Derivations on C^*-Algebras: A Revisit

Ilja Gogić

Department of Mathematics, Faculty of Science
University of Zagreb

Logic, Categories, and Applications Seminar
Department of Mathematics - University of Bologna
June 13, 2025

C^*-algebras - definition and basic properties

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- A is equipped with an involution, i.e. a map $*: A \to A$, $a \mapsto a^*$ satisfying the properties:

$$(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*, \quad (ab)^* = b^* a^*, \quad \text{and} \quad (a^*)^* = a,$$

  for all $a, b \in A$ and $\alpha, \beta \in \mathbb{C}$.

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  for all $a, b \in A$ and $\alpha, \beta \in \mathbb{C}$.
- Norm $\|\cdot\|$ satisfies the C^*-**identity**, i.e.

  $$\|a^* a\| = \|a\|^2$$

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Since the 1960s C^*-algebras serve as a natural mathematical framework for the quantum field theory.

The C^*-identity is a very strong requirement. For instance, for any $a \in A$ let $\sigma(a)$ denote the spectrum of a, i.e.

$$\sigma(a) := \{\lambda \in \mathbb{C} : \lambda 1 - a \text{ is not invertible in } A\}.$$

Then the C^*-identity combined with the spectral radius formula

$$r(a) := \max\{|\lambda| : \lambda \in \sigma(a)\} = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}},$$

implies that the C^*-norm is uniquely determined by the algebraic structure:

$$\|a\|^2 = \|a^*a\| = r(a^*a) = \max\{|\lambda| : \lambda \in \sigma(a^*a)\}.$$

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In the category of C^*-algebras, the natural candidates for morphisms are the $*$-**homomorphisms**, i.e. the algebra homomorphisms which which preserve the involution. Basic properties:

- they are automatically contractive (isometric if injective), and
- their image is a C^*-subalgebra of the codomain C^*-algebra.

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- To any LCH (locally compact Hausdorff) space one can associate a commutative C^*-algebra $C_0(X)$ of all continuous functions $f : X \to \mathbb{C}$ that vanish at infinity, with respect to the pointwise operations, involution $f^*(x) := \overline{f(x)}$, and sup-norm $\|f\|_\infty := \sup_{x \in X} |f(x)|$.

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- The set $\mathbb{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ becomes a C^*-algebra with respect to the standard operations, usual adjoint and operator norm. In particular, the complex matrix algebras $M_n(\mathbb{C})$ are C^*-algebras.

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- Any C^*-algebra can be isometrically embedded as a norm-closed self-adjoint subalgebra of $\mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (the noncommutative Gelfand-Naimark theorem).
- To any LC group G, one can associate a C^*-algebra $C^*(G)$. Everything about the representation theory of G is encoded in $C^*(G)$.
- The category of C^*-algebras is closed under the formation of direct products, direct sums, extensions, direct limits, pullbacks, pushouts, (C^*-)tensor products, etc.

In fact, all commutative C^*-algebras arise as in previous example:

Theorem (Commutative Gelfand-Naimark theorem, 1943)

The (contravariant) functor $X \rightsquigarrow C_0(X)$ defines an equivalence of categories of LCH spaces (with proper continuous maps as morphisms) and commutative C^*-algebras (with non-degenerate $*$-homomorphisms as morphisms).

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In other words: By passing from the space X the function algebra $C_0(X)$, no information is lost. In fact, X can be recovered from $C_0(X)$. Thus, topological properties of X can be translated into algebraic properties of $C_0(X)$, and vice versa. Therefore, the theory of C^*-algebras is often thought of as **noncommutative topology**.

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- **The multiplier algebra** of A is the C^*-subalgebra $M(A)$ of the enveloping von Neumann algebra A^{**} that consists of all $x \in A^{**}$ such that $ax \in A$ and $xa \in A$ for all $a \in A$. $M(A)$ is the largest unital C^*-algebra which contains A as an essential ideal.

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- We denote by $\hat{A}$ the spectrum of A (i.e. the set of unitary equivalence classes of irreducible representations of A), and by $\mathrm{Prim}(A)$ the primitive ideal space of A (i.e. kernels of irreducible representations). Both spaces are equipped with the Jacobson (hull-kernel) topology, i.e. the closure of a subset S of $\mathrm{Prim}(A)$ is given by

$$\overline{S} := \left\{ P \in \mathrm{Prim}(A) \ : \ker S = \bigcap_{Q \in S} Q \subseteq P \right\}.$$

Derivations on C^*-algebras

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$$\delta(xy) = \delta(x)y + x\delta(y), \quad \forall x, y \in A.$$

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- Any derivation on A vanishes on $Z(A)$. In particular, commutative C^*-algebras do not admit non-zero derivations.
- If δ is a derivation on A, then δ^{**} is a derivation on A^{**} and $\delta^{**}(M(A)) \subseteq M(A)$. In particular, any derivation on A extends to a derivation $\widetilde{\delta}$ of $M(A)$ of the same norm (i.e. $\widetilde{\delta} = \delta^{**}|_{M(A)}$).

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- AW^*-algebras (Olesen 1974);
- homogeneous C^*-algebras (Sproston 1976 - unital case; G. 2013 - extension to the non-unital case).

