When are the two-sided multiplication maps norm closed?

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(joint work in progress with Richard M. Timoney)

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- For any ideal I of A, ϕ induces a map $\phi_I : A/I \to A/I$ which sends $a + I$ to $\phi(a) + I$.
- If S is any subset of ideals of A with zero intersection, the norm of ϕ can be computed by the formula $\|\phi\| = \sup\{\|\phi_I\| \ : \ I \in S\}$.

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The most prominent class of maps $\phi \in \mathrm{IB}(A)$ are the **elementary operators**, i.e. those that can be expressed as finite sums of **two-sided multiplication maps** $M_{a,b} : x \mapsto axb$, where a and b are elements of $M(A)$.

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By $\mathrm{TM}(A)$ and $\mathcal{E}\ell(A)$ we denote, respectively, the set of all two-sided multiplication maps and all elementary operators on A.

In fact, elementary operators are completely bounded and

$$\left\| \sum_i M_{a_i,b_i} \right\|_{cb} \leq \left\| \sum_i a_i \otimes b_i \right\|_h , \tag{1}$$

where $\|\cdot\|_h$ is the Haagerup tensor norm on $M(A) \otimes M(A)$, i.e.

$$\|t\|_h = \inf \left\{ \left\| \sum_i a_i a_i^* \right\|^{\frac{1}{2}} \left\| \sum_i b_i^* b_i \right\|^{\frac{1}{2}} \ : \ t = \sum_i a_i \otimes b_i \right\}.$$

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Theorem (Haagerup 1980, Chatterjee-Sinclair 1992, Mathieu 2003)

The equality in (1) holds true for all elementary operators $\phi = \sum_i M_{a_i,b_i}$ if and only if A is a prime C^*-algebra.

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Remark

If the algebra A is not prime, then the map $a \otimes b \mapsto M_{a,b}$ is not even injective.

The **length** of an elementary operator $\phi \neq 0$ is the smallest positive integer $\ell = \ell(\phi)$ such that $\phi = \sum_{i=1}^{\ell} M_{a_i,b_i}$ for some $a_i, b_i \in M(A)$. We also define $\ell(0) = 0$.

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Theorem (Timoney 2003, 2007)

For every $\phi \in \mathcal{E}\ell(A)$ we have

$$\|\phi\|_{cb} = \|\phi \otimes \mathrm{id}_{\mathbb{M}_{\ell(\phi)}}\| \leq \sqrt{\ell(\phi)}\|\phi\|.$$

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Corollary

On each set $\mathcal{E}\ell_k(A)$ the cb-norm is equivalent to the operator norm.

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Theorem (Magajna 2009)

If A is a separable C^*-algebra A, then $\mathcal{E}\ell(A)$ is operator norm dense in $\mathrm{IB}(A)$ if and only if A can be decomposed as a finite direct sum $A = A_1 \oplus \cdots \oplus A_n$, where each summand A_i is homogeneous with the finite type property. In particular, in this case we have $\mathrm{IB}(A) = \mathcal{E}\ell(A)$.

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Remark

- Recall that a well-known theorem of Fell and Tomiyama-Takesaki asserts that for any n-homogeneous C^*-algebra A with (primitive) spectrum X there is a locally trivial bundle $\mathcal{E}$ over X with fibre $\mathbb{M}_n$ and structure group $PU(n) = \mathrm{Aut}(\mathbb{M}_n)$ such that A is isomorphic to the algebra $\Gamma_0(\mathcal{E})$ of sections of $\mathcal{E}$ which vanish at infinity.

Remark (continuation)

- Moreover, any two such algebras $A_i = \Gamma_0(\mathcal{E}_i)$ with spectra X_i are isomorphic if and only if there is a homeomorphism $f : X_1 \to X_2$ such that $\mathcal{E}_1 \cong f^*(\mathcal{E}_2)$ as bundles over X_1.
- An n-homogeneous C^*-algebra $\Gamma_0(\mathcal{E})$ with spectrum X is said to have the **finite type property** if $\mathcal{E}$ can be trivialized over some finite open cover of X.

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Theorem (G. 2011)

Let A be a separable C^*-algebra.

