The Dixmier property and weak centrality for C^*-algebras

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based on a joint work with Robert J. Archbold and Leonel Robert

C^*-algebras - definition and basic properties

C^*-algebra

A C^*-**algebra** is a complex Banach $*$-algebra A whose norm $\|\cdot\|$ satisfies the C^*-identity. More precisely:

- A is a Banach algebra with identity over the field $\mathbb{C}$.
- A is equipped with an involution, i.e. a map $* : A \to A$, $a \mapsto a^*$ satisfying the properties:

$$(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*, \quad (ab)^* = b^* a^*, \quad \text{and} \quad (a^*)^* = a,$$

  for all $a, b \in A$ and $\alpha, \beta \in \mathbb{C}$.
- Norm $\|\cdot\|$ satisfies the C^*-**identity**, i.e.

$$\|a^* a\| = \|a\|^2$$

  for all $a \in A$.

C^*-algebraic formulation of Quantum Mechanics

- C^*-algebras are historically associated with the development of QM through the groundbreaking work of Heisenberg, Jordan and von Neumann in the late 1920s.
- In QM a physical system can be described via a unital C^*-algebra A.
- The self-adjoint elements of A are thought of as the observables – the measurable quantities of the system.
- A state of the system is defined as a positive unital linear functional on A – if the system is in the state ω, then $\omega(a)$ is the expected value of the observable a.
- Automorphisms correspond to the symmetries, while one-parameter automorphism groups $\{\Phi_t\}_{t\in\mathbb{R}}$ describe the reversible time evolution of the system (in the Heisenberg picture).

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Since the 1960s C^*-algebras serve as a natural mathematical framework for the quantum field theory.

The C^*-identity is a very strong requirement. For instance, for any $a \in A$ let $\sigma(a)$ denote the spectrum of a, i.e.

$$\sigma(a) := \{\lambda \in \mathbb{C} : \lambda 1 - a \text{ is not invertible in } A\}.$$

Then the C^*-identity combined with the spectral radius formula

$$r(a) := \max\{|\lambda| : \lambda \in \sigma(a)\} = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}},$$

implies that the C^*-norm is uniquely determined by the algebraic structure:

$$\|a\|^2 = \|a^* a\| = r(a^* a) = \max\{|\lambda| : \lambda \in \sigma(a^* a)\}.$$

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In the category of C^*-algebras, the natural candidates for morphisms are the $*$-**homomorphisms**, i.e. the algebra homomorphisms which which preserve the involution. Basic properties:

- they are automatically contractive (isometric if injective), and
- their image is a C^*-subalgebra of the codomain C^*-algebra.

Basic examples

- To any LCH (locally compact Hausdorff) space one can associate a commutative C^*-algebra $C_0(X)$ of all continuous functions $f : X \to \mathbb{C}$ that vanish at infinity, with respect to the pointwise operations, involution $f^*(x) := \overline{f(x)}$, and sup-norm $\|f\|_\infty := \sup_{x \in X} |f(x)|$.
- The set $\mathrm{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ becomes a C^*-algebra with respect to the standard operations, usual adjoint and operator norm. In particular, the complex matrix algebras $M_n(\mathbb{C})$ are C^*-algebras. In fact, the finite direct sums of matrix algebras over $\mathbb{C}$ make up all finite-dimensional C*-algebras.
- To any locally compact group G, one can associate a C^*-algebra $C^*(G)$. Everything about the representation theory of G is encoded in $C^*(G)$.
- The category of C^*-algebras is closed under the formation of direct products, direct sums, extensions, direct limits, pullbacks, pushouts, (C^*-)tensor products, etc.

In fact, all commutative C^*-algebras arise as in previous example:

Theorem (Commutative Gelfand-Naimark theorem, 1943)

The (contravariant) functor $X \rightsquigarrow C_0(X)$ defines an equivalence of categories of LCH spaces (with proper continuous maps as morphisms) and commutative C^*-algebras (with non-degenerate $*$-homomorphisms as morphisms).

