The centre-quotient property and weak centrality for C^*-algebras

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The 48th Canadian Operator Symposium
May 25 - 29, 2020, The Fields Institute

joint work with Robert J. Archbold (to appear in IMRN)

This research was supported by the Croatian Science Foundation under the project IP-2016-06-1046.

Introduction

Let A be a C^*-algebra with centre $Z(A)$. If I is a (closed two-sided) ideal of A, it is immediate that

$$(Z(A)+I)/I = q_I(Z(A)) \subseteq Z(A/I), \tag{1}$$

where $q_I : A \to A/I$ is the canonical map.

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A C^*-algebra A is said to have the *centre-quotient property (CQ-property)* if for any ideal I of A, equality holds in (1).

In the unital case we have the following beautiful characterization of CQ-property due to Vesterstrøm.

Theorem (Vesterstrøm 1971)

If A is a unital C^*-algebra, then the following conditions are equivalent:

(i) *A has the CQ-property.*

(ii) *A is weakly central, that is for any pair of maximal ideals M and N of A, $M \cap Z(A) = N \cap Z(A)$ implies $M = N$.*

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- The most prominent examples of weakly central C^*-algebras A are those satisfying the *Dixmier property*, that is for each $x \in A$ the closure of the convex hull of the unitary orbit of x intersects $Z(A)$ (Archbold 1972). In particular, von Neumann algebras are weakly central (Dixmier 1949, Misonou 1952).

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- In 2008 Magajna gave a characterisation of weak centrality in terms of more general averaging that involves unital completely positive elementary operators.
- Finally, in 2017 Archbold, Robert and Tikuisis found the exact gap between weak centrality and the Dixmier property for unital C^*-algebras and showed that a postliminal C^*-algebra has the (singleton) Dixmier property if and only if it has the CQ-property.

Global Approach

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- $\mathrm{Max}(A)$ can be empty (e.g. the algebra $A = \mathrm{K}(\mathcal{H})$ of compact operators on a separable infinite-dimensional Hilbert space $\mathcal{H}$).

Definition

We say that a C^*-algebra A is *weakly central* if:

(a) no modular maximal ideal of A contains $Z(A)$, and

(b) for each pair of modular maximal ideals M_1 and M_2 of A, $M_1 \cap Z(A) = M_2 \cap Z(A)$ implies $M_1 = M_2$.

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Theorem (Archbold-G. 2020)

For a C^*-algebra A the following conditions are equivalent:

(i) *A has the CQ-property.*

(ii) *A is weakly central.*

(iii) *$A^\sharp$ is weakly central.*

(iv) *There is a weakly central ideal J of A such that all primitive ideals of A that contain J are non-modular.*

(v) *There is an ideal J of A such that both J and A/J have the CQ-property and $Z(A/J) = (Z(A)+J)/J$.*

It is possible to show that every C^*-algebra contains a largest ideal with the CQ-property by using Zorn's lemma and the fact that the sum of two ideals with the CQ-property has the CQ-property.

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- T_A^1 as the set of all $M \in \mathrm{Max}(A)$ such that $Z(A) \subseteq M$.
- T_A^2 as the set of all $M \in \mathrm{Max}(A)$ for which exists $N \in \mathrm{Max}(A)$ such that $M \neq N$, $Z(A) \nsubseteq M, N$ and $M \cap Z(A) = N \cap Z(A)$.
- $T_A := T_A^1 \cup T_A^2$.

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Theorem (Archbold-G. 2020)

If A is a C^*-algebra then $J_{wc}(A)$ is the largest weakly central ideal of A.

Example

(a) If A is the Dixmier's classic example of a C^*-algebra in which the Dixmier property fails, i.e. $A := \mathrm{K}(\mathcal{H}) + \mathbb{C}p + \mathbb{C}(1-p) \subset \mathrm{B}(\mathcal{H})$, where $\mathcal{H}$ is a separable infinite-dimensional Hilbert space and $p \in \mathrm{B}(\mathcal{H})$ a projection with infinite-dimensional kernel and image, then $J_{wc}(A) = \mathrm{K}(\mathcal{H})$.

(b) If A either the rotation algebra (the C^*-algebra of the discrete three-dimensional Heisenberg group, or $A = C^*(\mathbb{F}_2)$ (the full C^*-algebra of the free group on two generators, then $J_{wc}(A) = \{0\}$.

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The CQ-property/weak centrality is well-behaved with respect to the C^*-tensor products.

Theorem (Archbold 1971, Archbold-G. 2020)

Let A_1 and A_2 be C^*-algebras. The following conditions are equivalent:

(i) *Both A_1 and A_2 have the CQ-property.*

(ii) *$A_1 \otimes_\beta A_2$ has the CQ-property for every C^*-norm β.*

(iii) *$A_1 \otimes_\beta A_2$ has the CQ-property for some C^*-norm β.*

Local Approach

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Proposition

$\mathrm{CQ}(A)$ is a self-adjoint subset of A that is closed under scalar multiplication and contains $Z(A) + J_{wc}(A)$.
Moreover, $\mathrm{CQ}(A)$ contains all commutators $[a, b] = ab - ba$ $(a, b \in A)$, quasi-nilpotent elements and products by quasi-nilpotent elements.
In particular, $\mathrm{CQ}(A) = Z(A)$ if and only if A is abelian.

In order to identify the set $\mathrm{CQ}(A)$ we shall need the following result that for a unital C^*-algebra A gives a necessary and sufficient condition for a central element of A/I to lift to a central element of A.

