On unital $C(X)$-algebras and $C(X)$-valued conditional expectations of finite index

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C^*-algebras as noncommutative topology

Definition

A (unital) C^*-**algebra** is a complex Banach $*$-algebra A whose norm $\|\cdot\|$ satisfies the C^*-identity. More precisely:

- A is a Banach algebra with identity over the field $\mathbb{C}$.
- A is equipped with an involution, i.e. a map $* : A \to A$, $a \mapsto a^*$ satisfying the properties:

  $$(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*, \quad (ab)^* = b^* a^*, \quad \text{and} \quad (a^*)^* = a,$$

  for all $a, b \in A$ and $\alpha, \beta \in \mathbb{C}$.
- Norm $\|\cdot\|$ satisfies the C^*-**identity**, i.e.

  $$\|a^* a\| = \|a\|^2$$

  for all $a \in A$.

Remark

The C^*-identity is a very strong requirement. For instance, together with the spectral radius formula, it implies that the C^*-norm is uniquely determined by the algebraic structure: For all $a \in A$ we have

$$\|a\|^2 = \|a^*a\| = r(a^*a) = \sup\{|\lambda| \ : \ \lambda \in \mathrm{spec}(a^*a)\}.$$

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Example

Let X be a CH (compact Hausdorff) space and let $C(X)$ be the set of all continuous complex-valued functions on X. Then $C(X)$ becomes a commutative C^*-algebra with respect to the pointwise operations, involution $f^*(x) := \overline{f(x)}$, and max-norm $\|f\|_\infty := \sup\{|f(x)| \ : \ x \in X\}$.

In fact, all unital commutative C^*-algebras arise in this fashion:

Theorem (Commutative Gelfand-Naimark theorem)

The (contravariant) functor $X \rightsquigarrow C(X)$ defines an equivalence of categories of CH spaces (with continuous maps as morphisms) and commutative C^*-algebras (with $*$-homomorphisms as morphisms).

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In other words: By passing from the space X the function algebra $C(X)$, no information is lost. In fact, X can be recovered from $C(X)$. Thus, topological properties of X can be translated into algebraic properties of $C(X)$, and vice versa, so the theory of C^*-algebras is often thought of as **noncommutative topology**.

Basic examples

- If $\mathcal{H}$ is a Hilbert space, then the algebra $\mathbb{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$ with the operator norm and usual adjoint obeys the C^*-identity.
- In particular, the matrix algebras $M_n(\mathbb{C})$ over $\mathbb{C}$ with the euclidian norm are C^*-algebras. Moreover, the finite direct sums of matrix algebras over $\mathbb{C}$ make up all finite-dimensional C*-algebras.
- If A C^*-algebra and X is a CH space, then $C(X, A)$ becomes a C^*-algebra with respect to the pointwise operations and max-norm.
- To every locally compact group G, one can associate a C^*-algebra $C^*(G)$. Everything about the representation theory of G is encoded in $C^*(G)$.
- The category of C^*-algebras is closed under the formation of direct products, direct sums, extensions, direct limits, pullbacks, pushouts, (some) tensor products, etc.

Definition

A **representation** *of a C^*-algebra A is a $*$-homomorphism $\pi : A \to \mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. A representation π is said to be* **irreducible** *if it has no nontrivial closed invariant subspaces (i.e. if $\mathcal{K}$ is a closed subspace of $\mathcal{H}$ such that $\pi(A)\mathcal{K} \subseteq \mathcal{K}$, then $\mathcal{K} = \{0\}$ or $\mathcal{K} = \mathcal{H}$).*

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Noncommutative Gelfand-Naimark theorem

Every C^*-algebra admits an isometric representation on some Hilbert space.

Remark

Because of the previous theorem, C^*-algebras can be concretely defined to be norm closed self-adjoint subalgebras of bounded operators on some Hilbert space $\mathcal{H}$.

Definition

Let A be C^*-algebra.