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In other words: By passing from the space X the function algebra $C_0(X)$, no information is lost. In fact, X can be recovered from $C_0(X)$. Thus, topological properties of X can be translated into algebraic properties of $C_0(X)$, and vice versa. Therefore, the theory of C^*-algebras is often thought of as **noncommutative topology**.

Representations of C^*-algebras

A **representation** of a C^*-algebra A is a $*$-homomorphism $\pi : A \to \mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. A representation π is said to be **irreducible** if it has no nontrivial (closed) invariant subspaces (i.e. if $\mathcal{K}$ is a (closed) subspace of $\mathcal{H}$ such that $\pi(A)\mathcal{K} \subseteq \mathcal{K}$, then $\mathcal{K} = \{0\}$ or $\mathcal{K} = \mathcal{H}$).

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Theorem (General Gelfand-Naimark theorem, 1943)

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Because of the previous theorem, C^*-algebras can be concretely defined to be norm closed self-adjoint subalgebras of bounded operators on some Hilbert space $\mathcal{H}$.

The primitive spectrum of a C^*-algebra

Let A be C^*-algebra.

- A **primitive ideal** of A is an ideal which is the kernel of an irreducible representation of A.
- The **primitive spectrum** of A is the set $\mathrm{Prim}(A)$ of primitive ideals of A equipped with the **Jacobson** (**hull-kernel**) **topology**: if S is a set of primitive ideals, its closure is

$$\overline{S} := \left\{ P \in \mathrm{Prim}(A) \ : \ker S = \bigcap_{Q \in S} Q \subseteq P \right\}.$$

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Example - commutative case

If $A = C_0(X)$ and $x \in X$, let $P_x := \{f \in C_0(X) \ : \ f(x) = 0\}$. Then $\mathrm{Prim}(C_0(X)) = \{P_x \ : \ x \in X\}$. Moreover, the correspondence $x \mapsto P_x$ defines a homeomorphism between X and $\mathrm{Prim}(C_0(X))$.

Properties of Prim(A)

- $\mathrm{Prim}(A)$ is always a locally compact and is compact if A is unital.
- If A is separable, $\mathrm{Prim}(A)$ is second countable.
- However, as a topological space, $\mathrm{Prim}(A)$ is in general badly behaved and may satisfy only the T_0-separation axiom (e.g. if $\mathcal{H}$ is a separable infinite dimensional Hilbert space, then Prim$(\mathrm{B}(\mathcal{H}))=\{0, \mathrm{K}(\mathcal{H})\}$), so that $\{0\}$ is not closed in Prim$(\mathrm{B}(\mathcal{H}))$.

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When a C^*-algebra A is unital, the Jacobson topology on $\mathrm{Prim}(A)$ not only describes the ideal structure of A, but also allows us to completely describe its centre $Z(A) = \{z \in A : za = az\}$:

Dauns-Hofmann theorem, 1968

Let A be a unital C^*-algebra. Then there is a $*$-isomorphism $\Psi_A : Z(A) \to C(\mathrm{Prim}(A))$ such that

$$z + P = \Psi_A(z)(P)1 + P$$

for all $f \in C(\mathrm{Prim}(A))$, $a \in A$ and $P \in \mathrm{Prim}(A)$.

The Dixmier property and weak centrality

Preliminaries

- Throughout A will be a C^*-algebra with centre $Z(A)$ and unitary group $\mathcal{U}(A) = \{u \in A : u^*u = uu^* = 1\}$ (if A is unital).
- By an ideal of A we always mean a closed two-sided ideal. We denote by $\mathrm{Ideal}(A)$ the set of all (closed two-sided) ideals of A.
- By $\mathcal{S}(A)$ we denote the set of all states on A (i.e. positive linear functionals $\omega : A \to \mathbb{C}$ of norm 1) equipped with the relative w^*-topology.
- A state $\tau \in \mathcal{S}(A)$ is said to be **tracial** if $\tau(xy) = \tau(yx)$ $\forall x, y \in A$.
- By $\mathcal{T}(A)$ we denote the set of all tracial states on A. If A is unital then $\mathcal{T}(A)$ is a convex w^*-compact subset of $\mathcal{S}(A)$.
- By $\partial_e\mathcal{T}(A)$ we denote the extreme boundary of $\mathcal{T}(A)$, so that $\mathcal{T}(A)$ is equal to the closed convex hull of $\partial_e\mathcal{T}(A)$ (by the Krein-Milman theorem).

