CB-norm approximation of derivations by elementary operators

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(joint work in progress with Richard Timoney)

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- An ideal P of A is said to be **primitive** if P is the kernel of some irreducible representation of A.
- The **primitive spectrum** of A, which we denote by $\mathrm{Prim}(A)$, is the set of all primitive ideals of A equipped with the Jacobson topology. Hence, if S is some set of primitive ideals, its closure is

$$\overline{S} = \left\{ P \in \mathrm{Prim}(A) \ : \ P \supseteq \bigcap_{Q \in S} Q \right\}.$$

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  $\mathrm{Prim}(A)$ is a compact space which in general satisfies only T_0-separation axiom.
- **Dauns-Hofmann Theorem** (1968): There is a $*$-isomorphism Φ_A from $C(\mathrm{Prim}(A))$ onto $Z(A)$ such that

$$q_P(\Phi_A(f)a) = f(P)q_P(a)$$

  for all $f \in C(\mathrm{Prim}(A))$, $a \in A$ and $P \in \mathrm{Prim}(A)$.

- **Glimm ideals** of A are the ideals of A generated by the maximal ideals of $Z(A)$. By the Hewitt-Cohen Factorization Theorem, each Glimm ideal of A is of the form mA for some maximal ideal m of $Z(A)$. We denote the set of all Glimm ideals of A by $\mathrm{Glimm}(A)$.

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- Since the sum of two maximal ideals of $Z(A)$ contains the identity, it follows that the Glimm ideals of A are in one-to-one correspondence with the maximal ideals of $Z(A)$. Hence, we may equip $\mathrm{Glimm}(A)$ with the topology from the maximal ideal space of $Z(A)$ so that $\mathrm{Glimm}(A)$ becomes a compact Hausdorff space, homeomorphic to the maximal ideal space of $Z(A)$.

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- Each primitive ideal of A intersects $Z(A)$ in a maximal ideal, and therefore contains a (unique) Glimm ideal of A. In particular, Glimm ideals of A have zero intersection.
- For each $a \in A$ the norm-function $G \mapsto \|q_G(a)\|$ is upper semicontinuous on $\mathrm{Glimm}(A)$.

- An ideal Q of A is said to be *n*-**primal** $(n \geq 2)$ if whenever $I_1, \dots, I_n$ are ideals of A with $I_1 \cdots I_n = \{0\}$, then at least one I_i is contained in Q.

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- Also, one can show that an ideal Q of A is n-primal if and only if for all $P_1, \dots, P_n \in \mathrm{Prim}(A/Q)$ there exists a net in $\mathrm{Prim}(A)$ which converges simultaneously to each $P_1, \dots, P_n$.

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- In particular, $\mathrm{Prim}(A)$ is Hausdorff if and only if

$$\mathrm{Glimm}(A) = \mathrm{Primal}_2(A) \setminus \{A\} = \mathrm{Prim}(A).$$

- A linear map $\phi : A \to A$ is said to be **completely bounded** if

$$\|\phi\|_{cb} := \sup_{n \in \mathbb{N}} \|\phi_n\| < \infty,$$

where $\phi_n : \mathrm{M}_n(A) \to \mathrm{M}_n(A)$ denotes the induced map,

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- By $\mathrm{IB}(A)$ (resp. $\mathrm{ICB}(A)$) we denote the set of all bounded (resp. completely bounded) maps on A that preserve the ideals of A (i.e. $\phi(I) \subseteq I$ for all $I \in \mathrm{Id}(A)$).

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- Eery $\phi \in \mathrm{IB}(A)$ is $Z(A)$-(bi)modular. If S is any subset of $\mathrm{Id}(A)$ with zero intersection, then the norm of ϕ can be recovered via the formula

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where for each $I \in \mathrm{Id}(A)$, $\phi_I : A/I \to A/I$ denotes the induced map $q_I(a) \mapsto q_I(\phi(a))$.
- The analogues formula is valid for the cb-norm of maps in $\mathrm{ICB}(A)$.

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- A **state** of the system is defined as a positive functional on A (i.e. a linear map $\omega : A \to \mathbb{C}$ such that $\omega(a^*a) \geq 0$ for all $a \in A$) with $\omega(1_A) = 1$. If the system is in the state ω, then $\omega(a)$ is the expected value of the observable a.

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- Automorphisms correspond to the symmetries, while one-parameter automorphism groups $\{\Phi_t\}_{t \in \mathbb{R}}$ describe the reversible time evolution of the system (in the Heisenberg picture). Their infinitesimal generators

  $$\delta(x) := \lim_{t \to 0} \frac{1}{t}(\Phi_t(x) - x)$$

  are the $(*$-$)$**derivations**.

