The local multiplier algebra of a C^*-algebra with finite-dimensional irreducible representations

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$M(A)$ is the largest unital C^*-algebra which contains A as an essential ideal.

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Definition

The local multiplier algebra *of A is the direct limit C^*-algebra*

$$M_{\mathrm{loc}}(A) := (C^*-) \varinjlim \{M(I) \ : \ I \in \mathrm{Id}_{ess}(A)\}.$$

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Iterating the construction of the local multiplier algebra one obtains the following tower of C^*-algebras which, a priori, does not have the largest element:

$$A \subseteq M_{\mathrm{loc}}(A) \subseteq M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) \subseteq \cdots$$

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- He proved that every derivation of a separable C^*-algebra A becomes inner when extended to a derivation of $M_{\mathrm{loc}}(A)$. Moreover, it suffices to assume that every essential closed ideal of A is σ-unital.
- In particular, Pedersen's result entails Sakai's theorem that every derivation of a simple unital C^*-algebra is inner.
- Since $M_{\mathrm{loc}}(A) = M(A)$ if A is simple, and $M_{\mathrm{loc}}(A) = A$ if A is an AW^*-algebra, only an affirmative answer in the non-separable case would cover, extend and unify the results that every derivation of a simple C^*-algebra is inner in its multiplier algebra and that all derivations of AW^*-algebras are inner.

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Problem 2

Is $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$ for every C^*-algebra A?

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$I(A)$ is not an injective object in the category of C^*-algebras and $*$-homomorphisms, but in the category of operator spaces and complete contractions.

However, it turns out that (nevertheless) $I(A)$ is a C^*-algebra canonically containing A as a C^*-subalgebra. Moreover, $I(A)$ is monotone complete, so in particular, $I(A)$ is an AW^*-algebra.

Theorem (Frank and Paulsen, 2003)

Under this embedding of A into $I(A)$, $M_{\rm loc}(A)$ is the norm closure of the set of all $x \in I(A)$ which act as a multiplier on some $I \in \mathrm{Id}_{ess}(A)$, i.e.

$$M_{\rm loc}(A) = \left(\bigcup_{I \in \mathrm{Id}_{ess}(A)} \{x \in I(A) \ : xI + Ix \subseteq I\} \right)^{=}$$

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$$A \subseteq M_{\mathrm{loc}}(A) \subseteq \overline{A} \subseteq I(A),$$

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  where $\overline{A}$ is the regular monotone completion of A.
- Moreover, one has $I(M_{\mathrm{loc}}(A)) = I(A)$, so we have an additional sequence of inclusions of C^*-algebras:

$$A \subseteq M_{\mathrm{loc}}(A) \subseteq M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) \subseteq \cdots \subseteq \overline{A} \subseteq I(A).$$

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- However, for general AW^*-algebras we arrive at a long standing open problem dating back to the work of Kaplansky in 1951: *Are all AW^*-algebras monotone complete*?

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- Then $M_{\mathrm{loc}}(A)$ is a commutative AW^*-algebra. In particular, $M_{\mathrm{loc}}(A)$ is injective, so

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- The maximal ideal space of $M_{\mathrm{loc}}(A) = I(A)$ can be identified with the inverse limit $\varprojlim_U \beta U$ of Stone-Čech compactifications βU of dense open subsets U of X.

Problem 2 has a negative answer

- The first class of examples of C^*-algebras for which Problem 2 has a negative answer was given by Ara and Mathieu (2006): There exist unital separable approximately finite-dimensional primitive C^*-algebras A such that $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) \neq M_{\mathrm{loc}}(A)$.

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- After that, Argerami, Farenick and Massey (2009) showed that a relatively well-behaved C^*-algebra $C([0,1]) \otimes \mathbb{K}$ also fails to satisfy $M_{\text{loc}}(M_{\text{loc}}(A)) = M_{\text{loc}}(A)$.

