Derivations and local multipliers of $C^*$-algebras

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**Definition**

A **derivation** of an algebra $A$ is a linear map $\delta : A \to A$ satisfying the Leibniz rule

$$\delta(xy) = \delta(x)y + x\delta(y) \quad \text{for all } x, y \in A.$$
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**Some properties of derivations of $C^*$-algebras**

If $A$ is a $C^*$-algebra, then every derivation $\delta$ of $A$ satisfies the following properties:

- $\delta$ is completely bounded and its cb-norm coincides with its operator norm (i.e. $\|\delta\|_{cb} = \|\delta\|_\text{op}$).
- $\delta$ preserves the (closed two-sided) ideals of $A$ (i.e. $\delta(I) \subseteq I$ for every ideal $I$ of $A$).
- $\delta$ vanishes on the centre of $A$ (i.e. $\delta(z) = 0$ for all $z \in Z(A)$). In particular, commutative $C^*$-algebras don't admit non-zero derivations.
- $\delta$ extends uniquely and under preservation of the norm to a derivation of $M(A)$ (the multiplier algebra of $A$).
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- $\delta$ extends uniquely and under preservation of the norm to a derivation of $M(A)$ (the multiplier algebra of $A$).
If $A$ is a $C^*$-subalgebra of a $C^*$-algebra $B$, then each element $a \in B$ which derives $A$ (i.e. $ax - xa \in A$, for all $x \in A$) implements a derivation $\delta_a : A \to A$ given by

$$\delta_a(x) := ax - xa.$$ 

A derivation $\delta$ of $A$ is said to be an **inner derivation** if there exists a multiplier $a \in M(A)$ such that $\delta = \delta_a$. 

Main problem: Which $C^*$-algebras admit only inner derivations? 

Some classes of $C^*$-algebras which admit only inner derivations: 

- simple $C^*$-algebras (Sakai, 1968). 
- $AW^*$-algebras (Olesen, 1974). 
- $\sigma$-unital continuous-trace $C^*$-algebras (Akemann-Elliott-Pedersen-Tomiyama, 1976).
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- $\sigma$-unital continuous-trace $C^*$-algebras (Akemann-Elliott-Pedersen-Tomiyama, 1976).
Moreover, the separable case was completely solved in 1979:

**Theorem (Akemann, Elliott, Pedersen and Tomiyama, 1979)**

Let $A$ be a separable $C^*$-algebra, then $A$ admits only inner derivations if and only if $A = A_1 \oplus A_2$, where $A_1$ is a continuous-trace $C^*$-algebra, and $A_2$ is a direct sum of simple $C^*$-algebras.
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On the other hand, for inseparable $C^*$-algebras the problem of innerness of derivations remains widely open, even for the simplest cases such as subhomogeneous $C^*$-algebras (i.e. $C^*$-algebras which have finite-dimensional irreducible representations of bounded degree).
If $I$ and $J$ are two essential ideals of $A$ such that $J \subseteq I$, then there is an embedding $M(I) \hookrightarrow M(J)$.
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In this way, we obtain a directed system of $C^*$-algebras with isometric connecting morphisms, where $I$ runs through the directed set $\text{Id}_{\text{ess}}(A)$ of all essential ideals of $A$. 
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**Definition**

The local multiplier algebra of \( A \) is the direct limit \( C^* \)-algebra

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M_{loc}(A) := (C^* \rightarrow) \lim \{ M(I) : I \in \text{Id}_{ess}(A) \}.
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**Definition**

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$$M_{\text{loc}}(A) := (C^*-) \lim \{ M(I) : I \in \text{Id}_{\text{ess}}(A) \}.$$  

Iterating the construction of $M_{\text{loc}}(A)$, one obtains the following tower of $C^*$-algebras which, a priori, does not have the largest element:

$$A \subseteq M_{\text{loc}}(A) \subseteq M^{(2)}_{\text{loc}}(A) \subseteq \cdots \subseteq M^{(n)}_{\text{loc}}(A) \subseteq \cdots,$$

where $M^{(2)}_{\text{loc}}(A) = M_{\text{loc}}(M_{\text{loc}}(A))$, $M^{(3)}_{\text{loc}}(A) = M_{\text{loc}}(M^{(2)}_{\text{loc}}(A))$, etc.
Example

If $A$ is simple, then obviously $M_{10c}(A) = M(A)$. 

If $A$ is an $AW^*$-algebra, then $M_{10c}(A) = A$.

If $A = C_0(X)$ is a commutative $C^*$-algebra, then $M_{10c}(A)$ is a commutative $AW^*$-algebra whose maximal ideal space can be identified with the inverse limit $\lim_{\leftarrow} \beta U$ of Stone-Čech compactifications $\beta U$ of dense open subsets $U$ of $X$. 

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Every derivation of a $C^*$-algebra $A$ extends uniquely and under preservation of the norm to a derivation of $M_{\text{loc}}(A)$.

