Topologically finitely generated Hilbert $C(X)$-modules

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C^*-algebras

Definition

A C^*-**algebra** is a (complex) Banach $*$-algebra A whose norm $\|\cdot\|$ satisfies the C^*-identity. More precisely:

- A is a Banach algebra over the field $\mathbb{C}$.
- A is equipped with an involution, i.e. a map $* : A \to A$, $a \mapsto a^*$ satisfying the properties:

$$(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*, \quad (ab)^* = b^* a^*, \quad \text{and} \quad (a^*)^* = a,$$

  for all $a, b \in A$ and $\alpha, \beta \in \mathbb{C}$.
- Norm $\|\cdot\|$ satisfies the C^*-**identity**, i.e.

$$\|a^* a\| = \|a\|^2$$

  for all $a \in A$.

Remark

The C^*-identity is a very strong requirement. For instance, together with the spectral radius formula, it implies that the C^*-norm is uniquely determined by the algebraic structure: For all $a \in A$ we have

$$\|a\|^2 = \|a^*a\| = \sup\{|\lambda| \ : \ \lambda \in \sigma(a^*a)\},$$

where

$$\sigma(x) := \{\lambda \in \mathbb{C} \ : \lambda 1 - a \text{ is not invertible}\}$$

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In the category of C^*-algebras, the natural morphisms are the $*$-**homomorphisms**, i.e. the algebra homomorphisms which preserve the involution. They are automatically contractive.

Example

Let X be a CH (compact Hausdorff) space and let $C(X)$ be the set of all continuous complex-valued functions on X. Then $C(X)$ becomes a C^*-algebra with respect to the pointwise operations, involution $f^*(x) := \overline{f(x)}$, and max-norm $\|f\|_\infty := \sup\{|f(x)| \ : \ x \in X\}$. Obviously, $C(X)$ is a unital commutative C^*-algebra.

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In fact, all unital commutative C^*-algebras arise in this fashion:

Theorem (Gelfand-Naimark, 1943)

The (contravariant) functor $X \rightsquigarrow C(X)$ defines an equivalence of categories of CH spaces and unital commutative C^*-algebras.

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In other words: By passing from the space X the function algebra $C(X)$, no information is lost. In fact, X can be recovered from $C(X)$. Thus, topological properties of X can be translated into algebraic properties of $C(X)$, and vice versa, so the theory of C^*-algebras is often thought of as **noncommutative topology**.

Basic examples

- The set $\mathbb{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ becomes a C^*-algebra with respect to the standard operations, usual adjoint and operator norm. In particular, the complex matrix algebras $M_n(\mathbb{C})$ are C^*-algebras.
- In fact, every C^*-algebra can be isometrically embedded as a norm-closed self-adjoint subalgebra of $\mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (Gelfand-Naimark theorem).
- To every locally compact group G, one can associate a C^*-algebra $C^*(G)$. Everything about the representation theory of G is encoded in $C^*(G)$.
- The category of C^*-algebras is closed under the formation of direct products, direct sums, extensions, direct limits, pullbacks, pushouts, (some) tensor products, etc.

Hilbert C^*-modules

- Hilbert C^*-modules form a category in between Banach spaces (they have a little extra geometrical structure) and Hilbert spaces (they are not as well behaving as these).
- A Hilbert C^*-module obeys the same axioms as an ordinary Hilbert space, except that the inner product takes values in a more general C^*-algebras than $\mathbb{C}$.
- Hilbert C^*-modules were first introduced in the work of I. Kaplansky in 1953, who developed the theory for unital commutative C^*-algebras. In the 1970s the theory was extended to non-commutative C^*-algebras independently by W. Paschke and M. Rieffel.
- Hilbert C^*-modules appear naturally in many areas of C^*-algebra theory, such as KK-theory, Morita equivalence of C^*-algebras, and completely positive operators.

Definition

Let A be a C^*-algebra. A (left) **Hilbert A-module** is a left A-module V, equipped with an A-valued inner product $\langle \cdot, \cdot \rangle$ which is A-linear in the first and conjugate linear in the second variable, such that V is a Banach space with the norm

$$\|v\| := \sqrt{\|\langle v, v \rangle\|_A}.$$

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Similarly, the direct sum A^n of n-copies of A becomes a A-Hilbert module with respect to the pointwise operations and the inner product

$$\langle a_1 \oplus \cdots \oplus a_n, b_1 \oplus \cdots \oplus b_n \rangle := \sum_{k=1}^{n} a_k b_k^*.$$

Example

More generally, let

$$\mathcal{H}_A := \{(a_k) \in \prod_1^\infty A \ : \ \sum_{k=1}^\infty a_k a_k^* \text{ is norm convergent}\}.$$

Then the pointwise operations and the inner product

$$\langle (a_k), (b_k) \rangle := \sum_{k=1}^\infty a_k b_k^*$$

turn $\mathcal{H}_A$ into a Hilbert A-module; it is known as a **standard Hilbert A-module**.

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Definition

An **(F) Hilbert bundle** is a triple $\mathcal{E} := (p, E, X)$ where E and X are topological spaces with a continuous open surjection $p : E \to X$, together with operations and norms making each **fibre** $E_x := p^{-1}(x)$ $(x \in X)$ into a complex Hilbert space, such that the following conditions are satisfied:

- The maps $\mathbb{C} \times E \to E$, $E \times_X E \to E$ and $E \times_X E \to \mathbb{C}$ given in each fibre by scalar multiplication, addition, and the inner product, respectively, are continuous. Here $E \times_X E$ denotes the Whitney sum

$$\{(e, f) \in E \times E \ : \ p(e) = p(f)\}.$$

- If $x \in X$ and if (e_α) is a net in E such that $\|e_\alpha\| \to 0$ and $p(e_\alpha) \to x$ in X, then $e_\alpha \to 0_x$ in E (where 0_x is the zero-element of E_x).

