CB-norm approximation of derivations by elementary operators

Ilja Gogić

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(joint work in progress with Richard M. Timoney)
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**Definition**

A **derivation** on $A$ is a linear map $\delta : A \to A$ satisfying the **Leibniz rule**

$$\delta(xy) = \delta(x)y + x\delta(y) \quad \text{for all } x, y \in A.$$
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- They are completely bounded and their cb-norm coincides with their operator norm (i.e. $\|\delta\|_{cb} = \|\delta\|$).
- They preserve the (closed two-sided) ideals of $A$ (i.e. $\delta(I) \subseteq I$ for all ideals $I$ of $A$).
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- They preserve the (closed two-sided) ideals of $A$ (i.e. $\delta(I) \subseteq I$ for all ideals $I$ of $A$).
- They annihilate the centre of an underlying algebra. In particular, commutative $C^\ast$-algebras don’t admit non-zero derivations.
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Each element $a \in A$ induces an **inner derivation** $\delta_a$ on $A$ given by

$$\delta_a(x) := ax - xa \quad (x \in A).$$
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Some classes of $C^*$-algebras which admit only inner derivations:

- von Neumann algebras (Kadison-Sakai 1966)
- simple $C^*$-algebras (Sakai 1968)
- $AW^*$-algebras (Olesen 1974)
- homogeneous $C^*$-algebras (Sproston 1976)
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In fact, for separable $C^*$-algebras the above problem was completely solved back in 1979:

**Theorem (Akemann, Elliott, Pedersen and Tomiyama 1979)**

*For a separable $C^*$-algebra $A$ the following conditions are equivalent:*

- $A$ admits only inner derivations.
- $A$ is a direct sum of a finite number of $C^*$-subalgebras which are either homogeneous or simple;
- $\text{Der}(A)$ is separable in the operator norm.

On the other hand, for inseparable $C^*$-algebras the main problem remains widely open, even for the simplest cases such as subhomogeneous $C^*$-algebras (i.e. $C^*$-algebras which have finite-dimensional irreducible representations of bounded degree).
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Motivation

We often try to understand the structure of operators and spaces on which they act in terms of approximation by finite rank maps. On $C^*$-algebras, however, it is natural to regard two-sided multiplication maps $M_{a,b}: x \mapsto axb$ ($a, b \in A$) as basic building blocks (instead of rank one operators). We can therefore try to approximate a more general map on $A$, one that preserves ideals, by finite sums of two-sided multiplication maps, that is, by elementary operators.

By $\mathcal{E}_\ell(A)$ we denote the set of all elementary operators on $A$. It is easy to see that every elementary operator on $A$ is completely bounded, with

$$\|\sum_i M_{a_i,b_i}\|_h \leq \|\sum_i a_i \otimes b_i\|_h,$$

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Since the derivations of $C^*$-algebras preserve the ideals and are completely bounded, the approximation procedure by elementary operators in particular applies to derivations:

Which derivations of a $C^*$-algebra $A$ admit a completely bounded approximation by elementary operators? That is, which derivations of $A$ lie in the $\text{cb}$-norm closure $E_\ell(A)^{\text{cb}}$?

Let us denote $\text{Der}(A)$ and $\text{Inn}(A)$, respectively, the set of all derivations and the set of all inner derivations of $A$. Since each inner derivation is an elementary operator (of length 2) on $A$, $E_\ell(A)^{\text{cb}}$ includes the $\text{cb}$-norm closure of $\text{Inn}(A)$. Since the $\text{cb}$-norm of (inner) derivations coincides with their operator norm, the $\text{cb}$-norm closure of $\text{Inn}(A)$ coincides with the operator norm closure of $\text{Inn}(A)$. We denote this closure by $\text{Inn}(A)$. 
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- Since each inner derivation is an elementary operator (of length 2) on $A$, $\overline{\mathcal{E}\ell(A)}^{cb}$ includes the cb-corm closure of $\text{Inn}(A)$.
- Since the cb-norm of (inner) derivations coincides with their operator norm, the cb-norm closure of $\text{Inn}(A)$ coincides with the operator norm closure of $\text{Inn}(A)$. We denote this closure by $\overline{\text{Inn}(A)}$. 
Problem (G. 2013)

Does every $C^*$-algebra satisfy the condition

$$\text{Der}(A) \cap \overline{\mathcal{E}\ell(A)}^{cb} = \overline{\text{Inn}(A)}?$$

In many cases the set $\overline{\text{Inn}(A)}$ is already closed in the operator norm. However, this is not always true. In fact, we have the following beautiful characterization:

Theorem (Somerset 1993)

The set $\overline{\text{Inn}(A)}$ is closed in the operator norm, as a subset of $\text{Der}(A)$, if and only if $A$ has a finite connecting order.
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Connecting order of a $C^*$-algebra

The connecting order of a $C^*$-algebra is a constant in $\mathbb{N} \cup \{\infty\}$ arising from a certain graph structure on the primitive spectrum $\text{Prim}(A)$:

Two primitive ideals $P, Q$ of $A$ are said to be adjacent, if $P$ and $Q$ cannot be separated by disjoint open subsets of $\text{Prim}(A)$.

A path of length $n$ from $P$ to $Q$ is a sequence of points $P = P_0, P_1, \ldots, P_n = Q$ such that $P_{i-1}$ is adjacent to $P_i$ for all $1 \leq i \leq n$.

The distance $d(P, Q)$ from $P$ to $Q$ is defined as follows:

$\triangleq d(P, P) := 1$.

