}_n^{\leq k}$. Specifically:

- If $k \leq n-2$, the space L of possible n-tuples of eigenvalues always contains at least two zeros. In this case, $L \setminus \Delta_L$ is empty, so certainly not dense in L.
- If $k = n-1$, the connected components of $L \setminus \Delta_L$ are given by

$$\{(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n : \lambda_i \text{ pairwise distinct and } \lambda_r = 0\}, \quad 1 \leq r \leq n.$$

  The corresponding isotropy groups are the subgroups $S_{n-1} \subset S_n$ fixing the r-th symbol. These groups do not act transitively on $[n]$.

Moreover, if $k < n$, then for any $m \in \mathbb{Z}_{\geq 1}$, there *always* exist continuous maps $\phi : \mathcal{M}_n^{\leq k} \to \mathcal{M}_m$ that strictly shrink the spectrum:

Example

For any nilpotent matrix $N \in \mathcal{M}_m$ and continuous function $f : \mathcal{M}_n^{\leq k} \to \mathbb{C}$, the map

$$\phi : \mathcal{M}_n^{\leq k} \to \mathcal{M}_m, \qquad X \mapsto f(X)N$$

is clearly continuous and spectrum-shrinking, but not spectrum-preserving.

Question

What if we also require the injectivity of ϕ? That is, are all injective continuous spectrum-shrinking maps

$$\phi : \mathcal{M}_n^{\leq k} \to \mathcal{M}_m$$

(when they exist) necessarily spectrum-preserving?

In short: the answer varies substantially, heavily depending on the dimensions of the underlying spaces.

For simplicity, we state the result only in the case $m = n$.

Theorem (Chirvasitu-G.-Tomašević, LAA, 2025)

a. *If $k > \frac{n}{2}$, any continuous spectrum-shrinking map $\phi : \mathcal{M}_n^{\leq k} \to \mathcal{M}_n$ either preserves the characteristic polynomial or takes only nilpotent values.*
b. *Let $\phi : \mathcal{M}_n^{\leq k} \to \mathcal{M}_n$ be an injective continuous spectrum-shrinking map. If either*

$$k > n - \sqrt{n} \quad \text{or} \quad \phi(\mathcal{M}_n^{\leq k}) \subseteq \mathcal{M}_n^{\leq k},$$

then ϕ preserves the characteristic polynomial (and hence the spectrum).
c. *For every $k \in \mathbb{Z}_{\geq 1}$ and all sufficiently large $n \in \mathbb{Z}_{\geq 1}$ there exists a real-analytic embedding*

$$\phi : \mathcal{M}_n^{\leq k} \hookrightarrow \mathrm{Nil}(n)$$

of $\mathcal{M}_n^{\leq k}$ into the space $\mathrm{Nil}(n)$ of all nilpotent $n \times n$ matrices.

On analytical embeddings $\mathcal{M}_n^{\leq k} \hookrightarrow \mathrm{Nil}(n)$

Let us fix $n \in \mathbb{Z}_{\geq 2}$ and $1 \leq k \leq n$.

- Both spaces $\mathcal{M}_n^{\leq k}$ and $\mathrm{Nil}(n)$ are complex algebraic varieties of respective dimensions $k(2n-k)$ and $n^2 - n$.
- They are also *affine*, i.e. definable by polynomial equations as closed subsets of $\mathbb{C}^N$ for appropriate N.
- This means they are *Stein spaces*.
- In particular, if $n^2 - n \geq 2k(2n-k) + 1$, by a standard result there exists a proper holomorphic embedding

$$\mathcal{M}_n^{\leq k} \hookrightarrow \mathbb{C}^{n^2-n}.$$

- In turn, $\mathbb{C}^{n^2-n}$ is real-analytically isomorphic to a ball in the (n^2-n)-dimensional open subvariety of $\mathrm{Nil}(n)$ consisting of non-singular points.

Finite-dimensional algebras

We now turn our attention to spectrum-shrinking maps between arbitrary unital finite-dimensional complex algebras. We begin by recalling the notion of structural matrix algebras, introduced by van Wyk in 1988.