- A **unitary mixing operator** on A is a map $\phi\colon A \to A$ of the form

$$\phi(x) = \sum_{i=1}^{n} t_i u_i^* x u_i,$$

  where n is a positive integer, $u_1, \ldots, u_n \in \mathcal{U}(\mathcal{A})$ and $t_1, \ldots, t_n$ non-negative real numbers such that $t_1 + \ldots + t_n = 1$. The set of all such maps is denoted by $\mathrm{UM}(A)$.

The Dixmer property and weak centrality

Let A be a unital C^*-algebra.

- For an element $a \in A$ the **Dixmier set** $D_A(a)$ is defined as the norm-closure of the set $\{\phi(a) : \phi \in \mathrm{UM}(A)\}$ (i.e.the closed convex hull of the unitary orbit of a). Then A is said to have the **Dixmier property** (DP) if

$$D_A(a) \cap Z(A) \neq \emptyset \qquad \forall a \in A.$$

- A is said to be **weakly central** (WC) if for any pair of maximal ideals M_1 and M_2 of A, $M_1 \cap Z(A) = M_2 \cap Z(A)$ implies $M_1 = M_2$.

Important properties

- DP $\implies$ WC (Archbold 1972).
- All von Neumann algebras satisfy DP (Dixmier 1949, Misonou 1952).
- A unital simple C^*-algebra satisfies DP iff it admits at most one tracial state (Haagerup-Zsidó 1984). In particular, WC $\not\Longrightarrow$ DP.

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A complete characterization of C^*-algebras with DP was obtained recently.

Theorem (Archbold-Robert-Tikuisis, JFA 2017)

A unital C^*-algebra A has DP iff all of the following hold:

- *A is WC.*
- *Every simple quotient of A has at most one tracial state.*
- *Every extreme tracial state of A factors trough some simple quotient.*

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Corollary

A unital postliminal C^*-algebra has DP iff it is WC.

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For a unutal C^*-algebra A the following conditions are equivalent:

- $Z(A) = \mathbb{C}1$ *and A has DP.*
- *A has a unique maximal ideal M, A (or A/M) has at most one tracial state and M has no tracial states.*

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The Dixmier's example

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space and let $p \in \mathrm{B}(\mathcal{H})$ be any projection with infinite-dimensional kernel and image. Set

$$A := \mathrm{K}(\mathcal{H}) + \mathbb{C}p + \mathbb{C}(1-p) \subset \mathrm{B}(\mathcal{H}).$$

Then $Z(A) = \mathbb{C}1$, A has precisely two maximal ideals, namely

$$M_1 := \mathrm{K}(\mathcal{H}) + \mathbb{C}p \qquad \text{and} \qquad M_2 := \mathrm{K}(\mathcal{H}) + \mathbb{C}(1-p),$$

and obviously $M_1 \cap Z(A) = M_2 \cap Z(A) = \emptyset$. Hence, A is not WC.

In 2008 Magajna gave a characterisation of weak centrality in terms of more general averaging which are defined as follows.

EUCP operators and Magajna set

- By an **elementary unital completely positive operator** on a unital C^*-algebra A we mean a map $\phi : A \to A$ of the form

$$\phi(x) = \sum_{i=1}^{n} a_i^* x a_i,$$

  where $n \in \mathbb{N}$, $a_1, \ldots, a_n \in A$ such that $\sum_{i=1}^{n} a_i^* a_i = 1$. The set of all such maps on A is denoted by $\mathrm{EUCP}(A)$.
- For $a \in A$ we define the **Magajna set** $M_A(a)$ as the norm-closure of the set $\{\phi(a) : \phi \in \mathrm{EUCP}(A)\}$ (i.e. the closed C^*-convex hull of a). Obviously $D_A(a) \subseteq M_A(a)$ for any $a \in A$

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Theorem (Magajna, JMAA 2008)

A unital C^*-algebra A is WC if and only if $M_A(a) \cap Z(A) \neq \emptyset$ for all $a \in A$.

