Elementary Operators and Subhomogeneous C*-algebras

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Banach Algebras 2011
August 3–10, University of Waterloo
Ontario, Canada

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2. Induced contraction θ_A^Z
3. The surjectivity problem of θ_A
4. On equality $\operatorname{Im}\theta_A = \operatorname{E}(A)$

Elementary operators

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- Note that every $\phi \in \mathrm{IB}(A)$ iz Z-(bi)modular and its norm can be computed via the formula

$$\|\phi\| = \sup\{\|\phi_P\| \ : \ P \in \mathrm{Prim}(A)\}, \tag{1}$$

where for $J \in \mathrm{Id}(A)$, ϕ_J denotes the induced operator $A/J \to A/J$, $\phi_J : a + J \mapsto \phi(a) + J$.

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  where for $J \in \mathrm{Id}(A)$, ϕ_J denotes the induced operator $A/J \to A/J$, $\phi_J : a + J \mapsto \phi(a) + J$.
- The similar formula is valid for the cb-norm of operators in $\mathrm{ICB}(A)$.

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- The finite sums of two-sided multiplication operators are known as *elementary operators*.
- The set of all elementary operators on A is denoted by $\mathrm{E}(A)$. Hence, for each $T \in \mathrm{E}(A)$ there exists a finite number of elements $a_1, \dots, a_n \in A$ and $b_1, \dots, b_n \in A$ such that

$$Tx = \left(\sum_{i=1}^{n} M_{a_i,b_i}\right)(x) = \sum_{i=1}^{n} a_i x b_i \quad (x \in A). \tag{2}$$

Canonical contraction θ_A

- If $T \in \mathrm{E}(A)$ has a representation (2), it is easy to see that one has the following estimate for its cb-norm:

$$\|T\|_{cb} \leq \left\| \sum_{i=1}^{n} a_i a_i^* \right\|^{\frac{1}{2}} \left\| \sum_{i=1}^{n} b_i^* b_i \right\|^{\frac{1}{2}} .$$

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- Hence, if we endow the algebraic tensor product $A \otimes A$ with the Haagerup norm

$$\|t\|_h := \inf \left\{ \left\| \sum_{i=1}^{n} a_i a_i^* \right\|^{\frac{1}{2}} \left\| \sum_{i=1}^{n} b_i^* b_i \right\|^{\frac{1}{2}} \ : \ t = \sum_{i=1}^{n} a_i \otimes b_i \right\},$$

we obtain the well-defined contraction

$$(A \otimes A, \| \cdot \|_h) \to (\mathrm{E}(A), \| \cdot \|_{cb}),$$

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- Clearly, the range of θ_A lies in $\mathrm{ICB}(A)$.
- The two basic questions concerning the contraction θ_A are under which conditions on A is θ_A injective or isometric?
- Clearly, if A contains a pair of non-zero orthogonal ideals then θ_A cannot be injective.
- Hence, a necessary condition for the injectivity of θ_A is that A must be a prime C^*-algebra.

When is θ_A injective or isometric?

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This result was first proved by Haagerup (1980) for the case $A = \mathrm{B}(\mathcal{H})$ ($\mathcal{H}$ is a Hilbert space). Chatterjee and Sinclair (1992) showed that θ_A is isometric if A is a separably-acting von Neumann factor. Finally, Mathieu (2003) proved the result for all prime C^*-algebras.

Using Mathieu's theorem together with the cb-version of formula (1), one obtains the following important formula for the cb-norm of $\theta_A(t)$:

$$\|\theta_A(t)\|_{cb} = \sup\{\|t^P\|_h \ : \ P \in \mathrm{Prim}(A)\}, \qquad (3)$$

where for $J \in \mathrm{Id}(A)$, t^J denotes the canonical image of t in the quotient algebra $(A \otimes_h A)/(J \otimes_h A + A \otimes_h J)$ (which is isometrically isomorphic to $(A/J) \otimes_h (A/J)$, a result due to Allen, Sinclair and Smith).

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If A has a non-trivial center (so that A is certainly not prime), one can consider the closed ideal J_A of $A \otimes_h A$ generated by the tensors of the form

$$az \otimes b - a \otimes zb \quad (a, b \in A, z \in Z),$$

(note that $J_A \subseteq \ker \theta_A$) and the induced contraction

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Definition

The Banach algebra $(A \otimes_h A)/J_A$ with the quotient norm $\|\cdot\|_{Z,h}$ is known as the central Haagerup tensor product of A, and is denoted by $A \otimes_{Z,h} A$.