Given a quasi-order ρ on $[n]$ (i.e. a reflexive and transitive binary relation), the **structural matrix algebra (SMA) associated with** ρ is the subalgebra of $\mathcal{M}_n$ spanned by the matrix units E_{ij} corresponding to the pairs $(i,j) \in \rho$:

$$\mathcal{A}_\rho := \mathrm{span}\{E_{ij} : (i,j) \in \rho\}.$$

It is easy to see that SMAs are precisely the subalgebras of $\mathcal{M}_n$ that contain the algebra $\mathcal{D}_n$ of all diagonal matrices.

SMAs enjoy the following important property:

Theorem (G.–Tomašević, LAA, 2025)

Let $\mathcal{A}_\rho \subseteq \mathcal{M}_n$ be an SMA, and let $\mathcal{F} \subseteq \mathcal{A}_\rho$ be a commuting family of diagonalizable matrices. Then there exists an invertible matrix $S \in \mathcal{A}_\rho$ which simultaneously diagonalizes $\mathcal{F}$; that is, $S\mathcal{F}S^{-1} \subseteq \mathcal{D}_n$.

Theorem (G.-Tomašević, to appear in JAA, 2025)

Let $\mathcal{A}$ and $\mathcal{B}$ be unital finite-dimensional complex algebras, each equipped with the unique Hausdorff vector topology. Denote by

$$\mathrm{Max}(\mathcal{A}) = \{\mathcal{P}_1, \ldots, \mathcal{P}_p\} \quad \textit{and} \quad \mathrm{Max}(\mathcal{B}) = \{\mathcal{Q}_1, \ldots, \mathcal{Q}_q\}$$

the sets of all maximal ideals of $\mathcal{A}$ and $\mathcal{B}$, respectively.

For each $1 \le i \le p$ and $1 \le j \le q$ define the quantities

$$k_i := \sqrt{\dim(\mathcal{A}/\mathcal{P}_i)} \quad \textit{and} \quad m_j := \sqrt{\dim(\mathcal{B}/\mathcal{Q}_j)}.$$

which are positive integers by Wedderburn's structure theorem. Then:

(a) *There exists a continuous spectrum-shrinking map $\phi : \mathcal{A} \to \mathcal{B}$ if and only if for each $1 \le j \le q$ the linear Diophantine equation*

$$k_1 x_1^j + \cdots + k_p x_p^j = m_j \tag{1}$$

has a non-negative integer solution $(x_1^j, \ldots, x_p^j) \in \mathbb{Z}_{\ge 0}^p$.

Theorem (continuation)

(b) *There exists a continuous spectrum-preserving map* $\phi : \mathcal{A} \to \mathcal{B}$ *if and only if there exists a family* $\{(x_1^j, \ldots, x_p^j) : 1 \leq j \leq q\}$ *of non-negative integer solutions to* (1)*, with the property that for each* $1 \leq i \leq p$ *there exists some* $1 \leq j \leq q$ *with* $x_i^j > 0$*.*

(c) *If every continuous spectrum-shrinking map* $\phi : \mathcal{A} \to \mathcal{B}$ *is spectrum-preserving, then any family* $\{(x_1^j, \ldots, x_p^j) : 1 \leq j \leq q\}$ *of non-negative integer solutions to* (1) *satisfies that for each* $1 \leq i \leq p$ *there exists some* $1 \leq j \leq q$ *with* $x_i^j > 0$*. The converse holds when* $\mathcal{A}$ *is isomorphic to an SMA.*

Corollary

Let $\mathcal{A}$ *be a unital finite-dimensional complex algebra. Then* $\mathcal{A}$ *admits a continuous "eigenvalue selection" (i.e. a spectrum shrinker* $\mathcal{A} \to \mathbb{C}$*) if and only if* $\mathcal{A}$ *contains an ideal of codimension one.*

Remark

At present, it remains an open question whether the converse of (c) holds for arbitrary finite-dimensional algebras $\mathcal{A}$. When $\mathcal{A}$ is an SMA, our proof of the converse fundamentally relies on the density of the set

$$\left\{S\,\mathrm{diag}(\lambda_1,\ldots,\lambda_n)S^{-1} : S \in \mathcal{A} \text{ invertible}, \ \lambda_j \in \mathbb{C} \text{ pairwise distinct}\right\}$$

in $\mathcal{A}$. This density, in turn, depends on the simultaneous diagonalizability of commuting families of diagonalizable matrices within SMAs.