As usual, we say that p is the **projection**, E is the **bundle space** and X is the **base space** of $\mathcal{E}$.

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Example

The simplest example of an (F) Hilbert bundle is the **product bundle** over X with fibre H, $\epsilon(X, H) := (\mathrm{proj}_1, X \times H, H)$, where H is a Hilbert space.

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Example

Every locally trivial complex vector bundle $\mathcal{E}$ over a (para)compact Hausdorff space becomes an (F) Hilbert bundle for a choice of a Riemannian metric on $\mathcal{E}$. In fact, an (F) Hilbert bundle structure on $\mathcal{E}$ is essentially unique.

By a **section** of an (F) Hilbert bundle $\mathcal{E} = (p, E, X)$ we mean a map $s : X \to E$ such that

$$p(s(x)) = x \quad (x \in X).$$

By $\Gamma(\mathcal{E})$ we denote the set of all continuous of sections of $\mathcal{E}$.

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If X is compact, then $\Gamma(\mathcal{E})$ becomes a Hilbert $C(X)$-module with respect to the action

$$(\varphi s)(x) := \varphi(x)s(x)$$

and inner product

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In fact, all Hilbert $C(X)$-modules arise in this fashion:

Theorem

To every Hilbert $C(X)$-module V one can associate a natural (F) Hilbert bundle $\mathcal{E}_V$ such that $V \cong \Gamma(\mathcal{E}_V)$.

Homogeneous and subhomogeneous Hilbert $C(X)$-modules

An (F) Hilbert bundle $\mathcal{E} = (p, E, X)$ is said to be:

- **Trivial** if $\mathcal{E} \cong \epsilon(X, H)$ for some Hilbert space H.
- **Locally trivial** if there exists a Hilbert space H and an open cover $\mathcal{U}$ of X such that for each $U \in \mathcal{U}$ we have $\mathcal{E}|_U \cong \epsilon(U, H)$.
- n-**homogeneous**, if all fibres of $\mathcal{E}$ have the same finite dimension n.

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Every n-homogeneous (F) Hilbert bundle is automatically locally trivial.

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Theorem

Every n-homogeneous (F) Hilbert bundle is automatically locally trivial.

Hence, the category of n-homogeneous (F) Hilbert bundles over CH spaces is equivalent to the category of n-dimensional (locally trivial) complex vector bundles.

If all fibres of an (F) Hilbert bundle $\mathcal{E}$ are finite dimensional with

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If in addition every base space of such restriction bundle admits a finite trivializing open cover, then we say that $\mathcal{E}$ is n-**subhomogeneous of finite type**.

Algebraically finitely generated Hilbert $C(X)$-modules

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Theorem

Every A.F.G. Hilbert module over a unital C^*-algebra is automatically projective.

In particular, when $A = C(X)$, we get a Hilbert module version of the celebrated Serre-Swan theorem:

Theorem

Let V be a Hilbert $C(X)$-module, where X is a compact Hausdorff space, and let $\mathcal{E} := \mathcal{E}_V$. Then V is A.F.G. if and only if there exists a finite clopen partition $X = X_1 \sqcup \cdots \sqcup X_k$ such that each restriction bundle $\mathcal{E}|_{X_i}$ is homogeneous.

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The main difference between A.F.G. and T.F.G. Hilbert $C(X)$-modules is the fact that T.F.G. Hilbert $C(X)$-modules are not generally projective. Hence, the dimension of the fibres of the canonical (F) Hilbert bundle may vary, even if X is connected.

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Example

let X be the unit interval $[0,1]$ and let $V := C_0((0,1])$. Then V becomes a Hilbert $C([0,1])$-module with respect to the standard action and inner product $\langle f, g \rangle = f^* g$. Note that V is topologically singly generated (for instance, the identity function $f(x) = x$ is such generator, by the Weierstrass approximation theorem). On the other hand, each fibre E_x of $\mathcal{E}_V$ is one-dimensional, except E_0, which is zero.

However, this phenomenon is the only major difference between A.F.G. and T.F.G. Hilbert $C(X)$-modules (at least when X is metrizable):

Theorem (I.G. 2012)

Let X be a compact metrizable space and let V be a Hilbert $C(X)$-module with the canonical (F) Hilbert bundle $\mathcal{E}_V$. Then V is T.F.G. if and only if $\mathcal{E}_V$ is subhomogeneous of finite type.

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Here are some further characterizations of T.F.G. Hilbert $C(X)$-modules:

Theorem (I.G. 2012)

For a Hilbert $C(X)$-module V, where X is a compact metrizable space, the following conditions are equivalent:

(a) *V is T.F.G.*

(b) *V is weakly A.F.G., i.e. there exists $K \in \mathbb{N}$ such that every A.F.G. submodule of V can be generated with $k \le K$ generators.*

(c) *There exists $N \in \mathbb{N}$ such that for every Banach $C(X)$-module W, each tensor in the $C(X)$-projective tensor product $V \overset{\pi}{\otimes}_{C(X)} W$ is of (finite) rank at most N.*