On the other hand, in 1971 Vesterstrøm gave a different characterization of weak centrality of A in terms of the quotient images of $Z(A)$.

The centre-quotient property

- If $J \in \mathrm{Ideal}(A)$ it is immediate that

$$(Z(A) + J)/J = q_J(Z(A)) \subseteq Z(A/J),$$

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Example

If $A = \mathrm{K}(\mathcal{H}) + \mathbb{C}p + \mathbb{C}(1-p) \subset \mathrm{B}(\mathcal{H})$ is the Dixmier C^*-algebra, then $Z(A) = \mathbb{C}1$, while $Z(A/\mathrm{K}(\mathcal{H})) = A/\mathrm{K}(\mathcal{H}) \cong \mathbb{C} \oplus \mathbb{C}$.

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Show that any C^*-algebra A has the largest WC ideal $J_{wc}(A)$ and the largest ideal $J_{dp}(A)$ with DP, and obtain their concrete descriptions.

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Local apporach

Consider individual elements of A which witness DP and WC/CQP. We define an element $a \in A$ to be:

- a **Dixmier element** if $D_A(a) \cap Z(A) \neq \emptyset$;
- a **Magajna element** if $M_A(a) \cap Z(A) \neq \emptyset$;
- a **CQ-element** if for any ideal J of A, $a + J \in Z(A/J)$ implies $a \in Z(A) + J$.

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By $\mathrm{Dix}(A)$, $\mathrm{Mag}(A)$ and $\mathrm{CQ}(A)$ we respectively denote the sets of all Dixmier, Magajna and CQ-elements of A. Obviously A has DP iff $\mathrm{Dix}(A) = A$, while A is WC/has CQP iff $\mathrm{Mag}(A) = \mathrm{CQ}(A) = A$.

Global approach

We begin by extending the definition of WC and DP for non-unital C^*-algebras in the obvious way: We say that a non-unital C^*-algebra A is WC/has DP if its minimal unitization $A^\sharp = A \oplus \mathbb{C}1$ has the same property.

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Modular maximal ideals

- An ideal J of A is said to be **modular** if the algebra A/J is unital.
- Any proper modular ideal of A (if such exists) is contained in some modular maximal ideal of A and all modular maximal ideals of A are primitive. By $\mathrm{Max}(A)$ we denote the set of all modular maximal ideals of A, so that $\mathrm{Max}(A) \subseteq \mathrm{Prim}(A)$.
- $\mathrm{Max}(A)$ can be empty (e.g. the algebra $A = \mathrm{K}(\mathcal{H})$ of compact operators on a separable infinite-dimensional Hilbert space $\mathcal{H}$).
- If A is unital, both spaces $\mathrm{Prim}(A)$ and $\mathrm{Max}(A)$ are compact.
- For any $J \in \mathrm{Ideal}(A)$ we define $\mathrm{Max}^J(A)$ for the set of all modular maximal ideals of A that contain J. The space $\mathrm{Max}^J(A)$ is canonically homeomorphic to $\mathrm{Max}(A/J)$ via the assignment $M \mapsto M/J$.

Theorem (Archbold-G, IMRN 2022)

For any C^*-algebra A the following conditions are equivalent:

- *A is WC.*
- *No modular maximal ideal of A contains $Z(A)$ and for all $M_1, M_2 \in \mathrm{Max}(A)$, $M_1 \cap Z(A) = M_2 \cap Z(A)$ implies $M_1 = M_2$.*
- *A has CQP.*

Further, the class of WC C^*-algebras is closed under forming ideals, quotients, direct sums and C^*-tensor product. Moreover if A_1 and A_2 are C^*-algebras then $A_1 \otimes_\beta A_2$ is WC for some/every C^*-norm β if and only if both A_1 and A_2 are WC.

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It is possible to show that every C^*-algebra contains a largest ideal with CQP by using Zorn's lemma and the fact that the sum of two ideals with CQP has CQP.