**Theorem (Pedersen, 1978)**

Every derivation $\delta$ of a separable $C^*$-algebra $A$ is implemented by a local multiplier (i.e. $\delta$ becomes inner when extended to a derivation of $M_{\text{loc}}(A)$).
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**Theorem (Pedersen, 1978)**

*Every derivation $\delta$ of a separable $C^*$-algebra $A$ is implemented by a local multiplier (i.e. $\delta$ becomes inner when extended to a derivation of $M_{loc}(A)$).*

Moreover, it suffices to assume that every essential closed ideal of $A$ is $\sigma$-unital. In particular, Pedersen’s result entails Sakai’s theorem that every derivation of a simple unital $C^*$-algebra is inner.
Since $M_{\text{loc}}(A) = M(A)$ if $A$ is simple, and $M_{\text{loc}}(A) = A$ if $A$ is an $AW^*$-algebra, only an affirmative answer in the inseparable case would cover, extend and unify the results that all derivations of simple $C^*$-algebras and $AW^*$-algebras are inner.
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**Problem of innerness of derivations of $M_{loc}(A)$**

If $A$ is an arbitrary $C^*$-algebra, is every derivation of $M_{loc}(A)$ inner?
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**Problem of innerness of derivations of $M_{\text{loc}}(A)$**

If $A$ is an arbitrary $C^*$-algebra, is every derivation of $M_{\text{loc}}(A)$ inner?

**Stability problem of $M_{\text{loc}}(A)$**

Is $M_{\text{loc}}^{(2)}(A) = M_{\text{loc}}(A)$ for every $C^*$-algebra $A$?
There is another important characterisation of $M_{\text{loc}}(A)$, which was first obtained by Frank and Paulsen in 2003.

For a $C^*$-algebra $A$, let us denote by $I(A)$ its injective envelope as introduced by Hamana in 1979. $I(A)$ is not an injective object in the category of $C^*$-algebras and $*$-homomorphisms, but in the category of operator spaces and complete positive maps, i.e. for every inclusion $E \subseteq F$ of operator systems, each completely positive map $\phi : E \rightarrow I(A)$ has a completely positive extension $\tilde{\phi} : F \rightarrow I(A)$.

However, it turns out that (nevertheless) $I(A)$ is a $C^*$-algebra canonically containing $A$ as a $C^*$-subalgebra. Moreover, $I(A)$ is monotone complete, so in particular, $I(A)$ is an $AW^*$-algebra.
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\textbf{Theorem (Hamana, 1981)} All $AW^*$-algebras of type I are injective.
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**Theorem (Hamana, 1981)**

*All AW*-algebras of type I are injective.*
Theorem (Frank and Paulsen, 2003)

Under this embedding of \( A \) into \( I(A) \), \( M_{\text{loc}}(A) \) is the norm closure of the set of all \( x \in I(A) \) which act as a multiplier on some \( l \in \text{Id}_{\text{ess}}(A) \), i.e.

\[
M_{\text{loc}}(A) = \left( \bigcup_{l \in \text{Id}_{\text{ess}}(A)} \{ x \in I(A) : xl + lx \subseteq l \} \right)
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Using this result and the fact that \( I(M_{\text{loc}}(A)) = I(A) \), we obtain the following sequence of inclusions of \( C^* \)-algebras:

\[
A \subseteq M_{\text{loc}}(A) \subseteq M(2)_{\text{loc}}(A) \subseteq \cdots \subseteq A \subseteq I(A)
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where \( A \) is the regular monotone completion of \( A \).

Difficult problem

When is \( M_{\text{loc}}(A) = I(A) \), or at least \( M_{\text{loc}}(A) = A \)?
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Back to Pedersen’s questions, we have the following partial answers:

**Theorem (Somerset, 2000; Ara and Mathieu, 2011)**

If $A$ is a unital (or more generally quasi-central), separable $C^*$-algebra such that $\text{Prim}(A) (= \text{the primitive ideal space of } A)$ contains a dense $G_δ$ subset of closed points, then $M_{\text{loc}}^{(2)}(A) = M_{\text{loc}}(A)$. Moreover, in this case $M_{\text{loc}}(A)$ has only inner derivations.
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**Theorem (G., 2013)**

If all irreducible representations of a $C^*$-algebra $A$ are finite-dimensional, then $M_{loc}(A)$ is a finite or countable direct product of $C^*$-algebras of the form $C(X_n) \otimes \mathbb{M}_n$, where each space $X_n$ is Stonean. In particular, $M_{loc}(A)$ is an AW*-algebra of type I in this case, so $M_{loc}(A) = M_{loc}^{(2)}(A) = I(A)$ and $M_{loc}(A)$ admits only inner derivations.
We also have the following criterion for innerness of derivations of certain class of $C^*$-algebras

**Theorem (G., 2013)**

Let $A$ be a unital $C^*$-algebra in which every Glimm ideal (i.e. an ideal of the form $mA$, where $m$ is a maximal ideal of the centre of $A$) is prime. Then a derivation $\delta$ of $A$ is inner if and only if $\delta$ can be approximated by elementary operators in the cb-norm, i.e. for each $\varepsilon > 0$ there exists a natural number $n$ and elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ of $A$ such that for $\phi(x) := \sum_{i=1}^n a_i x b_i$ we have $\|\delta - \phi\|_{cb} < \varepsilon$. 