- *A* **primitive ideal** *of A is an ideal which is the kernel of an irreducible representation of A.*
- *The* **primitive spectrum** *of A is the set $\mathrm{Prim}(A)$ of primitive ideals of A equipped with the* **Jacobson topology***: If S is a set of primitive ideals, its closure is*

$$\overline{S} := \left\{ P \in \mathrm{Prim}(A) \ : \ P \supseteq \bigcap_{Q \in S} Q \right\}.$$

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Example

If $A = C(X)$, let $C_x(X) := \{f \in C(X) \; : \; f(x) = 0\}$ $(x \in X)$. Then $\mathrm{Prim}(C(X)) = \{C_x(X) \; : \; x \in X\}$. Moreover, the correspondence $x \mapsto C_x(X)$ defines a homeomorphism between X and $\mathrm{Prim}(C(X))$.

Remark

- $\mathrm{Prim}(A)$ is always a locally compact space. It is compact whenever A is unital.
- If A is separable, $\mathrm{Prim}(A)$ is second countable.
- However, as a topological space, $\mathrm{Prim}(A)$ is in general badly behaved and may satisfy only the T_0-separation axiom.

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When a C^*-algebra A is unital, the Jacobson topology on $\mathrm{Prim}(A)$ not only describes the ideal structure of A, but also allows us to completely describe the center $Z(A)$ of A:

Dauns-Hofmann theorem; 1968

Let A be a unital C^*-algebra. For each $P \in \mathrm{Prim}(A)$, let $q_P : A \to A/P$ be the quotient map. Then there is a $*$-isomorphism Φ_A of $C(\mathrm{Prim}(A))$ onto the center $Z(A)$ of A such that

$$q_P(\Phi_A(f)) = f(P)q_P(a)$$

for all $f \in C(\mathrm{Prim}(A))$, $a \in A$ and $P \in \mathrm{Prim}(A)$.

$C(X)$-algebras

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The natural candidate for the base space X is $\mathrm{Prim}(A)$, the primitive spectrum of A. However, since the topology on $\mathrm{Prim}(A)$ can be awkward to deal with, a natural alternative is to find a compact Hausdorff space X (which will turn out to be a continuous image of $\mathrm{Prim}(A)$) over which A fibres in a nice way.

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Such algebras are known as $C(X)$-algebras and were introduced by G. Kasparov in 1988:

Definition

*Suppose that X is a compact Hausdorff space. A unital C^*-algebra A is said to be a $C(X)$-***algebra** *if A is endowed with a unital $*$-homomorphism ψ_A from $C(X)$ to the centre of A.*

There is a natural connection between $C(X)$-algebras and upper semicontinuous C^*-bundles over X.

Definition

An **upper semicontinuous** C^***-bundle** *is a triple* $\mathfrak{A} = (p, \mathcal{A}, X)$ *where* $\mathcal{A}$ *is a topological space with a continuous open surjection* $p : \mathcal{A} \to X$, *together with operations and norms making each* **fibre** $\mathcal{A}_x := p^{-1}(x)$ *into a* C^**-algebra, such that the following conditions are satisfied:*

(A1) *The maps* $\mathbb{C} \times \mathcal{A} \to \mathcal{A}$, $\mathcal{A} \times_X \mathcal{A} \to \mathcal{A}$, $\mathcal{A} \times_X \mathcal{A} \to \mathcal{A}$ *and* $\mathcal{A} \to \mathcal{A}$ *given in each fibre by scalar multiplication, addition, multiplication and involution, respectively, are continuous (*$\mathcal{A} \times_X \mathcal{A}$ *denotes the Whitney sum over* X*).*

(A2) *The map* $\mathcal{A} \to \mathbb{R}$, *defined by norm on each fibre, is upper semicontinuous.*

(A3) *If* $x \in X$ *and if* (a_i) *is a net in* $\mathcal{A}$ *such that* $\|a_i\| \to 0$ *and* $p(a_i) \to x$ *in* X, *then* $a_i \to 0_x$ *in* $\mathcal{A}$ *(*0_x *denotes the zero-element of* $\mathcal{A}_x$*).*

If "upper semicontinuous" in (A2) is replaced by "continuous", then we say that $\mathfrak{A}$ is a **continuous** C^*-bundle.

