CB-norm approximation of derivations by elementary operators

Ilja Gogić

TRINITY COLLEGE DUBLIN
COLÁISTE NA TRÍONÓIDE, BAILE ÁTHA CLIATH
THE UNIVERSITY OF DUBLIN

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(joint work in progress with Richard M. Timoney)

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- They preserve the (closed two-sided) ideals of A (i.e. $\delta(I) \subseteq I$ for all ideals I of A).
- They annihilate the centre of an underlying algebra. In particular, commutative C^*-algebras don't admit non-zero derivations.

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- homogeneous C^*-algebras (Sproston 1976)

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On the other hand, for inseparable C^*-algebras the main problem remains widely open, even for the simplest cases such as subhomogeneous C^*-algebras (i.e. C^*-algebras which have finite-dimensional irreducible representations of bounded degree).

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By $\mathcal{E}\ell(A)$ we denote the set of all elementary operators on A. It is easy to see that every elementary operator on A is completely bounded, with

$$\left\| \sum_i M_{a_i,b_i} \right\|_{cb} \leq \left\| \sum_i a_i \otimes b_i \right\|_h ,$$

where $\| \cdot \|_h$ is the Haagerup tensor norm on $A \otimes A$.

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- Since each inner derivation is an elementary operator (of length 2) on A, $\overline{\mathcal{E}\ell(A)}^{cb}$ includes the cb-corm closure of $\mathrm{Inn}(A)$.

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- Since each inner derivation is an elementary operator (of length 2) on A, $\overline{\mathcal{E}\ell(A)}^{cb}$ includes the cb-corm closure of $\mathrm{Inn}(A)$.
- Since the cb-norm of (inner) derivations coincides with their operator norm, the cb-norm closure of $\mathrm{Inn}(A)$ coincides with the operator norm closure of $\mathrm{Inn}(A)$. We denote this closure by $\overline{\mathrm{Inn}(A)}$.

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In fact, we have the following beautiful characterization:

Theorem (Somerset 1993)

The set $\mathrm{Inn}(A)$ is closed in the operator norm, as a subset of $\mathrm{Der}(A)$, if and only if A has a finite connecting order.

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- The **distance** $d(P, Q)$ from P to Q is defined as follows:
  - ▷ $d(P, P) := 1$.
  - ▷ If $P \neq Q$ and there exists a path from P to Q, then $d(P, Q)$ is equal to the minimal length of a path from P to Q.
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  - ▷ If there is no path from P to Q, $d(P, Q) := \infty$.
- The **connecting order** $\mathrm{Orc}(A)$ of A is then defined by

$$\mathrm{Orc}(A) := \sup\{d(P, Q) : \ P, Q \in \mathrm{Prim}(A) \text{ such that } d(P, Q) < \infty\}.$$

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The equality $\mathrm{Der}(A) \cap \overline{\overline{\mathcal{E}\ell(A)}}^{cb} = \overline{\overline{\mathrm{Inn}(A)}}$ *holds true for all* C^**-algebras* A *in which every Glimm ideal is prime.*

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Recall that the **Glimm ideals** of a C^*-algebra A are the ideals generated by the maximal ideals of the centre of A.

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If a C^*-algebra A has only prime Glimm ideals, then $\mathrm{Orc}(A) = 1$, so Somerset's theorem yields that $\mathrm{Inn}(A)$ is closed in the operator norm. Hence:

Corollary

If every Glimm ideal of a C^**-algebra* A *is prime, then every derivation of* A *which lies in* $\overline{\overline{\mathcal{E}\ell(A)}}^{cb}$ *is inner.*

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Does there exist a C^*-algebra A which admits an outer elementary derivation?

Motivated by the previous discussion, it is natural to start looking for possible examples in the class of C^*-algebras with $\mathrm{Orc}(A) = \infty$.

Example (G. 2010, G. & Timoney 2015)

Let A be a C^*-algebra consisting of all elements $a \in C([0,\infty], \mathrm{M}_2(\mathbb{C}))$ such that

$$a(n) = \begin{bmatrix} \lambda_n(a) & 0 \\ 0 & \lambda_{n+1}(a) \end{bmatrix} \quad (n \in \mathbb{N}),$$

for some convergent sequence $(\lambda_n(a))$ of complex numbers. Then:

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Prposition (G. & Timoney 2015)

Every elementary derivation of length 2 on a C^*-algebra A is of the form $M_{a,b} - M_{b,a}$ for some $a, b \in A$.

I end this lecture with some problems of current interest:

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When is the set of all elementary operators of the form $M_{a,b} - M_{b,a}$ $(a, b \in A)$ closed in the operator ($\sim$ cb-)norm?