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However, we take a different approach that has the merit of obtaining a formula for this ideal in terms of the set of those modular maximal ideals of A which witness the failure of the weak centrality of A.

Theorem (Archbold-G, IMRN 2022)

Let A be a C^*-algebra and T_A the set of all $M \in \mathrm{Max}(A)$ such that either

- $Z(A) \subseteq M$, *or*
- *there is $N \in \mathrm{Max}(A)$ such that $M \neq N$, $Z(A) \nsubseteq M, N$ and $M \cap Z(A) = N \cap Z(A)$.*

Then $J_{wc}(A) := \ker T_A$ is the largest weakly central ideal of A.

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Example

- If $A = \mathrm{K}(\mathcal{H}) + \mathbb{C}p + \mathbb{C}(1-p) \subset \mathrm{B}(\mathcal{H})$ is the Dixmier's example, then $J_{wc}(A) = (\mathrm{K}(\mathcal{H}) + \mathbb{C}p) \cap (\mathrm{K}(\mathcal{H}) + \mathbb{C}(1-p)) = \mathrm{K}(\mathcal{H})$.
- If G is either the free group on two generators $\mathbb{F}_2$ or the discrete three-dimensional Heisenberg group

$$H_3 = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\},$$

then for $A = C^*(G)$ we have $J_{wc}(A) = \{0\}$.

By using Zorn's lemma, in 1972 Archbold showed that any unital C^*-algebra A contains the largest ideal $J_{dp}(A)$ with DP. We now describe $J_{dp}(A)$ more explicitly. But first we recall the notion of Glimm ideals.

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Glimm ideals in unital C^*-algebras

- For all $P, Q \in \mathrm{Prim}(A)$ we define $P \approx Q$ if $P \cap Z(A) = Q \cap Z(A)$.
- Let $\Psi_A : Z(A) \to C(\mathrm{Prim}(A))$ be the Dauns-Hofmann isomorphism, so that $z + P = \Psi_A(z)(P)1 + P$ for all $z \in Z(A)$ and $P \in \mathrm{Prim}(A)$. Then for all $P, Q \in \mathrm{Prim}(A)$,

$$P \approx Q \qquad \iff \qquad f(P) = f(Q) \quad \text{for all } f \in C(\mathrm{Prim}(A)).$$

- $\approx$ is an equivalence relation on $\mathrm{Prim}(A)$ and the equivalence classes are closed subsets of $\mathrm{Prim}(A)$ and there is is one-to-one correspondence between the quotient set $\mathrm{Prim}(A)/\approx$ and a set of ideals of A given by

$$[P]_\approx \longleftrightarrow \bigcap [P]_\approx,$$

where $[P]_\approx$ denotes the equivalence class of $P \in \mathrm{Prim}(A)$.

Glimm ideals in unital C^*-algebras (continuation)

- The set of ideals obtained in this way is denoted by $\mathrm{Glimm}(A)$, and its elements are called **Glimm ideals** of A.
- The quotient map $\mathrm{Prim}(A) \to \mathrm{Glimm}(A)$ given by $P \mapsto \bigcap [P]_{\approx}$ is known as the **complete regularization map**.
- We equip $\mathrm{Glimm}(A)$ with the quotient topology, which (since A is unital) coincides with the complete regularization topology. In this way $\mathrm{Glimm}(A)$ becomes a compact Hausdorff space.
- In fact, $\mathrm{Glimm}(A)$ is homeomorphic to $\mathrm{Max}(Z(A))$ via the assignment $\mathrm{Glimm}(A) \ni N \mapsto N \cap Z(A) \in \mathrm{Max}(Z(A))$, whose inverse is given by $\mathrm{Max}(Z(A)) \ni J \mapsto JA \in \mathrm{Glimm}(A)$ (the closures are not needed by the Hewitt-Cohen factorization theorem).
- For $z \in A$ we write $\hat{z}$ for the corresponding function in $C(\mathrm{Glimm}(A))$, so that the assignment $Z(A) \ni z \mapsto \hat{z} \in C(\mathrm{Glimm}(A))$ is an isomorphism such that