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Example

If A is a C^*-algebra, then the simplest example of a continuous C^*-bundle is the **product bundle** over X with fibre A,

$$\epsilon(X, A) := (\pi_1, X \times A, A).$$

where π_1 is a projection on the first coordinate.

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By a **section** of an upper semicontinuous C^*-bundle $\mathfrak{A}$ we mean a map $s : X \to \mathcal{A}$ such that $p(s(x)) = x$ for all $x \in X$. We denote by $\Gamma(\mathfrak{A})$ the set of all continuous sections of $\mathfrak{A}$. Then $\Gamma(\mathfrak{A})$ becomes a $C(X)$-algebra with respect to the natural pointwise operations and sup-norm.

On the other hand, we have the following important result:

Theorem (Fell & Lee)

For each $C(X)$-algebra A there exists an upper semicontinuous C^*-bundle $\mathfrak{A}$ over X such that $A \cong \Gamma(\mathfrak{A})$.

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For each $C(X)$-algebra A there exists an upper semicontinuous C^*-bundle $\mathfrak{A}$ over X such that $A \cong \Gamma(\mathfrak{A})$.

Definition

If all norm functions $x \mapsto \|a_x\|$ $(a \in A)$ are continuous on X, we say that A is a **continuous $C(X)$-algebra**. *This is equivalent to say that the associated bundle $\mathfrak{A}$ is continuous.*

Example

Let D be any unital C^*-algebra. Then $A := C(X, D)$ becomes a continuous $C(X)$-algebra in a natural way:

$$\psi_A(f)(x) := f(x) \cdot 1_A \qquad (f \in C(X)).$$

In this case, each fibre A_x is easily identified with D.

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Example (Degenerate example)

Let A be any unital C^*-algebra and let us fix a point $x_0 \in X$. Then A becomes a $C(X)$-algebra via the map

$$\psi_A(f) := f(x_0) \cdot 1_A \qquad (f \in C(X)).$$

In this example, every fibre A_x is zero, except for $x = x_0$, where $A_{x_0} = A$.

Remark

To avoid such pathological examples, we shall always assume that the $*$-homomorphism ψ_A is injective. Then we may identify $C(X)$ with the C^*-subalgebra $\psi_A(C(X))$ of $Z(A)$.

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Example

Let X and Y be two CH spaces. If $F : Y \to X$ is any continuous function, then $C(Y)$ becomes a $C(X)$-algebra with

$$\psi_{C(Y)}(f) := f \circ F.$$

- For each $x \in X$, every fibre $C(Y)_x$ is $*$-isomorphic to $C(F^{-1}(x))$.
- $C(Y)$ is a continuous $C(X)$-algebra if and only if F is an open map.

In fact, the previous example is not nearly as specialized as it might seem at first:

Theorem

Let A be a unital C^*-algebra and let X be a CH space.

- *If there exists a continuous map $F_A : \mathrm{Prim}(A) \to X$, then A becomes a $C(X)$-algebra with*

$$\psi_A(f) := \Phi_A \circ f \circ F_A \qquad (f \in C(X)),$$

  where $\Phi_A : C(\mathrm{Prim}(A)) \cong Z(A)$ is the Dauns-Hofmann isomorphism.
- *Moreover, every unital $C(X)$-algebra arises is this way.*
- *A $C(X)$-algebra A is continuous if end only if the associated map $F_A : \mathrm{Prim}(A) \to X$ is open.*

We will be particularly interested in the following classes of $C(X)$-algebras:

Definition

A unital $C(X)$-algebra A is said to be:

- **homogeneous** *all fibres of A are $*$-isomorphic to the same finite-dimensional C^*-algebra.*
- **subhomogeneous** *if there exists a positive integer N such that every fibre A_x of A is finite-dimensional with $\dim A_x \leq N$.*

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Example

- $C(X, \mathbb{M}_n)$ is a (continuous) homogeneous $C(X)$-algebra with fibre $\mathbb{M}_n$.
- Let

$$A := \{f \in C([0,1], \mathbb{M}_n) \ : \ f(0) \text{ is a diagonal matrix}\}.$$

  Then A is a (continuous) $C([0,1])$-algebra with $A_0 = \mathbb{C}^n$ and $A_x = \mathbb{M}_n$ for $0 < x \leq 1$.

