Derivations and local multipliers of C^*-algebras

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Definition

A **derivation** *of an algebra A is a linear map $\delta : A \to A$ satisfying the* **Leibniz rule**

$$\delta(xy) = \delta(x)y + x\delta(y) \qquad \text{for all } x, y \in A.$$

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- δ extends uniquely and under preservation of the norm to a derivation of $M(A)$ (the multiplier algebra of A).

If A is a C^*-subalgebra of a C^*-algebra B, then each element $a \in B$ which derives A (i.e. $ax - xa \in A$, for all $x \in A$) implements a derivation $\delta_a : A \to A$ given by

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- homogeneous C^*-algebras (Sproston, 1976; G., 2013).
- σ-unital continuous-trace C^*-algebras (Akemann-Elliott-Pedersen-Tomiyama, 1976).

Moreover, the separable case was completely solved in 1979:

Theorem (Akemann, Elliott, Pedersen and Tomiyama, 1979)

Let A be a separable C^*-algebra, Then A admits only inner derivations if and only if $A = A_1 \oplus A_2$, where A_1 is a continuous-trace C^*-algebra, and A_2 is a direct sum of simple C^*-algebras.

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On the other hand, for inseparable C^*-algebras the problem of innerness of derivations remains widely open, even for the simplest cases such as subhomogeneous C^*-algebras (i.e. C^*-algebras which have finite-dimensional irreducible representations of bounded degree).

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Iterating the construction of $M_{\mathrm{loc}}(A)$, one obtains the following tower of C^*-algebras which, a priori, does not have the largest element:

$$A \subseteq M_{\mathrm{loc}}(A) \subseteq M_{\mathrm{loc}}^{(2)}(A) \subseteq \cdots \subseteq M_{\mathrm{loc}}^{(n)}(A) \subseteq \cdots ,$$

where $M_{\mathrm{loc}}^{(2)}(A) = M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$, $M_{\mathrm{loc}}^{(3)}(A) = M_{\mathrm{loc}}(M_{\mathrm{loc}}^{(2)}(A))$, etc.

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If $A = C_0(X)$ is a commutative C^*-algebra, then $M_{\mathrm{loc}}(A)$ is a commutative AW^*-algebra whose maximal ideal space can be identified with the inverse limit $\varprojlim \beta U$ of Stone-Čech compactifications βU of dense open subsets U of X.

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Every derivation δ of a separable C^*-algebra A is implemented by a local multiplier (i.e. δ becomes inner when extended to a derivation of $M_{\mathrm{loc}}(A)$).

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Every derivation δ of a separable C^*-algebra A is implemented by a local multiplier (i.e. δ becomes inner when extended to a derivation of $M_{\mathrm{loc}}(A)$).

Moreover, it suffices to assume that every essential closed ideal of A is σ-unital. In particular, Pedersen's result entails Sakai's theorem that every derivation of a simple unital C^*-algebra is inner.

Since $M_{\mathrm{loc}}(A) = M(A)$ if A is simple, and $M_{\mathrm{loc}}(A) = A$ if A is an AW^*-algebra, only an affirmative answer in the inseparable case would cover, extend and unify the results that all derivations of simple C^*-algebras and AW^*-algebras are inner.

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Problem of innerness of derivations of $M_{\mathrm{loc}}(A)$

If A is an arbitrary C^*-algebra, is every derivation of $M_{\mathrm{loc}}(A)$ inner?

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Problem of innerness of derivations of $M_{\mathrm{loc}}(A)$

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Stability problem of $M_{\mathrm{loc}}(A)$

Is $M_{\mathrm{loc}}^{(2)}(A) = M_{\mathrm{loc}}(A)$ for every C^*-algebra A?

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$I(A)$ is not an injective object in the category of C^*-algebras and $*$-homomorphisms, but in the category of operator spaces and complete positive maps, i.e. for every inclusion $E \subseteq F$ of operator systems, each completely positive map $\phi : E \to I(A)$ has a completely positive extension $\widetilde{\phi} : F \to I(A)$.

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However, it turns out that (nevertheless) $I(A)$ is a C^*-algebra canonically containing A as a C^*-subalgebra. Moreover, $I(A)$ is monotone complete, so in particular, $I(A)$ is an AW^*-algebra.

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Theorem (Hamana, 1981)

All AW^*-algebras of type I are injective.