(a) *If $\mathcal{E}\ell(A)$ is norm closed, then A is necessarily subhomogeneous and each homogeneous sub-quotient of A has the finite type property.*

(b) *The converse is also true if $\mathrm{Prim}(A)$ is Hausdorff.*

(c) *There exists a compact subset X of $\mathbb{R}$ and a unital C^*-subalgebra A of $C(X, \mathbb{M}_2)$ with trivial homogeneous sub-quotients such that $\mathcal{E}\ell(A)$ is not norm closed.*

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Describe the operator norm closure $\overline{\overline{\mathrm{TM}(A)}}$.

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Lemma (G., Timoney 2015)

Let a, b, c and d be norm-one elements of an operator space V. If

$$\|a \otimes b - c \otimes d\|_h < \varepsilon \leq 1/9,$$

then there exists a complex number λ such that $|\lambda| = 1$ and

$$\max\{\|a - \lambda c\|, \|b - \overline{\lambda} d\|\} < 9\varepsilon.$$

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Corollary

If A is a prime C^*-algebra, then $\mathrm{TM}(A)$ is norm closed.

Let us now consider what happens when $A = C_0(X, \mathbb{M}_n)$, where X is a locally compact Hausdorff space.

- In this case $\mathrm{Prim}(A) = X$ (via $x \leftrightarrow C_0(X \setminus \{x\}, \mathbb{M}_n)$. As usual we write A_x for $A/(C_0(X \setminus \{x\}, \mathbb{M}_n) \cong \mathbb{M}_n$ and q_x for the corresponding quotient map.
- $\mathrm{IB}(A) = \mathcal{E}\ell(A)$ can be identified with $C_b(X, \mathrm{B}(\mathbb{M}_n))$ by mapping which sends $\phi \in \mathrm{IB}(A)$ to $x \mapsto \phi_x = q_x \circ \phi$.

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Notation

- $\mathrm{IB}_1(A) := \{\phi \in \mathrm{IB}(A) : \phi_x \in \mathrm{TM}(A_x) \text{ for all } x \in X\}$.
- $\mathrm{IB}_1^{\mathrm{nv}}(A) := \{\phi \in \mathrm{IB}_1(A) : \phi_x \neq 0 \text{ for all } x \in X\}$.

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Corollary

If $A = C_0(X, \mathbb{M}_n)$, *then* $\overline{\overline{\mathrm{TM}(A)}} \subseteq \mathrm{IB}_1(A)$.

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Corollary

If $A = C_0(X, \mathbb{M}_n)$, then $\overline{\overline{\mathrm{TM}(A)}} \subseteq \mathrm{IB}_1(A)$.

Question

Do we always have $\overline{\overline{\mathrm{TM}(A)}} = \mathrm{IB}_1(A)$?

Theorem (G., Timoney 2015)

Let $A = C_0(X, \mathbb{M}_n)$, where X is a locally compact Hausdorff space.

(a) *To every operator $\phi \in \mathrm{IB}_1^{\mathrm{nv}}(A)$ we can associate a complex line subbundle $\mathcal{L}_\phi$ of $X \times \mathbb{M}_n$ with the property that $\phi \in \mathrm{TM}(A)$ if and only if $\mathcal{L}_\phi$ is a trivial bundle.*

(b) *To every complex line subbundle $\mathcal{E}$ of $X \times \mathbb{M}_n$ we can associate an operator $\phi_{\mathcal{E}} \in \mathrm{IB}_1^{\mathrm{nv}}(A)$ such that $\mathcal{L}_{\phi_{\mathcal{E}}} \cong \mathcal{E}$.*

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Corollary

If X is a paracompact (locally compact Hausdorff) space such that $H^2(X; \mathbb{Z}) = 0$ (the second Čech cohomology), then for $A = C_0(X, \mathbb{M}_n)$ we have the inclusion $\mathrm{IB}_1^{\mathrm{nv}}(A) \subseteq \mathrm{TM}(A)$.