The class of $C^*$-algebras in which every Glimm ideal is prime is fairly large. It includes all prime $C^*$-algebras, $C^*$-algebras with Hausdorff primitive spectrum, quotients of $AW^*$-algebras, and local multiplier algebras. In particular, if there exists a $C^*$-algebra $A$ such that $M_{loc}(A)$ admits an outer derivation $\delta$, then $\delta$ cannot be approximated by elementary operators in the cb-norm.
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In particular, if there exists a $C^*$-algebra $A$ such that $M_{\text{loc}}(A)$ admits an outer derivation $\delta$, then $\delta$ cannot be approximated by elementary operators in the cb-norm.
On the other hand, the stability problem of $M_{\text{loc}}(A)$ has a negative solution:

The first class of examples of $C^*$-algebras for which the stability problem of local multiplier algebras has a negative answer was given by Ara and Mathieu (2006): There exist unital separable primitive AF-algebras $A$ such that $M_{\text{loc}}(A)^{(2)} \neq M_{\text{loc}}(A)$.

Soon after, Argerami, Farenick and Massey (2009) showed that a relatively well-behaved $C^*$-algebra $C([0,1]) \otimes K$ also fails to satisfy $M_{\text{loc}}(A)^{(2)} = M_{\text{loc}}(A)$.

Moreover, Ara and Mathieu (2011) showed that whenever $X$ is a perfect, second countable locally compact Hausdorff space, and $A = C_0(X) \otimes B$ for some non-unital separable simple $C^*$-algebra $B$, then $M_{\text{loc}}(A)^{(2)} \neq M_{\text{loc}}(A)$.
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- The first class of examples of $C^*$-algebras for which the stability problem of local multiplier algebras has a negative answer was given by Ara and Mathieu (2006): There exist unital separable primitive AF-algebras $A$ such that $M_{loc}^{(2)}(A) \neq M_{loc}(A)$.
- Soon after, Argerami, Farenick and Massey (2009) showed that a relatively well-behaved $C^*$-algebra $C([0, 1]) \otimes \mathbb{K}$ also fails to satisfy $M_{loc}^{(2)}(A) = M_{loc}(A)$.
- Moreover, Ara and Mathieu (2011) showed that whenever $X$ is a perfect, second countable locally compact Hausdorff space, and $A = C_0(X) \otimes B$ for some non-unital separable simple $C^*$-algebra $B$, then $M_{loc}^{(2)}(A) \neq M_{loc}(A)$. 
This leads to the following two restatements of the stability problem of $M_{\text{loc}}(A)$:

**Problem**

When is $M_{\text{loc}}^{(2)}(A) = M_{\text{loc}}(A)$?
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Whether for each positive integer $n$ there exists a $C^*$-algebra $A$ such that $M_{\text{loc}}^{(n)}(A) \neq M_{\text{loc}}^{(n+1)}(A)$?
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Besides the $C^*$-algebras $A$ which satisfy $M_{\text{loc}}^{(2)}(A) = M_{\text{loc}}(A)$, we know that $M_{\text{loc}}^{(3)}(A) = M_{\text{loc}}^{(2)}(A)$ for a certain class of type $I$ $C^*$-algebras, such as:

- separable $C^*$-algebras of type $I$ (Somerset, 2000);
- (not necessarily separable) spatial Fell algebras (Argerami, Farenick and Massey, 2010).

Moreover, in these two cases $M_{\text{loc}}^{(2)}(A)$ is a type $I$ $AW^*$-algebra.
Problem

Is $M_{\text{loc}}^{(2)}(A)$ an $AW^*$-algebra of type $I$ whenever $A$ is a $C^*$-algebra of type $I$?
Problem

Is $M_{\text{loc}}^{(2)}(A)$ an $AW^*$-algebra of type $I$ whenever $A$ is a $C^*$-algebra of type $I$?

Summary

- We have no example in which $M_{\text{loc}}^{(2)}(A) = M_{\text{loc}}(A)$ and we do not know that every derivation of $M_{\text{loc}}(A)$ is inner.
- We have no example in which $M_{\text{loc}}^{(2)}(A) \neq M_{\text{loc}}(A)$ and we know every derivation of $M_{\text{loc}}(A)$ is inner.
- We have no example in which $M_{\text{loc}}^{(3)}(A) \neq M_{\text{loc}}^{(2)}(A)$.