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- All type I AW^*-algebras are monotone complete (Hamana 1981), but it is unknown whether all AW^*-algebras are monotone complete; this is a long standing open problem dating back to the work of Kaplansky.

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- More generally, if E is an algebraic $\mathbb{M}_n$-bundle over a LCH space X, i.e. E is a locally trivial fibre bundle with fibre $\mathbb{M}_n$ and structure group $\mathrm{Aut}(\mathbb{M}_n) \cong \mathrm{PU}(n)$ (the projective unitary group), then the set $\Gamma_0(E)$ of all continuous sections of E vanishing at infinity is an n-homogeneous C^*-algebra, with respect to the fiberwise operations and sup-norm.

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- By a famous theorem due to Fell and Tomiyama-Takesaki from 1961, every n-homogeneous C^*-algebra A can be realized as $A = \Gamma_0(E)$ for some algebraic $\mathbb{M}_n$-bundle E over $\mathrm{Prim}(A)$ (which is always a LCH space when A is homogeneous).

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Back to the main problem, the separable case was completely solved in 1979:

Theorem (Akemann, Elliott, Pedersen and Tomiyama 1979)

Let A be a separable C^*-algebra, Then the following conditions are equivalent:

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- the (n-)**subhomogeneous C^*-algebras**, i.e. C^*-algebras such that $n = \sup_{\pi \in \hat{A}} \dim \pi < \infty$.

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- If $\overline{X_2}$ is the Stone-Čech compactification of X_2, A is said to have the **Stone-Čech property**.

- Let $T := X_1 \cup \overline{X_2}$, topologized so that a subset S is open in T iff $S \cap X_1$ is open in X_1 and $S \cap \overline{X_2}$ is open in $\overline{X_2}$.

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Theorem (Sproston, 1981)

Let A be a unital 2-subhomogeneous C^*-algebra. Then A admits outer derivations in either of the following cases:

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Problem

Is the problem on the innerness of derivations on unital non-Fell 2-subhomogeneous C^*-algebras decidable within ZFC?

The local multiplier algebra

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Example

If A is simple, then obviously $M_{\mathrm{loc}}(A) = M(A)$. If A is an AW^*-algebra, then $M_{\mathrm{loc}}(A) = A$.

Example

If $A = C_0(X)$ is a commutative C^*-algebra, then $M_{\mathrm{loc}}(A)$ is a commutative AW^*-algebra whose maximal ideal space can be identified with the inverse limit $\varprojlim \beta U$ of Stone-Čech compactifications βU of dense open subsets U of X.

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Theorem (Pedersen 1978)

Every derivation on a C^*-algebra A extends uniquely and under preservation of the norm to a derivation on $M_{\mathrm{loc}}(A)$. Moreover, if A is separable (or more generally, if every essential closed ideal of A is σ-unital), this extension becomes inner in $M_{\mathrm{loc}}(A)$.

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In particular, Pedersen's result entails Sakai's theorem that every derivation on a simple unital C^*-algebra is inner.

Pedersen's problem

Since $M_{\mathrm{loc}}(A) = M(A)$ if A is simple, and $M_{\mathrm{loc}}(A) = A$ if A is an AW^*-algebra, an affirmative answer in the inseparable case would cover, extend and unify the results that all derivations on simple C^*-algebras and AW^*-algebras are inner.

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- C^*-algebras with finite-dimensional irreducible representations (G. 2013).

The stability problem of local multiplier algebras

Problem A

Is $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$ for every C^*-algebra A?

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However, it turns out that (nevertheless) $I(A)$ is a C^*-algebra canonically containing A as a C^*-subalgebra. Moreover, $I(A)$ is monotone complete, so in particular, $I(A)$ is an AW^*-algebra.

Theorem (Frank and Paulsen, 2003)

Under this embedding of A into $I(A)$, $M_{\mathrm{loc}}(A)$ is the norm closure of the set of all $x \in I(A)$ which act as a multiplier on some $I \in \mathrm{Id}_{ess}(A)$, i.e.

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- Moreover, one has $I(M_{\mathrm{loc}}(A)) = I(A)$, so we have an additional sequence of inclusions of C^*-algebras:

$$A \subseteq M_{\mathrm{loc}}(A) \subseteq M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) \subseteq \cdots \subseteq \overline{A} \subseteq I(A).$$

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- This is true if A is of type I; in this case A is injective (Hamana, 1981).
- However, for arbitrary AW^*-algebras this brings us back to the long standing open problem, originating with the work of Kaplansky (1951): *Are all AW^*-algebras monotone complete*?

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- Soon after, Argerami, Farenick and Massey (2009) showed that a relatively well-behaved C^*-algebra $C([0,1]) \otimes \mathbb{K}$ also fails to satisfy $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$.

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- This example was further developed by Ara and Mathieu (2011), who showed that if X is a perfect, second countable LCH space, and $A = C_0(X) \otimes B$ for some non-unital separable simple C^*-algebra B, then $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) \neq M_{\mathrm{loc}}(A)$.