Remark

If $p \geq 2$ and the numbers k_i's are coprime, then the largest $m \in \mathbb{N}$ for which there is no continuous spectrum-shrinking map $\phi : \mathcal{A} \to \mathcal{M}_m$ is precisely the *Frobenius number* $g(k_1,\ldots,k_p)$ (related to the coin problem), i.e. the largest integer that cannot be expressed as a sum

$$k_1x_1 + \cdots + k_px_p,$$

where $x_1,\ldots,x_p$ are non-negative integers.

Šemrl's preserver-type characterizations of Jordan homomorphisms

Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras, and suppose that $\mathcal{B}$ is *central*, i.e. its centre consists only of scalar multiples of the identity.

If $\phi : \mathcal{A} \to \mathcal{B}$ is a Jordan homomorphism whose image has a trivial commutant in $\mathcal{B}$ (e.g. if ϕ is surjective). Then, in addition to being spectrum-shrinking, ϕ is also **commutativity preserving**; that is,

$$\forall a, b \in \mathcal{A}, \quad ab = ba \quad \implies \quad \phi(a)\phi(b) = \phi(b)\phi(a).$$

In general, this implication fails when $\phi(\mathcal{A})$ has a non-trivial commutant:

Example

Consider the unital algebra

$$\mathcal{A} := \left\{ \begin{bmatrix} a & b & -c & d \\ 0 & a & 0 & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{bmatrix} : a, b, c, d \in \mathbb{C} \right\} \subset \mathcal{M}_4.$$

Example (Continuation)

Define the map

$$\phi : \mathcal{A} \to \mathcal{M}_4, \qquad \phi\left(\begin{bmatrix} a & b & -c & d \\ 0 & a & 0 & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{bmatrix}\right) := \begin{bmatrix} a & b & -d & c \\ 0 & a & 0 & d \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{bmatrix}.$$

One easily verifies that ϕ is a unital Jordan homomorphism.

However, ϕ does not preserve commutativity. Namely, we have

$$(E_{12} + E_{34})E_{14} = E_{14}(E_{12} + E_{34}) = 0,$$

but

$$\phi(E_{12} + E_{34})\phi(E_{14}) = (E_{12} + E_{34})(-E_{13} + E_{24}) = E_{14},$$
$$\phi(E_{14})\phi(E_{12} + E_{34}) = (-E_{13} + E_{24})(E_{12} + E_{34}) = -E_{14}.$$

We have the following important result by Šemrl:

Theorem (Šemrl, 2008)

If $n \geq 3$, any continuous map $\phi : \mathcal{M}_n \to \mathcal{M}_n$ preserving commutativity and spectrum is a Jordan automorphism of $\mathcal{M}_n$, and hence of the form

$$\phi(X) = TXT^{-1} \quad \text{or} \quad \phi(X) = TX^t T^{-1},$$

for some invertible matrix $T \in \mathrm{GL}(n)$.

Remark

- The first version of this result was formulated by Petek and Šemrl in 1998, with an additional assumption that ϕ either preserves rank-one matrices or preserves commutativity in both directions.
- Unlike the 1998 version, which relied entirely on direct computation, the proof of the current variant is based on a clever application of the *Fundamental Theorem of Projective Geometry*.
- The necessity of all assumptions in Šemrl's theorem was demonstrated via counterexamples.

Remark

By our previous result, the spectrum-preserving assumption in Šemrl's theorem can be further relaxed to spectrum-shrinking.