$$z + N = \hat{z}(N)1 + N \quad \forall z \in Z(A),\ N \in \mathrm{Glimm}(A).$$

- Now consider the set $X \subseteq \mathrm{Glimm}(A)$ of Glimm ideals N such that A/N has DP and a trivial centre.
- This is equivalent to saying that N is contained in a unique maximal ideal M_N of A, that A/N has at most one tracial state and that if A/N does have a tracial state then it factors through A/M_N.
- For $N \in \mathrm{Glimm}(A) \backslash X$ define

$$I_N := \ker\{M : M \in \mathrm{Max}^N(A)\} \cap \ker\{I_\tau : \tau \in \mathcal{T}(A/N)\}$$

where, for $\tau \in \mathcal{T}(A/N)$,

$$I_\tau := \{a \in A : \tau(a^* a + N) = 0\}.$$

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Theorem (Archbold-G-Robert, IMRN 2023)

We have

$$J_{dp}(A) = \ker\{I_N : N \in \mathrm{Glimm}(A) \setminus X\}.$$

Local approach

Recall, if $a \in A$ then:

- $a \in \mathrm{Dix}(A)$ if $D_A(A) \cap Z(A) \neq \emptyset$
- $a \in \mathrm{Mag}(A)$ if $M_A(A) \cap Z(A) \neq \emptyset$.
- $a \in \mathrm{CQ}(A)$ if for any ideal J of A, $a + J \in Z(A/J)$ implies $a \in Z(A) + J$.

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We always have

$$\mathrm{Dix}(A) \subseteq \mathrm{Mag}(A) \subseteq \mathrm{CQ}(A).$$

- $\mathrm{Dix}(A)$ always contains $Z(A) + J_{dp}(A)$, all self-commutators $[a^*, a]$ ($a \in A$) and all quasinilpotents. In particular, $\mathrm{Dix}(A) = Z(A)$ iff A is abelian.
- $\mathrm{Mag}(A)$ always contains $Z(A) + J_{wc}(A)$ and all products ab where a or b is quasinilpotent.
- $\mathrm{CQ}(A)$ always contains all commutators $[a, b]$ ($a, b \in A$). There are C^*-algebras A such that $[a, b] \notin \mathrm{Mag}(A)$ for some $a, b \in A$.

We always have

$$\overline{\mathrm{span}(\mathrm{Mag}(A))} = \overline{\mathrm{span}(\mathrm{CQ}(A))} = Z(A) + \mathrm{Ideal}([A, A]).$$

On the other hand, when sets $\mathrm{CQ}(A)$ and $\mathrm{Mag}(A)$ coincide, A is not far from being WC. On the other hand, when this fails, both sets dramatically fail to be C^*-subalgebras of A:

Theorem (Archbold-G & Archbold-G-Robert, IMRN 2022 & 2023)

The following conditions are equivalent:

- $\mathrm{Mag}(A) = \mathrm{CQ}(A)$.
- $\mathrm{Mag}(A) = \mathrm{CQ}(A) = Z(A) + J_{wc}(A)$.
- $A/J_{wc}(A)$ *is abelian.*
- $\mathrm{Mag}(A)$ *and/or* $\mathrm{CQ}(A)$ *is closed under addition.*
- $\mathrm{Mag}(A)$ *and/or* $\mathrm{CQ}(A)$ *is closed under multiplication.*
- $\mathrm{Mag}(A)$ *is closed under EUCP operators.*
- $\mathrm{CQ}(A)$ *is norm-closed.*

We also exhibited examples of (separable continuous trace) C^*-algebras A for which $J_{wc}(A) = \{0\}$, while $\mathrm{CQ}(A)$ is norm-dense in A.

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In order to identify the set $\mathrm{CQ}(A)$ we shall need the following result:

Theorem (Archbold-G, IMRN 2022)

Let A be a unital C^*-algebra and let J be an ideal of A. A central element $\dot{z}$ of A/I can be lifted to a central element of A iff

$$\Psi_{A/J}(\dot{z})(P_1/J) = \Psi_{A/J}(\dot{z})(P_2/J)$$

for all $P_1, P_2 \in \mathrm{Prim}(A)$ that contain J and $P_1 \cap Z(A) = P_2 \cap Z(A)$.