If D is a finite-dimensional C^*-algebra, recall that A is isomorphic to the finite direct sums of matrix algebras $\mathbb{M}_{n_i}$. We define the **rank** of D as

$$r(D) := \sum_i n_i.$$

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Let A be a unital $C(X)$-algebra.

- A is subhomogeneous if and only if

$$r(A) := \sup\{r(A_x) \ : x \in X\} < \infty.$$

  As in the finite-dimensional case, we call this number as **rank** of A.
- If A is continuous and homogeneous with fibre D, then by an important result of J. Fell from 1961, A is automatically locally trivial. This intuitively means that for every point $x \in X$ there exists a compact neighborhood U of x such that the restriction of A on U looks like $C(U, D)$.

Definition

Let $B \subseteq A$ be two C^*-algebras with common identity element. A **conditional expectation** *(abbreviated C.E.) from A onto B is a completely positive (c.p.) contraction $E : A \to B$ which satisfies the following conditions:*

- $E(b) = b$ *for all* $b \in B$.
- E *is* ${}_BA_B$*-linear, i.e.* $E(b_1 a b_2) = b_1 E(a) b_2$ *for all* $a \in A$ *and* $b_1, b_2 \in B$.

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Remark

The C^*-algebraic conditional expectations are the noncommutative analogues of classical conditional expectations from probability theory.

Theorem (Tomiyama; 1957)

A map $E : A \to B$ is a C.E. if and only if E is a projection of norm one.

Definition

A C.E. $E : A \to B$ *is said to be of* **finite index** *(abbreviated C.E.F.I.) if there exists a constant* $K \geq 1$ *such that the map* $(K \cdot E - \mathrm{id}_A) : A \to A$ *is positive.*

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However, attempts to describe the more general situation of conditional expectations on C^*-algebras with arbitrary centers to be "of finite index" in some sense(s) went into difficulties. In fact, M. Baillet, Y. Denizeau and J.-F. Havet showed that even in the case of normal faithful conditional expectations E on W^*-algebras M with non-trivial centres, the index value can be calculated only in situations when there exists a number $L \geq 1$ such that the mapping $(L \cdot E - \mathrm{id}_A)$ is completely positive.

However, the following important result resolved this issue, and consequently justified the given definition for C.E. on general C^*-algebras to be of finite index:

Theorem (Frank & Kirchberg; 1998)

For a C.E. $E : A \to B$ the following conditions are equivalent:

(a) *There exists $K \geq 1$ such that the map $K \cdot E - \mathrm{id}_A$ is positive.*

(b) *There exists $L \geq 1$ such that the map $L \cdot E - \mathrm{id}_A$ is c.p.*

(c) *A becomes a (complete) Hilbert B-module when equipped with the inner product $\langle a_1, a_2 \rangle := E(a_1^* a_2)$.*

Moreover, if

$$K(E) := \inf\{K \geq 1 \ : \ K \cdot E - \mathrm{id}_A \text{ is positive}\},$$

$$L(E) := \inf\{L \geq 1 \ : \ L \cdot E - \mathrm{id}_A \text{ is c.p.}\},$$

with $K(E) = \infty$ or $L(E) = \infty$ if no such number K or L exists, then

$$K(E) \leq L(E) \leq \lfloor K(E) \rfloor K(E).$$

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For a unital inclusion $A \subseteq B$ of unital C^*-algebras we can now introduce the following constant, which plays an important role in our research:

$$K(A, B) := \inf\{K(E) \ : \ E : A \to B \text{ is C.E.F.I.}\},$$

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with $K(A, B) = \infty$, if no such E exists.