If $P \neq Q$ and there exists a path from $P$ to $Q$, then $d(P, Q)$ is equal to the minimal length of a path from $P$ to $Q$.

If there is no path from $P$ to $Q$, $d(P, Q) := \infty$.

The connecting order $\text{Orc}(A)$ of $A$ is then defined by $\text{Orc}(A) := \sup\{d(P, Q) : P, Q \in \text{Prim}(A) \text{ such that } d(P, Q) < \infty\}$.
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Connecting order of a C*-algebra

The connecting order of a C*-algebra is a constant in \( \mathbb{N} \cup \{\infty\} \) arising from a certain graph structure on the primitive spectrum \( \text{Prim}(A) \):

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- A **path** of length \( n \) from \( P \) to \( Q \) is a sequence of points \( P = P_0, P_1, \ldots, P_n = Q \) such that \( P_{i-1} \) is adjacent to \( P_i \) for all \( 1 \leq i \leq n \).
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- The **connecting order** \( \text{Orc}(A) \) of \( A \) is then defined by

\[
\text{Orc}(A) := \sup\{d(P, Q) : P, Q \in \text{Prim}(A) \text{ such that } d(P, Q) < \infty\}.
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Theorem (G. 2013)

The equality $\text{Der}(A) \cap \mathcal{E}_\ell(A)^{cb} = \text{Inn}(A)$ holds true for all $C^*$-algebras $A$ in which every Glimm ideal is prime.
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**Glimm ideals**

Recall that the **Glimm ideals** of a $C^*$-algebra $A$ are the ideals generated by the maximal ideals of the centre of $A$. 
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Glimm ideals

Recall that the **Glimm ideals** of a $C^*$-algebra $A$ are the ideals generated by the maximal ideals of the centre of $A$.

If a $C^*$-algebra $A$ has only prime Glimm ideals, then $Orc(A) = 1$, so Somerset’s theorem yields that $\text{Inn}(A)$ is closed in the operator norm. Hence:

Corollary

If every Glimm ideal of a $C^*$-algebra $A$ is prime, then every derivation of $A$ which lies in $\mathcal{E}\ell(A)^{cb}$ is inner.
Example

The class of $C^*$-algebras in which every Glimm ideal is prime is fairly large. It includes:

- Prime $C^*$-algebras
- $C^*$-algebras with Hausdorff primitive spectrum
- Quotients of $AW^*$-algebras
- Local multiplier algebras

By elementary derivation on $A$ we mean every derivation on $A$ which is also an elementary operator on $A$.

Question

Does there exist a $C^*$-algebra $A$ which admits an outer elementary derivation?

Motivated by the previous discussion, it is natural to start looking for possible examples in the class of $C^*$-algebras with $\text{Orc}(A) = \infty$. 
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Ilja Gogić (TCD)  Derivations and elem. operators  GPOTS 2015, May 26, 2015  10 / 12
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Motivated by the previous discussion, it is natural to start looking for possible examples in the class of $C^*$-algebras with $Orc(A) = \infty$. 
Example (G. 2010, G. & Timoney 2015)

Let $A$ be a $C^*$-algebra consisting of all elements $a \in C([0, \infty], M_2(\mathbb{C}))$ such that

$$a(n) = \begin{bmatrix} \lambda_n(a) & 0 \\ 0 & \lambda_{n+1}(a) \end{bmatrix} \quad (n \in \mathbb{N}),$$

for some convergent sequence $(\lambda_n(a))$ of complex numbers. Then:

\[ \text{d}(\ker \lambda_1, \ker \lambda_n) = n \quad \text{for all} \quad n \in \mathbb{N}. \]

In particular, $\mathcal{O}(A) = \infty$. $\text{E}_\lambda(A)$ is closed in the cb-norm. In particular, $A$ admits outer elementary derivations.

More recently, we showed that $A$ admits outer derivations of the form $M_a, b - M_b, a$ for some $a, b \in A$. In particular $A$ has outer elementary derivations of length 2. Also, $A$ satisfies $\text{Inn}(A) = \text{Der}(A) \cap \text{E}_\lambda(A)$.

Proposition (G. & Timoney 2015)

Every elementary derivation of length 2 on a $C^*$-algebra $A$ is of the form $M_a, b - M_b, a$ for some $a, b \in A$. 

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- $d(\ker \lambda_1, \ker \lambda_n) = n$ for all $n \in \mathbb{N}$. In particular, $\text{Orc}(A) = \infty$. 

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- $d(\ker \lambda_1, \ker \lambda_n) = n$ for all $n \in \mathbb{N}$. In particular, $\text{Orc}(A) = \infty$.
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for some convergent sequence $(\lambda_n(a))$ of complex numbers. Then:

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- $\mathcal{E}\ell(A)$ is closed in the cb-norm.

In particular, $A$ admits outer elementary derivations.
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Proposition (G. & Timoney 2015)

Every elementary derivation of length 2 on a $C^*$-algebra $A$ is of the form $M_{a,b} - M_{b,a}$ for some $a, b \in A$. 
I end this lecture with some problems of current interest:

**Problem**

What can be said about the lengths of outer elementary derivations? In particular, is it possible for each $n \geq 2$ find a $C^*$-algebra $A$ which admits an (outer) elementary derivation of length $n$?
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When is the set of all elementary operators of the form $M_{a,b} - M_{b,a}$ ($a, b \in A$) closed in the operator ($\sim cb$-)norm?