Let us now focus on the algebra $\mathcal{T}_n$ of all upper-triangular $n \times n$ complex matrices. Similar to the $\mathcal{M}_n$ case, it is well-known that all Jordan automorphisms ϕ of $\mathcal{T}_n$ are of the form

$$\phi(X) = TXT^{-1} \quad \text{or} \quad \phi(X) = TX^t T^{-1},$$

for *suitable* $T \in \mathrm{GL}(n)$. A similar description holds for Jordan monomorphisms $\phi : \mathcal{T}_n \to \mathcal{M}_n$, with $T \in \mathrm{GL}(n)$ *arbitrary*.

The structure of non-injective Jordan homomorphism on $\mathcal{T}_n$ is more subtle:

Example

The map $\phi : \mathcal{T}_n \to \mathcal{T}_n$ that preserves the diagonal entries and annihilates all strictly upper-triangular entries is a non-injective unital Jordan homomorphism, preserving both commutativity and spectrum.

Theorem (Petek, 2002)

If $n \geq 3$, any injective continuous map $\phi : \mathcal{T}_n \to \mathcal{M}_n$ preserving commutativity and spectrum is a Jordan homomorphism.

Remark

- The same conclusion holds if the injectivity of ϕ is replaced by the condition $\phi(\mathcal{T}_n) = \mathcal{T}_n$. In this case, ϕ is a Jordan automorphism of $\mathcal{T}_n$.
- At present, it remains unclear whether the spectrum-preserving condition in Petek's theorem can be weakened to spectrum-shrinking, due to topological challenges. For instance, this relaxation holds for maps $\phi : \mathcal{T}_n \to \mathcal{T}_n$ by the *Invariance of Domain theorem*.

Problem

Given a multiplicatively closed subset $\mathcal{X} \subseteq \mathcal{M}_n$, under what conditions does every (injective) continuous map $\phi : \mathcal{X} \to \mathcal{M}_n$ that preserves commutativity and spectrum (or merely shrinks the spectrum) extend to a Jordan homomorphism $\widetilde{\phi} : [\mathcal{X}]_{\mathrm{alg}} \to \mathcal{M}_n$ on the unital subalgebra of $\mathcal{M}_n$ generated by $\mathcal{X}$?

Jordan monomorphisms between SMAs

Given a unital complex algebra $\mathcal{A}$, we denote its centre by $Z(\mathcal{A})$ and its group of invertible elements by $\mathcal{A}^\times$.

Let ρ be a quasi-order on $[n]$. A map $g : \rho \to \mathbb{C}^\times$ is called **transitive** if

$$g(i,j)g(j,k) = g(i,k), \quad \text{for all } (i,j),(j,k) \in \rho.$$

A transitive map $g : \rho \to \mathbb{C}^\times$ is said to be **trivial** if there exists a map $s : [n] \to \mathbb{C}^\times$ such that

$$g(i,j) = \frac{s(i)}{s(j)}, \quad \text{for all } (i,j) \in \rho.$$

Every transitive map g induces an algebra automorphism g^* of $\mathcal{A}_\rho$, defined on the matrix units by

$$g^*(E_{ij}) = g(i,j)E_{ij}, \quad \text{for all } (i,j) \in \rho.$$

It is easy to see that the trivial transitive maps correspond to inner automorphisms of SMAs, via the assignment $g \mapsto g^*$.

The structure of the automorphism group of SMAs was completely described by Coelho in 1993.

More recently, we extended Coelho's classification to all Jordan automorphisms of SMAs $\mathcal{A}_\rho \subseteq \mathcal{M}_n$, as well as to all Jordan monomorphisms $\phi : \mathcal{A}_\rho \to \mathcal{M}_n$. In fact:

Theorem (G.-Tomašević, 2025, LAA)

Let $\mathcal{A}_\rho \subseteq \mathcal{M}_n$ be an SMA and let $\phi : \mathcal{A}_\rho \to \mathcal{M}_n$ be a Jordan homomorphism such that $\phi(E_{ij}) \neq 0$ for all $(i,j) \in \rho$. Then there exists an invertible matrix $S \in \mathcal{M}_n$, a central idempotent $P \in Z(\mathcal{A}_\rho)$, and a transitive map $g : \rho \to \mathbb{C}^\times$ such that

$$\phi(X) = S(Pg^*(X) + (I-P)g^*(X)^t)S^{-1}.$$

In particular, ϕ is injective (i.e. a Jordan monomorphism).

