On automorphisms, derivations and elementary operators of C^*-algebras

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Derivation *of A is a linear map $d : A \to A$ satisfying the Leibniz rule*

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If there exists a multiplier $a \in M(A)$ such that $d(x) = ax - xa$ for all $x \in A$, d is said to be an **inner derivation**. Otherwise, d is said to be an **outer derivation**.

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For separable C^*-algebras the problem was completely solved in 1979:

Theorem (Akemann, Elliott, Pedersen and Tomiyama 1979)

Let A be a separable C^*-algebra. TFAE:

(i) *A admits only inner derivations.*

(ii) *$A = A_1 \oplus A_2$, where A_1 is a continuous-trace C^*-algebra, and A_2 is a direct sum of simple C^*-algebras.*

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For inseparable C^*-algebras the problem of innerness of derivations remains widely open, even for the simplest cases such as subhomogeneous C^*-algebras.

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If $A = C_0(X)$ is a commutative C^*-algebra, then $M_{\mathrm{loc}}(A)$ is a commutative AW^*-algebra whose maximal ideal space can be identified with the inverse limit $\varprojlim \beta U$ of Stone-Čech compactifications βU of dense open subsets U of X.

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Theorem (Pedersen 1978)

Every derivation of a C^*-algebra A extends uniquely and under preservation of the norm to a derivation of $M_{\mathrm{loc}}(A)$. Moreover, if A is separable (or more generally, if every essential closed ideal of A is σ-unital), this extension becomes inner in $M_{\mathrm{loc}}(A)$.

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In particular, Pedersen's result implies Sakai's theorem that every derivation of a simple unital C^*-algebra is inner.

Since $M_{\mathrm{loc}}(A) = M(A)$ if A is simple, and $M_{\mathrm{loc}}(A) = A$ if A is an AW^*-algebra, an affirmative answer in the inseparable case would cover, extend and unify the results that all derivations of simple C^*-algebras and AW^*-algebras are inner.

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- quasi-central separable C^*-algebras such that $\mathrm{Prim}(A)$ contains a dense G_δ subset consisting of closed points (Somerset 2000, Ara-Mathieu 2011);
- C^*-algebras with finite-dimensional irreducible representations; in this case $M_{\mathrm{loc}}(A)$ coincides with the injective envelope of A (G. 2013).

The cb-norm approximation by elementary operators

Let A be a C^*-algebra. An attractive and fairly large class of bounded linear maps $\phi : A \to A$ that preserve all ideals of A is the class of **elementary operators**, that is, those that can be expressed as a finite sum

$$\phi = \sum_i M_{a_i,b_i}$$

of **two-sided multiplications** $M_{a_i,b_i} : x \mapsto a_i x b_i$, where $a_i, b_i \in M(A)$.

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$$\|\phi\|_{cb} := \sup_{n\in\mathbb{N}} \|\phi_n\| < \infty,$$

where for each n, ϕ_n is an induced map on $M_n(A)$, i.e.

$$\phi_n([a_{ij}]) = [\phi(a_{ij})].$$

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Let us denote by $\mathcal{E}\ell(A)$ the set of all elementary operators on A and by $\overline{\mathcal{E}\ell(A)}_{cb}$ its cb-norm closure.

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Theorem (G. 2013)

If A is a unital C^*-algebra whose every Glimm ideal is prime, then a derivation d of A lies in $\overline{\mathcal{E}\ell(A)}_{cb}$ if and only if d is an inner derivation.

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The **Glimm ideals** of A are the ideals of A generated by the maximal ideals of $Z(A)$.

Example

The class of C^*-algebras whose every Glimm ideal is prime includes:

- prime C^*-algebras;
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Corollary

The Pederesen's problem has a positive solution if and only if for each C^*-algebra A, every derivation of $M_{\mathrm{loc}}(A)$ lies in $\overline{\overline{\mathcal{E}\ell(M_{\mathrm{loc}}(A))}}_{cb}$.

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For prime C^*-algebras we also established the following result:

Theorem (G. 2019)

If A is a prime C^*-algebra then an (algebra) epimorphism $\sigma : A \to A$ lies in $\overline{\mathcal{E}\ell(A)}_{cb}$ if and only if σ is an (algebra) inner automorphism of A.

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On the other hand:

Proposition

Let A be a separable n-homogeneous C^*-algebra whose primitive spectrum X is locally contractable. Then every $Z(M(A))$-linear automorphism of A becomes inner when extended to $M_{\mathrm{loc}}(A)$.

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In particular, all (outer) elementary automorphisms of $A_n = C(PU(n), \mathbb{M}_n)$ become inner in $M_{\mathrm{loc}}(A_n)$.

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Example

Let A be a C^*-subalgebra of $B = C([1, \infty], \mathbb{M}_2)$ that consists of all $a \in B$ such that If

$$a(n) = \begin{bmatrix} \lambda_n(a) & 0 \\ 0 & \lambda_{n+1}(a) \end{bmatrix} \qquad (n \in \mathbb{N})$$

for some convergent sequence $(\lambda_n(a))$ of complex numbers. Then A admits outer derivations and outer automorphisms that are also elementary operators. In fact, there are outer derivations of A of the form $d = M_{a,b} - M_{b,a}$ for suitable $a, b \in A$.

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Problem

Does every automorphism of a C^*-algebra A that is also an elementary operator become inner when extended to $M_{\mathrm{loc}}(A)$?