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for all $P_1, P_2 \in \mathrm{Prim}(A)$ that contain J and $P_1 \cap Z(A) = P_2 \cap Z(A)$.

Theorem (Archbold-G, IMRN 2022)

An element $a \in A$ belongs to $A \setminus \mathrm{CQ}(A)$ iff one of the following holds:

- *there exists $M \in \mathrm{Max}(A)$ such that $Z(A) \subseteq M$ and $a + M$ is a non-zero scalar in A/M;*
- *there exist $M_1, M_2 \in \mathrm{Max}(A)$ and scalars $\lambda_1 \neq \lambda_2$ such that $Z(A) \neq M_1 \cap Z(A) = M_2 \cap Z(A)$ and $a + M_i = \lambda_i 1_{A/M_i}$ $(i = 1, 2)$.*

On the other hand, a complete description of $\mathrm{Dix}(A)$ and $\mathrm{Mag}(A)$ is in general difficult to obtain. This has led us to also consider the sets

$$\overline{\mathrm{Mag}}(A) = \{a \in A : \mathrm{dist}(M_A(a), Z(A)) = 0\},$$
$$\overline{\mathrm{Dix}}(A) = \{a \in A : \mathrm{dist}(D_A(a), Z(A)) = 0\}.$$

These are more tractable sets (e.g. they are norm-closed). We have

$$\begin{array}{ccccc} & & \overline{\mathrm{Dix}}(A) & & \\ & \subsetneq & & \subsetneq & \\ \mathrm{Dix}(A) & & & & \overline{\mathrm{Mag}}(A) \subseteq \mathrm{CQ}(A). \\ & \subsetneq & & \subsetneq & \\ & & \mathrm{Mag}(A) & & \end{array}$$

Also note that A has DP iff $\overline{\mathrm{Dix}}(A) = A$ and A is WC iff $\overline{\mathrm{Mag}}(A) = A$.

Numerical range

Given $a \in A$ the **(algebraic) numerical range** of a is defined as $W_A(a) := \{\omega(a) : \omega \in \mathcal{S}(A)\}$. It is a compact convex subset of $\mathbb{C}$ that contains $\sigma(a)$. If a is normal then $W_A(a)$ is the convex hull of $\sigma(a)$.

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Theorem (Magajna, Canad. Math. Bull. 2000)

Let $a \in A$. A normal element $b \in A$ belongs to $M_A(a)$ iff $W_{A/P}(b+P) \subseteq W_{A/P}(a+P)$ for each $P \in \mathrm{Prim}(A)$.

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Let $a \in A$. A normal element $b \in A$ belongs to $M_A(a)$ iff $W_{A/P}(b+P) \subseteq W_{A/P}(a+P)$ for each $P \in \mathrm{Prim}(A)$.

Theorem (Archbold-G-Robert, IMRN 2023)

For any $a \in A$ we have $a \in \overline{\mathrm{Mag}}(A)$ iff for all $N \in \mathrm{Glimm}(A)$,

$$\Lambda_a(N) := \bigcap_{M \in \mathrm{Max}^N(A)} W_{A/M}(a+M) \neq \emptyset.$$

Further, if a is selfadjoint, then $a \in \overline{\mathrm{Mag}}(A)$ iff $a \in \mathrm{Mag}(A)$.

Example

Let $B = \mathrm{K}(\mathcal{H}) + \mathbb{C}p + \mathbb{C}(1-p)$ be the Dixmier C^*-algebra and let $A = C([-1,1], M_2(\mathbb{C})) \otimes B$. We define elements $a, b \in A$ as:

$$a(t) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b(t) := \begin{pmatrix} \alpha(t) & 0 \\ 0 & \beta(t) \end{pmatrix},$$

where $\alpha(t)$ and $\beta(t)$ are curves in the plane such that:

- From $t = -1$ to $t = 0$ the interval $[\alpha(t), \beta(t)]$ starts at $[-1, -1+2i]$, remains pinned at -1 while rotating till it is flat and equal to $[-1, 1]$ at $t = 0$.
- Then from $t = 0$ to $t = 1$ the interval $[\alpha(t), \beta(t)]$ is pinned at 1, and rotates till it stops at $[1, 1+2i]$.