In contrast to the cases of $\mathcal{M}_n$ and $\mathcal{T}_n$, Jordan automorphisms of SMAs are not necessarily multiplicative or antimultiplicative.

Example

Let $n = p + q$ for some $p, q \in \mathbb{Z}_{\geq 2}$. Consider the algebra $\mathcal{A} := \mathcal{M}_p \oplus \mathcal{M}_q$, which can be naturally identified with an SMA inside $\mathcal{M}_n$. Then the map

$$\phi : \mathcal{A} \to \mathcal{A}, \qquad \phi(X, Y) := (X, Y^t)$$

defines a Jordan automorphism of $\mathcal{A}$ that is neither multiplicative nor antimultiplicative.

Corollary

Let $\mathcal{A}_\rho \subseteq \mathcal{M}_n$ be an SMA. The following conditions are equivalent:

(a) *Every Jordan monomorphism $\phi : \mathcal{A}_\rho \to \mathcal{M}_n$ is multiplicative or antimultiplicative.*

(b) *$\mathcal{A}_\rho$ cannot be decomposed as a direct sum of two noncommutative subalgebras.*

Likewise, (Jordan) automorphisms of SMAs generally do not preserve rank.

Example

Consider the quasi-order

$$\rho = \{(1,1),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(4,4)\},$$

and define a transitive map

$$g : \rho \to \mathbb{C}^\times, \qquad g(i,j) := \begin{cases} 2, & \text{if } (i,j) = (1,4), \\ 1, & \text{otherwise.} \end{cases}$$

Then the induced algebra automorphism $g^* : \mathcal{A}_\rho \to \mathcal{A}_\rho$,

$$g^*\left(\begin{bmatrix} x_{11} & 0 & x_{13} & x_{14} \\ 0 & x_{22} & x_{23} & x_{24} \\ 0 & 0 & x_{33} & 0 \\ 0 & 0 & 0 & x_{44} \end{bmatrix}\right) = \begin{bmatrix} x_{11} & 0 & x_{13} & 2x_{14} \\ 0 & x_{22} & x_{23} & x_{24} \\ 0 & 0 & x_{33} & 0 \\ 0 & 0 & 0 & x_{44} \end{bmatrix},$$

does not preserve rank-one matrices.

The following results link Jordan homomorphisms with rank-one and rank-preserving maps on an SMA $\mathcal{A}_\rho \subseteq \mathcal{M}_n$.

Theorem (G.-Tomašević, 2025, LAA)

For a linear map unital $\phi : \mathcal{A}_\rho \to \mathcal{M}_n$, the following is equivalent:

(a) *ϕ is a rank-one preserver.*

(b) *ϕ is a Jordan monomorphism, and the associated transitive map $g : \rho \to \mathbb{C}^\times$ satisfies*

$$\begin{vmatrix} g(i,j) & g(i,l) \\ g(k,j) & g(k,l) \end{vmatrix} = 0$$

for every rectangle $(i,j),(i,l),(k,j),(k,l)$ in ρ.