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Here is a brief historical overview:

- Chatterjee and Smith (1993) first showed that θ_A^Z is isometric if A is a von Neumann algebra or if $\mathrm{Prim}(A)$ is Hausdorff.

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- Ara and Mathieu (1994) showed that θ_A^Z is isometric if A is boundedly centrally closed.
- A further generalization was obtained by Somerset (1998):

Theorem (Somerset)

(i) *The formula (3) is also valid if we replace* $\mathrm{Prim}(A)$ *by the larger set* $\mathrm{Primal}(A)$. *Hence,*

$$\|\theta_A(t)\|_{cb} = \sup\{\|t^Q\|_h \;:\; Q \in \mathrm{Primal}(A)\}.$$

(ii) $\|t\|_{Z,h} = \sup\{\|t^G\|_h \;:\; G \in \mathrm{Glimm}(A)\}$. *Hence,*

$$J_A = \bigcap\{G \otimes_h A + A \otimes_h G \;:\; G \in \mathrm{Glimm}(A)\}.$$

(iii) $Q \in \mathrm{Id}(A)$ *is 2-primal if and only if* $\ker\theta_A \subseteq Q \otimes_h A + A \otimes_h Q$, *so*

$$\ker\theta_A = \bigcap\{Q \otimes_h A + A \otimes_h Q \;:\; Q \in \mathrm{Primal}_2(A)\}. \quad (4)$$

Hence, θ_A^Z *is isometric if every Glimm ideal of* A *is primal, and* θ_A^Z *is injective if and only if every Glimm ideal of* A *is 2-primal.*

After some time, Archbold, Somerset and Timoney (2005) proved that the primality of Glimm ideals of A is also a necessary condition for θ_A^Z to be isometric, so that the isometry problem of θ_A^Z was also solved in terms of the ideal structure of A:

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Theorem (Archbold, Somerset and Timoney)

θ_A^Z is isometric if and only if every Glimm ideal of A is primal.

Glimm and primal ideals

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- The set of all Glimm ideals of A is denoted by $\mathrm{Glimm}(A)$, and is equipped with the topology from the maximal ideal space of Z, such that $\mathrm{Glimm}(A)$ is a compact Hausdorff space homeomorphic to the maximal ideal space of Z.

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- Thus, by the Dauns-Hofmann theorem we can identify Z with the C^*-algebra $C(\mathrm{Glimm}(A))$ of continuous complex valued functions on $\mathrm{Glimm}(A)$.
- For $P \in \mathrm{Prim}(A)$ let $\phi_A(P)$ be the unique Glimm ideal of A such that $\phi_A(P) \subseteq P$. The map $\phi_A : \mathrm{Prim}(A) \to \mathrm{Glimm}(A)$, $\phi_A : P \mapsto \phi_A(P)$ is continuous and is known as the *complete regularization map*.

On Glimm and primal ideals

- An ideal Q of A is said to be *n-primal* ($n \geq 2$) if whenever $J_1, \ldots, J_n$ are ideals of A with $J_1 \cdots J_n = \{0\}$, then at least one J_i is contained in Q.

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- It is not difficult to see that every 2-primal ideal contains a unique Glimm ideal.
- Also, one can show that an ideal Q of A is n-primal if for all $P_1, \dots, P_n \in \mathrm{Prim}(A/Q)$ there exists a net (P_α) in $\mathrm{Prim}(A)$ which converges to each element of $\{P_1, \dots, P_n\}$.

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- Hence, $\mathrm{Prim}(A)$ is Hausdorff if and only if

$$\mathrm{Glimm}(A) = \mathrm{Primal}_2(A) \setminus \{A\} = \mathrm{Prim}(A).$$

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On the other hand, in order to understand the structure of operators lying in $\mathrm{Im}\,\theta_A$, Magajna (2009) considered the problem of when $\mathrm{Im}\,\theta_A$ is as large as possible, hence equal to $\mathrm{ICB}(A)$. He obtained the following result:

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Theorem (Magajna)

Let A be a unital separable C^*-algebra. Then $\mathrm{Im}\,\theta_A = \mathrm{ICB}(A)$ if and only if A is a finite sum of (unital separable) homogeneous C^*-algebras. Moreover, in this case we have $\mathrm{IB}(A) = \mathrm{ICB}(A) = \mathrm{E}(A)$.