If $c \in A$ defined as

$$c := a \otimes p + b \otimes (1-p),$$

then c is a normal element of A such that $c \in \overline{\mathrm{Mag}}(A) \setminus \mathrm{Mag}(A)$.

We now describe the set $\overline{\mathrm{Dix}}(A)$. Define

$$Y := \{N \in \mathrm{Glimm}(A) : \mathcal{T}(A/N) \neq \emptyset\}.$$

It is not difficult to se that Y is a closed subset of $\mathrm{Glimm}(A)$.

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Theorem (Archbold-G-Robert, IMRN 2023)

For an element $a \in A$ consider the following conditions:

(i) $a \in \mathrm{Dix}(A)$.

(ii) $a \in \overline{\mathrm{Dix}}(A)$.

(iii) **(a)** *There is a function $f_a \colon Y \to \mathbb{C}$ such that*

- **(a1)** *for all $N \in Y$ and $\tau \in \mathcal{T}(A/N)$, $f_a(N) = \tau(a + N)$,*
- **(a2)** *for all $N \in Y$, $f_a(N) \in \Lambda_a(N)$.*

(b) *For all $N \in \mathrm{Glimm}(A) \setminus Y$, $\Lambda_a(N) \neq \emptyset$.*

Then (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii). Further, if (iii) holds then f_a is unique and it is continuous on Y. Finally, if a is selfadjoint, then (i), (ii), and (iii) are equivalent.

Example

Let $B = \mathrm{K}(\mathcal{H}) + \mathbb{C}p + \mathbb{C}(1-p)$ be the Dixmier C^*-algebra and $A := C([-1,1], \mathcal{O}_2) \otimes B$ ($\mathcal{O}_2$ is the Cuntz algebra). Then $\mathcal{T}(A) = \emptyset$, so that $\mathrm{Dix}(A) = \mathrm{Mag}(A)$ and there is $a \in \overline{\mathrm{Dix}(A)} \setminus \mathrm{Dix}(A)$.

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Theorem (Archbold-G-Robert, IMRN 2023)

The set $Z(A) + \overline{[A,A]}$ *contains* $\overline{\mathrm{Dix}}(A)$ *and is equal to the closed linear span of* $\mathrm{Dix}(A)$. *Further, the following conditions are equivalent:*

- **(i)** $\mathrm{Dix}(A) = Z(A) + \overline{[A,A]}$.
- **(ii)** $\mathrm{Dix}(A)$ *is closed under unitary mixing operators.*
- **(iii)** $\mathrm{Dix}(A)$ *is closed under addition.*
- **(iv)** **(a)** *For all* $N \in Y$ *and* $M \in \mathrm{Max}^N(A)$, $\mathcal{T}(A/M) \neq \varnothing$.
  (b) *For all* $N \in \mathrm{Glimm}(A) \setminus Y$, $\mathrm{Max}^N(A)$ *is a singleton set.*

Moreover, when these equivalent conditions hold, $\mathrm{Dix}(A) = \overline{\mathrm{Dix}}(A)$.

Problem

Is $\overline{\mathrm{Mag}(A)} = \overline{\mathrm{Mag}}(A)$ and $\overline{\mathrm{Dix}(A)} = \overline{\mathrm{Dix}}(A)$?

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Theorem (Archbold-G-Robert, IMRN 2023)

The following conditions are equivalent:

(i) $A/J_{dp}(A)$ *is abelian.*

(ii) $\mathrm{Dix}(A) = Z(A) + J_{dp}(A)$,

(iii) $\mathrm{Dix}(A)$ *is closed under multiplication.*

Moreover, under these equivalent conditions $J_{dp}(A) = J_{wc}(A)$, *so that*

$$\mathrm{Dix}(A) = \mathrm{Mag}(A) = Z(A) + J_{dp}(A) = Z(A) + \overline{[A, A]}.$$