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If A is unital, first recall that by the Dauns-Hofmann theorem there exists an isomorphism

$$\Psi_A : Z(A) \to C(\mathrm{Prim}(A)) \qquad \text{such that} \qquad z + P = \Psi_A(z)(P)1 + P$$

for all $z \in Z(A)$ and $P \in \mathrm{Prim}(A)$ (as A is unital, $\mathrm{Prim}(A)$ is compact).

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Theorem (Archbold-G. 2020)

Let A be a unital C^*-algebra and let I be an ideal of A. A central element $\dot{z}$ of A/I can be lifted to a central element of A if and only if

$$\Psi_{A/I}(\dot{z})(P_1/I) = \Psi_{A/I}(\dot{z})(P_2/I)$$

for all $P_1, P_2 \in \mathrm{Prim}(A)$ that contain I and $P_1 \cap Z(A) = P_2 \cap Z(A)$.

Theorem (Archbold-G. 2020)

If A is a C^*-algebra then $A \setminus \mathrm{CQ}(A) = V_A^1 \cup V_A^2$, where:

- *V_A^1 is the set of all $a \in A$ for which there exists $M \in \mathrm{Max}(A)$ such that $Z(A) \subseteq M$ and $a + M$ is a non-zero scalar in A/M,*
- *V_A^2 is the set of all $a \in A$ for which there exist $M_1, M_2 \in \mathrm{Max}(A)$ and scalars $\lambda_1 \neq \lambda_2$ such that $Z(A) \nsubseteq M_i$, $M_1 \cap Z(A) = M_2 \cap Z(A)$ and $a + M_i = \lambda_i 1_{A/M_i}$ $(i = 1, 2)$.*

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If A is a C^*-algebra then all commutators belong to $\mathrm{CQ}(A)$. Let $[A, A]$ be the linear span of all commutators of A and $\overline{[A, A]}$ its norm-closure. We now characterise when $\mathrm{CQ}(A)$ contains $\overline{[A, A]}$ (using a result of Pop 2002).

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Theorem (Archbold-G. 2020)

Let A be a C^*-algebra that is not weakly central.

(a) *If for all $M \in T_A$, A/M admits a tracial state then $\overline{[A, A]} \subseteq \mathrm{CQ}(A)$.*

(b) *If there is $M \in T_A$ such that A/M does not admit a tracial state, then $[A, A] \nsubseteq \mathrm{CQ}(A)$.*

Corollary

If A is a postliminal C^*-algebra or an AF-algebra, then $\overline{[A, A]} \subseteq \mathrm{CQ}(A)$.

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As already mentioned, $\mathrm{CQ}(A)$ always contains $Z(A) + J_{wc}(A)$. The next result in particular demonstrates that $\mathrm{CQ}(A)$ is a C^*-subalgebra of A if and only if $\mathrm{CQ}(A) = Z(A) + J_{wc}(A)$. In fact, when this does not hold, $\mathrm{CQ}(A)$ fails dramatically to be a C^*-algebra.

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Theorem (Archbold-G. 2020)

Let A be a C^*-algebra. The following conditions are equivalent:

(i) $\mathrm{CQ}(A)$ *is closed under addition.*

(ii) $\mathrm{CQ}(A)$ *is closed under multiplication.*

(iii) $\mathrm{CQ}(A)$ *is norm-closed.*

(iv) $\mathrm{CQ}(A) = Z(A) + J_{wc}(A)$.

(v) $A/J_{wc}(A)$ *is abelian.*

Corollary

If A is a postliminal C^*-algebra or an AF-algebra, then the conditions (i)-(v) of previous theorem are also equivalent to:

(vi) *For any $x \in \mathrm{CQ}(A)$, $x^n \in \mathrm{CQ}(A)$ for all $n \in \mathbb{N}$.*

Corollary

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Example

Let $B \neq \mathbb{C}$ be any unital simple projectionless C^*-algebra (e.g. the Jiang-Su algebra $\mathcal{Z}$) and let A be the C^*-algebra of all cts. functions $x : [0,1] \to \mathrm{M}_2(B)$, such that $x(1) = \mathrm{diag}(b(x), 0)$, for some $b(x) \in B$. If $M := C_0([0,1), \mathrm{M}_2(B))$, then $M \in \mathrm{Max}(A)$ is (weakly) central so

$$T_A = T_A^1 = \{M\}, \qquad J_{wc}(A) = M \qquad \text{and}$$

$$\mathrm{CQ}(A) = \{x \in A : \ b(x) \text{ is not a non-zero scalar}\}.$$

As $A/J_{wc}(A) \cong B$ is non-abelian, $\mathrm{CQ}(A)$ is not norm-closed and is neither closed under addition nor under multiplication. On the other hand (as B is projectionless), for each $x \in \mathrm{CQ}(A)$ and $n \in \mathbb{N}$ we have $x^n \in \mathrm{CQ}(A)$.

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An algebra A over a field $\mathbb{F}$ is said to be *centrally stable (CS)* if for any ideal I of A we have $(Z(A)+I)/I = Z(A/I)$.

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- Central stability do not passes to ideals and we do not know if the tensor product of two unital CS algebras is always CS.
- If A is a finite-dimensional unital algebra over a perfect field $\mathbb{F}$, then A is CS if and only if there exist finite field extensions $\mathbb{F}_1, \ldots, \mathbb{F}_r$ of $\mathbb{F}$, commutative unital $\mathbb{F}_i$-algebras $C_1, \ldots, C_r$, and central simple $\mathbb{F}_i$-algebras $A_1, \ldots, A_r$ such that $A \cong (C_1 \otimes_{\mathbb{F}_1} A_1) \times \cdots \times (C_r \otimes_{\mathbb{F}_r} A_r)$ (Brešar-G. 2019).