Definition

A **derivation** *of an algebra A is a linear map $\delta : A \to A$ satisfying the* **Leibniz rule**

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Each element $a \in A$ induces an **inner derivation** δ_a on A given by

$$\delta_a(x) := ax - xa \qquad (x \in A).$$

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Stampfli's formula, 1970

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- homogeneous C^*-algebras (Sproston, 1976).

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On the other hand, for inseparable C^*-algebras the main problem remains widely open, even for the simplest cases such as subhomogeneous C^*-algebras (i.e. C^*-algebras which have finite-dimensional irreducible representations of bounded degree).

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- We can therefore try to approximate a more general map on A, one that preserves ideals, by finite sums of two-sided multiplication maps, that is, by **elementary operators**.

By $\mathcal{E}\ell(A)$ we denote the set of all elementary operators on A. It is easy to see that every elementary operator on A is completely bounded, with the following estimate for its cb-norm:

$$\left\| \sum_i M_{a_i,b_i} \right\|_{cb} \le \left\| \sum_i a_i a_i^* \right\|^{\frac{1}{2}} \left\| \sum_i b_i^* b_i \right\|^{\frac{1}{2}}.$$

- Hence, if we endow the algebraic tensor product $A \otimes A$ with the Haagerup norm

$$\|t\|_h := \inf \left\{ \left\|\sum_i a_i a_i^*\right\|^{\frac{1}{2}} \left\|\sum_i b_i^* b_i\right\|^{\frac{1}{2}} \ : \ t = \sum_i a_i \otimes b_i \right\},$$

we obtain a well-defined contraction

$$(A \otimes A, \|\cdot\|_h) \to (\mathcal{E}\ell(A), \|\cdot\|_{cb}),$$

given by

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- Its continuous extension to the completed Haagerup tensor product $A \otimes_h A$ is known as a **canonical contraction** from $A \otimes_h A$ to $\mathrm{ICB}(A)$ and is denoted by θ_A.

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$$\sum_i a_i \otimes b_i \mapsto \sum_i M_{a_i, b_i}.$$

- Its continuous extension to the completed Haagerup tensor product $A \otimes_h A$ is known as a **canonical contraction** from $A \otimes_h A$ to $\mathrm{ICB}(A)$ and is denoted by θ_A.
- If A contains a pair of non-zero orthogonal ideals, then θ_A cannot be injective. Hence, the necessary condition for the injectivity of θ_A is that A must be a prime C^*-algebra.

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Theorem (Mathieu, 2003)

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A is prime $\iff$ θ_A *is injective* $\iff$ θ_A *is isometric.*

This result was first proved by Haagerup in 1980 when $A = \mathrm{B}(\mathcal{H})$. Chatterjee and Sinclair in 1992 showed that θ_A is isometric if A is a separably-acting von Neumann factor. Finally, Mathieu extended this result to all prime C^*-algebras.

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If A is a general C^*-algebra, then using the Mathieu's theorem we obtain the following formula for the cb-norm of $\theta_A(t)$:

$$\|\theta_A(t)\|_{cb} = \sup\{\|t^P\|_h \ : \ P \in \mathrm{Prim}(A)\},$$

where for each $I \in \mathrm{Id}(A)$ by t^I we denote the quotient image of t in $(A \otimes_h A)/(I \otimes_h A + A \otimes_h I)$, which is isometrically isomorphic to $(A/I) \otimes_h (A/I)$ (a result due to Allen, Sinclair and Smith), so that $\|t^I\| = \|(q_I \otimes q_I)(t)\|_h$.

If A has a non-trivial centre, one can consider the closed ideal J_A of $A \otimes_h A$ generated by the tensors of the form $az \otimes b - a \otimes zb$ ($a, b \in A, z \in Z(A)$) (note that $J_A \subseteq \ker \theta_A$), the induced contraction $\theta_A^Z : (A \otimes_h A)/J_A \to \mathrm{ICB}(A)$, and ask when is θ_A^Z is injective or isometric.