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- This example was further developed by Ara and Mathieu (2011), who showed that if X is a perfect, second countable LCH space, and $A = C_0(X) \otimes B$ for some non-unital separable simple C^*-algebra B, then $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) \neq M_{\mathrm{loc}}(A)$.

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We have the following partial answer:

Theorem (Somerset, 2000; Ara and Mathieu, 2011)

If A is a unital (or more generally quasi-central), separable C^*-algebra such that $\mathrm{Prim}(A)$(= the primitive ideal space of A) contains a dense G_δ subset of closed points, then $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$. Moreover, in this case $M_{\mathrm{loc}}(A)$ has only inner derivations.

On the other hand, $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$ is always a type I AW^*-algebra, whenever A is separable and liminal. More generally:

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Theorem (Somerset, 2000; Argerami and Farenick, 2005)

If the injective envelope of a C^*-algebra A is of type I, then A has a liminal essential ideal. The converse is also true if A is separable. Moreover, in this case $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$ is an AW^*-algebra of type I.

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There is also a partial converse in a non-separable direction:

Theorem (Argerami, Farenick and Massey, 2010)

If A is a spatial Fell algebra, then $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$ is an AW^*-algebra of type I.

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This result applies in particular to algebras of the form $A = C_0(X) \otimes \mathbb{K}$, for any LCH space X.

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Theorem (G., 2013)

If A belongs to **FIN***, then $M_{\mathrm{loc}}(A)$ is a finite or countable direct product of C^*-algebras of the form $C(X_n) \otimes \mathbb{M}_n$, where each space X_n is Stonean. In particular, $M_{\mathrm{loc}}(A)$ is an AW^*-algebra of type I, so it coincides with the injective envelope of A and it admits only inner derivations.*

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Recall that a space X is said to be **Stonean** if it is an extremally disconnected CH space. It is well known that a commutative C^*-algebra $A = C_0(X)$ is an AW^*-algebra if and only if X is a Stonean space.

Proof, Step 1

- We first show that every C^*-algebra in **FIN** contains an essential ideal J which can be expressed as a direct sum of a sequence (J_n) of C^*-algebras, where each J_n is either zero, or n-homogeneous (i.e. all irreducible representations of J_n are n-dimensional).

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- This reduces the problem to the homogeneous case.

Homogeneous C^*-algebras can be represented in a following way:

Theorem (Fell, 1961)

If J_n is an n-homogeneous C^*-algebra, then it is a continuous-trace C^*-algebra, and there exists a locally trivial C^*-bundle E_n over $\mathrm{Prim}(J_n)$ with fibres $\mathbb{M}_n$ such that J_n is isomorphic to the C^*-algebra $\Gamma_0(E_n)$ of all continuous sections of E_n which vanish at infinity.

Proof, Step 2

- If $J_n = \Gamma_0(E_n)$ is as above, we use Zorn's lemma to find a dense open subset $O_n \subseteq \mathrm{Prim}(J_n)$ such that the restriction bundle $E_n|_{O_n}$ is trivial.

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- Hence, $I_n := \Gamma_0(E_n|_{O_n}) \cong C_0(O_n) \otimes \mathbb{M}_n$ is an essential ideal of J_n.

Proof, Step 3

Putting all together, $\bigoplus_{n=1}^{\infty} I_n$ is an essential ideal of A, so we have

$$\begin{aligned} M_{\mathrm{loc}}(A) &= M_{\mathrm{loc}}\left(\bigoplus_{n=1}^{\infty} I_n\right) = \prod_{n=1}^{\infty} M_{\mathrm{loc}}(I_n) = \prod_{n=1}^{\infty} M_{\mathrm{loc}}(C_0(O_n)) \otimes \mathbb{M}_n \\ &= \prod_{n=1}^{\infty} C(X_n) \otimes \mathbb{M}_n, \end{aligned}$$

where X_n is the maximal ideal space of $M_{\mathrm{loc}}(C_0(O_n))$. Finally since $M_{\mathrm{loc}}(C_0(O_n))$ is a commutative AW^*-algebra for all n, each X_n is a Stonean space.