Example

Let A be a homogeneous $C(X)$-algebra $C(X, \mathbb{M}_n)$ and let $\mathrm{tr}(\cdot)$ be the standard trace on $\mathbb{M}_n$. Then

$$E(f)(x) := \frac{1}{n}\mathrm{tr}(f(x))$$

defines a C.E.F.I. from A onto $C(X)$. In this case we have $K(A, C(X)) = K(E) = n$.

Noncommutative branched coverings

Definition

Let X and Y be two CH spaces. A **branched coverings** *is an open continuous surjection $\sigma : Y \to X$ with uniformly bounded number of pre-images, i.e.*

$$\sup_{x \in X} |\sigma^{-1}(x)| < \infty.$$

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Find an equivalent formulation of the existence of a branched covering $\sigma : Y \to X$ in terms of their associated C^*-algebras $C(X)$ i $C(Y)$.

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Theorem (Pavlov & Troitsky; 2011)

A pair (X, Y) admits a branched covering $\sigma : Y \to X$ if and only if there exists a C.E.F.I. $E : C(Y) \to C(X)$.

In light of noncommutative topology, A. Pavlov and E. Troitsky introduced the following definition:

Definition

A **noncommutative branched covering** *is a pair* (A, B) *consisting of a* C^**-algebra* A *and its* C^**-subalgebra* B *with common identity element, such that there exists a C.E.F.I. from* A *onto* B*.*

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Reinterpretation in terms of $C(X)$-algebras

If $\sigma : Y \to X$ is a continuous surjection, then (as already described) $C(Y)$ becomes a $C(X)$-algebra via

$$\psi_A(f) = f \circ \sigma \qquad (f \in C(X)).$$

Then:

- σ is an open map if and only if $C(Y)$ is a continuous $C(X)$-algebra.
- $\sup_{x \in X} |\sigma^{-1}(x)| < \infty$ if and only if $C(Y)$ is a subhomogeneous $C(X)$-algebra.

Therefore, if A is a unital commutative $C(X)$-algebra, then a pair $(A, C(X))$ defines a noncommutative branched covering if and only if A is a continuous subhomogeneous $C(X)$-algebra.

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- Is the above result also valid for noncommutative $C(X)$-algebras A?
- What can be said about the weak index $K(A, C(X))$?

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- What can be said about the weak index $K(A, C(X))$?

We managed to prove one direction:

Theorem (Blanchard & G.; 2016)

Let A be a unital $C(X)$-algebra. If a pair $(A, C(X))$ defines a noncommutative branched covering, then A is necessarily a continuous subhomogeneous $C(X)$-algebra. Moreover, in this case we have $K(A, C(X)) \geq r(A)$.

We also established the partial converse when:

(A) A is a homogeneous $C(X)$-algebra (our proof essentially relies on the local triviality of the underlying bundle of A).

(B) A is a subhomogeneous $C(X)$-algebra of rank 2 (our proof cannot be generalized for subhomogeneous $C(X)$-algebras of higher rank).

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Moreover, in both this cases the equality $K(A, C(X)) = r(A)$ is achieved.

As a direct consequence of part (A), we get:

Corollary

If a unital $C(X)$-algebra A admits a $C(X)$-linear embedding into some unital continuous homogeneous $C(X)$-algebra A', then $(A, C(X))$ defines a noncommutative branched covering with $K(A, C(X)) \leq K(A', C(X))$.

This leads to the following question:

Problem

If a pair $(A, C(X))$ defines a noncommutative branched covering, is it possible to embed A as a $C(X)$-subalgebra of some unital continuous homogeneous $C(X)$-algebra?

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The answer is (unfortunately) negative. In fact:

- We exhibited an example of a continuous $C(X)$-algebra A with fibres $\mathbb{M}_2$ i $\mathbb{C}$, where X is the Alexandroff compactification of the disjoint union $\bigsqcup_{n=1}^{\infty} \mathbb{C}P^n$ of complex projective n-dimensional spaces, which does not admit a $C(X)$-linear embedding into any unital continuous homogeneous $C(X)$-algebra.
- On the other hand, since A is of rank 2, the part (B) implies that the pair $(A, C(X))$ defines a noncommutative branched covering, with $K(A, C(X)) = 2$.