Theorem (Frank and Paulsen, 2003)

Under this embedding of A into $I(A)$, $M_{\mathrm{loc}}(A)$ is the norm closure of the set of all $x \in I(A)$ which act as a multiplier on some $I \in \mathrm{Id}_{ess}(A)$, i.e.

$$M_{\mathrm{loc}}(A) = \left(\bigcup_{I \in \mathrm{Id}_{ess}(A)} \{x \in I(A) \ : xI + Ix \subseteq I\} \right)^{=}$$

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Using this result and the fact that $I(M_{\mathrm{loc}}(A)) = I(A)$, we obtain the following sequence of inclusions of C^*-algebras:

$$A \subseteq M_{\mathrm{loc}}(A) \subseteq M_{\mathrm{loc}}^{(2)}(A) \subseteq \cdots \subseteq \overline{A} \subseteq I(A).$$

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Difficult problem

When is $M_{\mathrm{loc}}(A) = I(A)$, or at least $M_{\mathrm{loc}}(A) = \overline{A}$?

Back to Pedersen's questions, we have the following partial answers:

Theorem (Somerset, 2000; Ara and Mathieu, 2011)

If A is a unital (or more generally quasi-central), separable C^*-algebra such that $\mathrm{Prim}(A)$(= the primitive ideal space of A) contains a dense G_δ subset of closed points, then $M_{\mathrm{loc}}^{(2)}(A) = M_{\mathrm{loc}}(A)$. Moreover, in this case $M_{\mathrm{loc}}(A)$ has only inner derivations.

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Theorem (G., 2013)

If all irreducible representations of a C^*-algebra A are finite-dimensional, then $M_{\mathrm{loc}}(A)$ is a finite or countable direct product of C^*-algebras of the form $C(X_n) \otimes \mathbb{M}_n$, where each space X_n is Stonean. In particular, $M_{\mathrm{loc}}(A)$ is an AW^*-algebra of type I in this case, so $M_{\mathrm{loc}}(A) = M_{\mathrm{loc}}^{(2)}(A) = I(A)$ and $M_{\mathrm{loc}}(A)$ admits only inner derivations.

We also have the following criterion for innerness of derivations of certain class of C^*-algebras

Theorem (G., 2013)

Let A be a unital C^*-algebra in which every Glimm ideal (i.e. an ideal of the form mA, where m is a maximal ideal of the centre of A) is prime. Then a derivation δ of A is inner if and only if δ can be approximated by elementary operators in the cb-norm, i.e. for each $\varepsilon > 0$ there exists a natural number n and elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ of A such that for $\phi(x) := \sum_{i=1}^{n} a_i x b_i$ we have $\|\delta - \phi\|_{cb} < \varepsilon$.

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In particular, if there exists a C^*-algebra A such that $M_{\mathrm{loc}}(A)$ admits an outer derivation δ, then δ cannot be approximated by elementary operators in the cb-norm.

On the other hand, the stability problem of $M_{\mathrm{loc}}(A)$ has a negative solution:

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- Soon after, Argerami, Farenick and Massey (2009) showed that a relatively well-behaved C^*-algebra $C([0,1]) \otimes \mathbb{K}$ also fails to satisfy $M_{\mathrm{loc}}^{(2)}(A) = M_{\mathrm{loc}}(A)$.

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- Moreover, Ara and Mathieu (2011) showed that whenever X is a perfect, second countable locally compact Hausdorff space, and $A = C_0(X) \otimes B$ for some non-unital separable simple C^*-algebra B, then $M_{\mathrm{loc}}^{(2)}(A) \neq M_{\mathrm{loc}}(A)$.

This leads to the following two restatements of the stability problem of $M_{\mathrm{loc}}(A)$:

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Besides the C^*-algebras A which satisfy $M_{\rm loc}^{(2)}(A) = M_{\rm loc}(A)$, we know that $M_{\rm loc}^{(3)}(A) = M_{\rm loc}^{(2)}(A)$ for a certain class of type I C^*-algebras, such as:

- separable C^*-algebras of type I (Somerset, 2000);
- (not necessarily separable) spatial Fell algebras (Argerami, Farenick and Massey, 2010).

Moreover, in these two cases $M_{\rm loc}^{(2)}(A)$ is a type I AW^*-algebra.

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Summary

- We have no example in which $M_{\mathrm{loc}}^{(2)}(A) = M_{\mathrm{loc}}(A)$ and we do not know that every derivation of $M_{\mathrm{loc}}(A)$ is inner.
- We have no example in which $M_{\mathrm{loc}}^{(2)}(A) \neq M_{\mathrm{loc}}(A)$ and we know every derivation of $M_{\mathrm{loc}}(A)$ is inner.
- We have no example in which $M_{\mathrm{loc}}^{(3)}(A) \neq M_{\mathrm{loc}}^{(2)}(A)$.