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- A further generalization was obtained by Somerset in 1998:

Theorem (Somerset, 1998)

(a)

$$\|\theta_A(t)\|_{cb} = \sup\{\|t^Q\|_h \ : \ Q \in \mathrm{Primal}(A)\}.$$

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(c) $Q \in \mathrm{Id}(A)$ *is 2-primal if and only if* $\ker \theta_A \subseteq Q \otimes_h A + A \otimes_h Q$*, so*

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In particular, θ_A^Z *is isometric if every Glimm ideal of A is primal and* θ_A^Z *is injective if and only if every Glimm ideal of A is 2-primal.*

Finally, Archbold, Somerset and Timoney proved in 2005 that the primality condition of Glimm ideals of A is also a necessary one for θ_A^Z being isometric. In particular, the isometry problem of θ_A^Z was completely solved in terms of the ideal structure of A:

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Which derivations of a C^*-algebra A admit a completely bounded approximation by elementary operators? That is, which derivations of A lie in the cb-norm closure $\overline{\mathcal{E}\ell(A)}^{cb}$?

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- Since each inner derivation is an elementary operator (of length 2) on A, $\overline{\mathcal{E}\ell(A)}^{cb}$ includes the cb-corm closure of $\mathrm{Inn}(A)$.

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- Since each inner derivation is an elementary operator (of length 2) on A, $\overline{\mathcal{E}\ell(A)}^{cb}$ includes the cb-corm closure of $\mathrm{Inn}(A)$.
- Since the cb-norm of (inner) derivations coincides with their operator norm, the cb-norm closure of $\mathrm{Inn}(A)$ coincides with the operator norm closure of $\mathrm{Inn}(A)$. We denote this closure by $\overline{\mathrm{Inn}(A)}$.

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In fact, we have the following beautiful characterization:

Theorem (Somerset, 1993)

The set $\mathrm{Inn}(A)$ is closed in the operator norm, as a subset of $\mathrm{Der}(A)$, if and only if A has a finite connecting order.

Connecting order of a C^*-algebra

The connecting order of a C^*-algebra is a constant in $\mathbb{N} \cup \{\infty\}$ arising from a certain graph structure on the primitive spectrum $\mathrm{Prim}(A)$:

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- The **distance** $d(P, Q)$ from P to Q is defined as follows:
  - ▷ $d(P, P) := 1$.
  - ▷ If $P \neq Q$ and there exists a path from P to Q, then $d(P, Q)$ is equal to the minimal length of a path from P to Q.
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  - If there is no path from P to Q, $d(P, Q) := \infty$.
- The **connecting order** $\mathrm{Orc}(A)$ of A is then defined by

$$\mathrm{Orc}(A) := \sup\{d(P, Q) : \ P, Q \in \mathrm{Prim}(A) \text{ such that } d(P, Q) < \infty\}.$$

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The equality $\mathrm{Der}(A) \cap \overline{\mathcal{E}\ell(A)}^{cb} = \overline{\mathrm{Inn}(A)}$ *holds true for all* C^**-algebras* A *in which every Glimm ideal is prime.*

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- Using Somerset's Theorem from 1998, θ_A is isometric in our case, so

$$\overline{\overline{\mathcal{E}\ell(A)}}^{cb} = \mathrm{Im}\,\theta_A.$$

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- First assume that A is prime. In this case, we can use Mathieu's Theorem to identify $\mathrm{Im}\,\theta_A$ with $A \otimes_h A$ and then work inside $A \otimes_h A$. Using the Leibniz rule, appropriate decompositions of the tensors (due to R. Smith) and the partition of unity argument, it is not difficult to see that δ is inner in this (prime) case.

Proof (continuation)

- The next step is to show that the norm function $G \mapsto \|\delta_G\|$ is upper semicontinuous on $\mathrm{Glimm}(A)$. To do this, we first fix some $G \in \mathrm{Glimm}(A)$. It is easy to see that the following diagram

$$\begin{array}{ccc} A \otimes_h A & \xrightarrow{\theta_A} & \mathrm{ICB}(A) \\ {\scriptstyle q_G \otimes q_G}\downarrow & & {\scriptstyle Q_G}\downarrow \\ (A/G) \otimes_h (A/G) & \xrightarrow{\theta_{A/G}} & \mathrm{ICB}(A/G) \end{array}$$

commutes, where $Q_G : \mathrm{ICB}(A) \to \mathrm{ICB}(A/G)$ is a map given by $Q_G(\phi)(q_G(a)) = q_G(\phi(a))$ ($\phi \in \mathrm{ICB}(A)$, $a \in A$), so that

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$$\begin{aligned} \|\delta_G\| &= \|\delta_G\|_{cb} = \|\theta_{A/G}((q_G \otimes q_G)(t))\|_{cb} = \|(q_G \otimes q_G)(t)\|_h \\ &= \|t^G\|_h. \end{aligned}$$

Here we used again the Mathieu's Theorem (A/G is prime by assumption).