Theorem (G.-Tomašević, 2025, LAA)

A linear unital map $\phi : \mathcal{A}_\rho \to \mathcal{M}_n$ is a rank preserver if and only if there exists an invertible matrix $S \in \mathrm{GL}(n)$ and a central idempotent $P \in Z(\mathcal{A})$ such that

$$\phi(X) = S\left(P(X) + (I-P)X^{\mathrm{t}}\right)S^{-1}.$$

Šemrl-type theorem for SMAs

Given a quasi-order ρ on $[n]$, for a fixed $i \in [n]$ denote

$$\rho(i) := \{j \in [n] : (i,j) \in \rho\}, \qquad \rho^{-1}(i) := \{j \in [n] : (j,i) \in \rho\}.$$

Theorem (G.-Tomašević, JMAA, 2025)

For an SMA $\mathcal{A}_\rho \subseteq \mathcal{M}_n$ the following conditions are equivalent:

(a) *For each $(i,j) \in \rho$, $i \neq j$, we have*

$$|(\rho(i) \cup \rho^{-1}(i)) \cap (\rho(j) \cup \rho^{-1}(j))| \geq 3.$$

(b) *Every injective continuous map $\phi : \mathcal{A}_\rho \to \mathcal{M}_n$ that preserves both spectrum and commutativity is a Jordan homomorphism.*

(c) *Every injective continuous map $\phi : \mathcal{A}_\rho \to \mathcal{A}_\rho$ that preserves commutativity and is spectrum-shrinking is a Jordan automorphism.*

Remark

For any $(i,j) \in \rho$, $i \neq j$, we have $i,j \in (\rho(i) \cup \rho^{-1}(i)) \cap (\rho(j) \cup \rho^{-1}(j))$.

Discussion of the (i,j)-condition in the theorem

Assume that for some $(i,j) \in \rho$, $i \neq j$, we have

$$(\rho(i) \cup \rho^{-1}(i)) \cap (\rho(j) \cup \rho^{-1}(j)) = \{i,j\}.$$

We distingush two cases.

Example: the case $(j,i) \in \rho$

Then $\mathcal{A}_\rho$ contains a direct summand $\mathcal{B}$ isomorphic to $\mathcal{M}_2$.
Let $f : [0,+\infty) \to \mathbb{S}^1$ be a non-constant continuous map such that $\lim_{t\to+\infty} f(t) = 1$ (e.g. $f(t) := e^{\frac{i}{t+1}}$). Then $\psi : \mathcal{M}_2 \to \mathcal{M}_2$,

$$\psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) := \begin{cases} \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}, & \text{if } b = 0, \\ \begin{bmatrix} a & b\, f\left(\left|\frac{c}{b}\right|\right) \\ \dfrac{c}{f\left(\left|\frac{c}{b}\right|\right)} & d \end{bmatrix}, & \text{otherwise,} \end{cases}$$

defines an injective continuous map that preserves both spectrum and commutativity, but is not linear.

Example: the case $(j, i) \notin \rho$

Consider the SMA

$$\mathcal{A}_\rho := \begin{bmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} \subset \mathcal{M}_3.$$

Then the (i, j)-condition fails for $(i, j) = (1, 2)$. Define a map

$$f : \mathbb{C} \times \mathbb{C} \to \mathbb{C}, \qquad f(u, v) := \begin{cases} v, & \text{if } |u| \leq |v|, \\ v\left|\dfrac{v}{u}\right|, & \text{if } |u| > |w|, \end{cases}$$

which is continuous and homogeneous. Then

$$\phi : \mathcal{A}_\rho \to \mathcal{A}_\rho, \qquad \phi\left(\begin{bmatrix} \alpha & x & y \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}\right) := \begin{bmatrix} \alpha & f(\beta - \alpha, x) & y \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$$

defines an injective continuous map that preserves both spectrum and commutativity, but is not linear.

Šemrl-type theorem for singular matrices

First recall the notation. For $n \in \mathbb{Z}_{\geq 2}$ and $1 \leq k \leq n$,

$$\mathcal{M}_n^{\leq k} = \{\text{all } n \times n \text{ matrices of rank } \leq k\}.$$

Clearly, the linear span of $\mathcal{M}_n^{\leq k}$ is the entire algebra $\mathcal{M}_n$. In particular, we have $[\mathcal{M}_n^{\leq k}]_{\mathrm{alg}} = \mathcal{M}_n$.