Example

Let $\mathcal{E}$ be the Hopf fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \twoheadrightarrow \mathbb{S}^2$ and let $n \geq 2$. We consider $\mathbb{S}^2$ as the unit sphere in $\mathbb{C}^2$ (where $\mathbb{C}^2$ is equipped with the standard euclidian metric) and we realise $\mathbb{S}^3 \subset \mathbb{M}_n$ as $\{z_1 e_{11} + z_2 e_{12} \ : \ |z_1|^2 + |z_2|^2 = 1\}$. For a local section $e : U \to \mathbb{S}^3$ of the bundle $\mathcal{E}$ (U is an open subset of $\mathbb{S}^2$) and $x \in X$ we define $\phi_x \in \mathcal{E}\ell_1(\mathbb{M}_n)$ by

$$\phi_x(y) := e(x) y e(x)^* \quad (y \in \mathbb{M}_n).$$

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Then $\phi \in \mathrm{IB}_1^{\mathrm{nv}}(A) \setminus \mathrm{TM}(A)$.

Corollary

If X is a second countable locally compact Hausdorff space, then for $A = C_0(X, \mathbb{M}_n)$ the following conditions are equivalent:

(a) $\mathrm{IB}_1(A) = \mathrm{TM}(A)$.

(b) *For every open subset U, each complex line subbundle of $U \times \mathbb{M}_n$ is trival.*

Theorem (G., Timoney 2015)

Let X be a second countable locally compact Hausdorff space and let $A = C_0(X, \mathbb{M}_n)$. For an operator $\phi \in \mathrm{IB}(A)$ the following two conditions are equivalent:

(a) $\phi \in \overline{\overline{\mathrm{TM}(A)}}$.

(b) *If $U = \{x \in X : \phi_x \neq 0\}$, then $\mathcal{L}_{\phi|_U}$ is trivial on each compact subset of U.*

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Definition

A locally trivial fibre bundle $\mathcal{F}$ over a locally compact Hausdorff space X is said to be a **phantom bundle** if $\mathcal{F}$ is not globally trivial, but is trivial on each compact subset of X.

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Corollary

If $A = C_0(X, \mathbb{M}_n)$ as above, then $\mathrm{TM}(A)$ is not uniformly closed if and only if there exists an open subset U of X and a phantom complex line subbundle of $U \times \mathbb{M}_n$.

Example

If $A = C(\mathbb{S}^2, \mathbb{M}_n)$ ($n \geq 2$), then the operator ϕ defined by the Hopf fibration shows that in general $\overline{\overline{\mathrm{TM}(A)}} \subsetneq \mathrm{IB}_1(A)$.

Example

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Example

Let X be the Eilenberg-MacLane space $K(\mathbb{Q}, 1)$.

- The standard model of X is a mapping telescope of the sequence

$$\mathbb{S}^1 \xrightarrow{z} \mathbb{S}^1 \xrightarrow{z^2} \mathbb{S}^1 \xrightarrow{z^3} \cdots .$$

- Applying $H_1(-;\mathbb{Z})$ to the levels of this mapping telescope gives the system

$$\mathbb{Z} \xrightarrow{\times 1} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 3} \cdots .$$

  The colimit of this system is $H_1(X;\mathbb{Z}) = \mathbb{Q}$ and all other (integral) homology is trivial.

- By the universal coefficient theorem, each integral cohomology group of X is trivial except for $H^2(X;\mathbb{Z})$ which is isomorphic to $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$.

Example (continuation)

- In particular, $H^2(X;\mathbb{Z})$ is non-trivial. Let $\mathcal{E}$ be a line bundle defined by some non-zero class of $H^2(X;\mathbb{Z})$. Then $\mathcal{E}$ is a phantom bundle, since the restriction of $\mathcal{E}$ to each finite subcomplex of X is trivial.
- Since (the standard model of) X is a 2-complex, $\mathcal{E}$ is a direct summand of a trivial bundle $X \times \mathbb{C}^2$. Hence, $\mathrm{TM}(C_0(X, \mathbb{M}_2))$ is not uniformly closed.

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Moreover, Prof. Mladen Bestvina (University of Utah) informed us that $K(\mathbb{Q}, 1)$ is homotopy equivalent to an open subset of $\mathbb{R}^3$. As a consequence of this we get:

Corollary

(a) *For any open subset U of $\mathbb{R}^3$, $\mathrm{TM}(C_0(U, \mathbb{M}_2))$ is not uniformly closed.*

(b) *In fact, $d = 3$ is the smallest possible dimension with the following property: there exists an open subset U of $\mathbb{R}^d$ such that $\mathrm{TM}(C_0(U, \mathbb{M}_n))$ is not uniformly closed for some n.*