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We have the following partial answer:

Theorem (Somerset, 2000; Ara and Mathieu, 2011)

If A is a unital (or more generally quasi-central), separable C^*-algebra such that $\mathrm{Prim}(A)$ contains a dense G_δ subset of closed points, then $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$. Moreover, in this case $M_{\mathrm{loc}}(A)$ has only inner derivations.

On the other hand, $M_{\rm loc}(M_{\rm loc}(A))$ is always a type I AW^*-algebra, whenever A is separable and liminal. More generally:

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There is also a partial converse in a non-separable direction:

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This result applies in particular to algebras of the form $A = C_0(X) \otimes \mathbb{K}$, for any LCH space X.

On the other hand, for (not necessarily separable) C^*-algebras A that admit only finite-dimensional irreducibe representations we obtained a full description of $M_{\mathrm{loc}}(A)$:

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Theorem (G., 2013)

If all irreducible representations of A are finite-dimensional, then $M_{\mathrm{loc}}(A)$ is a finite or countable direct product of C^*-algebras of the form $C(X_n) \otimes \mathbb{M}_n$, where each space X_n is Stonean. In particular, $M_{\mathrm{loc}}(A)$ is an AW^*-algebra of type I, so it coincides with the injective envelope of A and therefore admits only inner derivations.

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Recall that a space X is said to be **Stonean** if it is an extremally disconnected CH space. It is well known that a commutative C^*-algebra $A = C_0(X)$ is an AW^*-algebra if and only if X is a Stonean space.

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Problem

Is Pedersen's problem on the innerness of derivations on local multiplier algebras decidable within ZFC?

Derivations in the cb-norm closure of elementary operators

An attractive and fairly large class of bounded linear maps $\phi : A \to A$ that preserve all ideals of A consists of of **elementary operators**, that is, those that can be expressed as a finite sum

$$\phi = \sum_i M_{a_i,b_i}$$

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$$\|\phi\|_{cb} := \sup_{n\in\mathbb{N}} \|\phi_n\| < \infty,$$

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Let us denote by $\mathcal{E}\ell(A)$ the set of all elementary operators on A and by $\overline{\mathcal{E}\ell(A)}_{cb}$ its cb-norm closure.

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Obviously $*$-automorphisms of C^*-algebras A are completely bounded. The same holds true for derivations: if δ is a derivation on A, it extends to a derivation δ^{**} of A^{**}, and thus δ^{**} is inner in A^{**} by the Kadison-Sakai theorem.

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Theorem (G. 2013)

If A is a unital C^*-algebra whose every Glimm ideal is prime, then a derivation δ of A lies in $\overline{\mathcal{E}\ell(A)}_{cb}$ if and only if δ is an inner derivation.

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The **Glimm ideals** of A are the ideals of A generated by the maximal ideals of $Z(A)$.

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Corollary

The Pederesen's problem has a positive solution if and only if for each C^*-algebra A, every derivation on $M_{\mathrm{loc}}(A)$ lies in $\overline{\overline{\mathcal{E}\ell(M_{\mathrm{loc}}(A))}}_{cb}$.

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Corollary

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For prime C^*-algebras we also established the following result:

Theorem (G. 2021)

If A is a prime C^*-algebra then an algebra epimorphism $\sigma : A \to A$ lies in $\overline{\mathcal{E}\ell(A)}_{cb}$ if and only if σ is an inner automorphism of A.

In a contrast to the similar result for derivations, the above result cannot be extended even to homogeneous C^*-algebras, which admit only inner derivations (by Sproston's Theorem):

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Example

For $n \geq 2$ let $A_n = C(\mathrm{PU}(n), \mathbb{M}_n)$. Then A_n admits outer automorphisms that are simultaneously elementary operators.

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On the other hand:

Proposition (G. 2021)

Let A be a separable n-homogeneous C^*-algebra whose primitive spectrum X is locally contractable. Then every $Z(M(A))$-linear automorphism of A becomes inner when extended to $M_{\mathrm{loc}}(A)$. In particular, all elementary automorphisms on $A_n = C(PU(n), \mathbb{M}_n)$ become inner in $M_{\mathrm{loc}}(A_n)$.

Moreover, if the primitive spectrum of a C^*-algebra A is rather pathological, it can happen that A admits both outer derivations and outer automorphisms that are simultaneously elementary operators:

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Example

Let A be a C^*-subalgebra of $B = C([1,\infty], \mathbb{M}_2)$ consisting of all $a \in B$ such that

$$a(n) = \begin{bmatrix} \lambda_n(a) & 0 \\ 0 & \lambda_{n+1}(a) \end{bmatrix}, \quad n \in \mathbb{N},$$

for some convergent sequence $(\lambda_n(a))_{n\in\mathbb{N}}$ of complex numbers. Then A admits outer derivations and outer automorphisms that are also elementary operators. In fact, there are outer derivations of A of the form $\delta = M_{a,b} - M_{b,a}$ for suitable $a, b \in A$.

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Problem

Characterize the class of unital C^*-algebras A with the property that any derivation in $\overline{\mathcal{E}\ell(A)}_{cb}$ (or $\overline{\mathcal{E}\ell(A)}$) is necessarily inner.