Theorem (Chirvasitu-G.-Tomašević, LAA, 2025)

Let $n \geq 3$ and $1 \leq k < n$. Any injective continuous commutativity-preserving and spectrum-shrinking map $\phi : \mathcal{M}_n^{\leq k} \to \mathcal{M}_n^{\leq k}$ extends to a Jordan homomorphism (hence automorphism) of $\mathcal{M}_n$.

Remark

In contrast to the previously discussed results, the injectivity condition above is *indispensable*, as demonstrated by maps $\phi : \mathcal{M}_n^{\leq k} \to \mathcal{M}_n^{\leq k}$ of the form $\phi(X) = f(X)N$, where N is a fixed nilpotent matrix in $\mathcal{M}_n^{\leq k}$ and $f : \mathcal{M}_n^{\leq k} \to \mathbb{C}$ is an arbitrary continuous function.

Let $k = n - 1$, which corresponds to the set $\mathrm{Sing}(n)$.

Leveraging the fact that any injective continuous spectrum-shrinking map $\phi : \mathrm{Sing}(n) \to \mathcal{M}_n$ is automatically spectrum-preserving, so that in particular $\phi(\mathrm{Sing}(n)) \subseteq \mathrm{Sing}(n)$, we obtain the following stronger variant of the theorem, when ϕ takes values in $\mathcal{M}_n$:

Corollary

If $n \geq 3$, any injective continuous commutativity-preserving and spectrum-shrinking map $\phi : \mathrm{Sing}(n) \to \mathcal{M}_n$ extends to a Jordan automorphism of $\mathcal{M}_n$.

When $k = 1$, our result cannot be extended to maps with values in $\mathcal{M}_n$:

Example

The map $\phi : \mathcal{M}_n^{\leq 1} \to \mathcal{M}_n$, defined by $\phi(X) := \mathrm{Tr}(X) I_n - X$, is continuous and injective, preserving both spectrum and commutativity. However, it does not extend to a Jordan automorphism of $\mathcal{M}_n$.

For $2 \leq k \leq n - 2$, it remains unclear whether the singular variant of Šemrl's preserver theorem extends for maps $\phi : \mathcal{M}_n^{\leq k} \to \mathcal{M}_n$, even if $k > n - \sqrt{n}$.

Šemrl-type theorem for other matrix domains

Let $\mathcal{X}$ be any of the matrix Lie groups $\mathsf{GL}(n)$, $\mathsf{SL}(n)$, or $\mathsf{U}(n)$, or the space of normal matrices N_n.

Again, the linear span of $\mathcal{X}$ in $\mathcal{M}_n$ is the entire algebra $\mathcal{M}_n$, so that $[\mathcal{X}]_{\mathrm{alg}} = \mathcal{M}_n$.

Theorem (Chirvasitu-G.-Tomašević, preprint, 2025)

If $\mathcal{X} \in \{\mathsf{GL}(n), \mathsf{SL}(n), \mathsf{U}(n), N_n\}$ or consists of the diagonalizable matrices in either $\mathsf{GL}(n)$ or $\mathsf{SL}(n)$, then any continuous, commutativity-preserving and spectrum-shrinking map extends to a Jordan automorphism of $\mathcal{M}_n$.

Remark

Our proof of the semisimple branch does not rely on Šemrl's preserver characterization of Jordan automorphisms of $\mathcal{M}_n$. In fact, a simple continuity argument recovers Šemrl's theorem.

Remark

In contrast to previous cases, the techniques used to prove the above result involve minimal direct computation. They are:

- partly algebraic/geometric (heavily dependent on the Fundamental Theorem of Projective Geometry),
- partly reliant on the topology of the matrix spaces involved,
- and partly operator-theoretic. In particular, during the analysis, we encounter a qualitatively new candidate map:

$$SNS^{-1} \mapsto S^{-1}NS, \quad \forall \text{ normal } N \in N_n \text{ and positive } S \in \mathrm{GL}(n).$$

  To show that it satisfies all requirements except continuity, we make crucial use of the celebrated *Putnam-Fuglede theorem*.

Thank you for your attention!