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- Using the fact that the norm functions $G \mapsto \|q_G(a)\|$ $(a \in A)$ are upper semicontinuous on $\mathrm{Glimm}(A)$, one can now show that the map $G \mapsto \|t^G\|_h$ is also upper semicontinuous on $\mathrm{Glimm}(A)$.

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- The next step is to show that δ can be approximated in the (cb-)norm by inner derivations. Indeed, let $\varepsilon > 0$. Since each Glimm quotient A/G is prime, by the first part of the proof, the upper semicontinuity of the norm function $G \mapsto \|\delta_G\| = \|t^G\|_h$ and a simple compactness argument, we obtain a finite number of elements $\{a_i\}$ and a finite open cover $\{U_i\}$ of $\mathrm{Glimm}(A)$ such that $\|(\delta_G - (\delta_{a_i})_G\| < \varepsilon$ for all $G \in U_i$. Choose a partition of unity $\{f_i\}$ of $\mathrm{Glimm}(A)$ subordinated to the cover $\{U_i\}$ and define $a := \sum_i f_i a_i \in A$ (here we used the identification $C(\mathrm{Glimm}(A)) = Z(A)$). Using the fact that Glimm ideals have zero intersection, it is easy to verify that $\|\delta - \delta_a\| < \varepsilon$.

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- By the Somerset's Theorem from 1993, $\mathrm{Inn}(A)$ is (cb-)closed in our case (since $\mathrm{Orc}(A) = 1$), which completes the proof.

- Unfortunately, the presented proof cannot be generalized for some larger reasonable class of C^*-algebras (e.g. for those in which every Glimm ideal is primal). There are two main obstacles in the proof: The first one is that we do not know whether each Glimm quotient A/G admits only inner derivations lying in $\mathrm{Im}\,\theta_{A/G}$. The second one is that for $\delta \in \mathrm{Der}(A) \cap \overline{\overline{\mathcal{E}\ell(A)}}^{cb}$, the function $G \mapsto \|\delta_G\|$ does not need be upper semicontinuous on Glimm(A), even if δ is already inner.

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- Indeed, let A be a C^*-algebra consisting of all functions $a \in C([0,1], \mathrm{M}_2(\mathbb{C}))$ such that $a(1)$ is a diagonal matrix. Then Glimm(A) is canonically homeomorphic to $[0,1]$ and let us denote this correspondence by $x \leftrightarrow G(x)$. Further, each Glimm ideal of A is primal. On the other hand, let a be an element of A defined by $a(x) := e_{1,1}$ for all $x \in [0,1]$ (where $e_{1,1}$ is the matrix unit which has a non-zero entry 1 at $(1,1)$-position) and let $\delta := \delta_a$. By Stampfli's formula we have $\|\delta_{G(x)}\| = 1$ for all $0 \leq x < 1$ and $\|\delta_{G(1)}\| = 0$ (since $A/G(1) \cong \mathbb{C} \oplus \mathbb{C}$). Therefore, the function $G \mapsto \|\delta_G\|$ is not upper semicontinuous on $[0,1] = \mathrm{Glimm}(A)$.

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Motivated by our previous discussion, it is natural to start looking for possible examples in the class of C^*-algebras with $\mathrm{Orc}(A) = \infty$.

Example (G., 2010)

Let A be a C^*-algebra consisting of all elements $a \in C([0,\infty], \mathrm{M}_2(\mathbb{C}))$ such that

$$a(n) = \begin{bmatrix} \lambda_n(a) & 0 \\ 0 & \lambda_{n+1}(a) \end{bmatrix} \quad (n \in \mathbb{N}),$$

for some convergent sequence $(\lambda_n(a))$ of complex numbers. Then:

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for some convergent sequence $(\lambda_n(a))$ of complex numbers. Then:

- $d(\ker \lambda_1, \ker \lambda_n) = n$ for all $n \in \mathbb{N}$. In particular, $\mathrm{Orc}(A) = \infty$.

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for some convergent sequence $(\lambda_n(a))$ of complex numbers. Then:

- $d(\ker \lambda_1, \ker \lambda_n) = n$ for all $n \in \mathbb{N}$. In particular, $\mathrm{Orc}(A) = \infty$.
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Let A be a C^*-algebra consisting of all elements $a \in C([0,\infty], \mathrm{M}_2(\mathbb{C}))$ such that

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More recently, R. Timoney showed that the above C^*-algebra A admits outer derivations δ of the form $\delta = M_{a,b} - M_{b,a}$ for some $a, b \in A$. In particular A has outer elementary derivations of length 2. Further, this C^*-algebra satisfies $\overline{\overline{\mathrm{Inn}(A)}} = \mathrm{Der}(A) \cap \mathcal{E}\ell(A)$.

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