Homogeneous C^*-algebras

- Recall that (a not necessarily unital) C^*-algebra B is said to be *n-homogeneous* if its irreducible representations are of the same finite dimension n. In this case $X := \mathrm{Prim}(B)$ is a (locally compact) Hausdorff space, so its canonical C^*-bundle $\mathfrak{B}$ over X, (whose fibres are just matrix algebras $\mathrm{M}_n(\mathbb{C})$) is continuous, and moreover locally trivial (a result due to Fell (1961)).

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- If X admits a finite cover $\{U_j\}$ such that each restriction bundle $\mathfrak{B}|_{U_j}$ is trivial as a vector (resp. C^*-bundle) we say that $\mathfrak{B}$ (and hence B) is of finite type as a vector bundle (resp. C^*-bundle).

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- If X admits a finite cover $\{U_j\}$ such that each restriction bundle $\mathfrak{B}|_{U_j}$ is trivial as a vector (resp. C^*-bundle) we say that $\mathfrak{B}$ (and hence B) is of finite type as a vector bundle (resp. C^*-bundle).
- Fortunately, every continuous $\mathrm{M}_n(\mathbb{C})$-bundle is of finite type as a vector bundle if and only if it is of finite type as a C^*-bundle (a result due to Phillips (2007)).

Two remarks

- Magajna's theorem is also valid in a non-unital case, but then θ_A is defined on $M(A) \otimes_h M(A)$, and theorem then says that $\mathrm{Im}\,\theta_A = \mathrm{ICB}(A)$ if and only if A is a finite direct sum of homogeneous C^*-algebras of finite type.

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- We note that in the inseparable case the problem remains open.

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Using Somerset's description (4) of $\ker\theta_A$ and some additional calculations inside $A \otimes_h A$, we obtained the following result:

Theorem (G. 2011)

Suppose that A satisfies the equality $\operatorname{Im}\theta_A = \mathrm{E}(A)$. Then A is necessarily subhomogeneous. Moreover, if A is separable then there exists a finite number of elements $a_1, \ldots, a_n \in A$ whose canonical images linearly generate every two-primal quotient of A, i.e.

$$\mathrm{span}\{a_1 + Q, \ldots, a_n + Q\} = A/Q \quad \text{for all } Q \in \mathrm{Primal}_2(A). \tag{5}$$

- Recall, A is said to be subhomogeneous if the dimensions of its irreducible representations are bounded by some finite constant.

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- Recall, A is said to be subhomogeneous if the dimensions of its irreducible representations are bounded by some finite constant.
- The condition (5) seems to be rather technical, but it has a nice interpretation in some cases.

- For example, Phillips (2007) introduced the class of *recursively subhomogeneous C^*-algebras*, which play an important role in K-theory. In separable case, those are just subhomogeneous C^*-algebras satisfying the following condition: If

$$0 = J_0 \trianglelefteq J_1 \trianglelefteq \cdots \trianglelefteq J_n = A$$

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- We proved that a unital separable C^*-algebra A is recursively subhomogeneous if and only if there exists a finite number of elements $a_1, \ldots, a_n \in A$ whose canonical images linearly generate every primitive quotient of A.

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- We proved that a unital separable C^*-algebra A is recursively subhomogeneous if and only if there exists a finite number of elements $a_1, \ldots, a_n \in A$ whose canonical images linearly generate every primitive quotient of A.
- Since $\mathrm{Primal}_2(A)$ contains $\mathrm{Prim}(A)$, (5) implies that every unital separable C^*-algebras satisfying $\mathrm{Im}\,\theta_A = \mathrm{E}(A)$ must be recursively subhomogeneous (the converse is not true in general).

Bundles

- In order to prove the partial converse, recall that to every unital (or more generally quasi-central) C^*-algebra A one can associate the canonical upper semicontinuous C^*-bundle $\mathfrak{A}$ over $X := \operatorname{Max}(Z)$, such that $A \cong \Gamma(\mathfrak{A})$, where $\Gamma(\mathfrak{A})$ denotes the algebra of all continuous sections of $\mathfrak{A}$ (fibres of $\mathfrak{A}$ are just the Glimm quotients).

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- The similar statement is true for Hilbert $C(X)$-modules, but it is an important fact that their canonical Hilbert bundles are automatically continuous.
- Using this canonical duality between Hilbert $C(X)$-modules and continuous Hilbert bundles over X, we obtained the following result:

Theorem (G. 2011)

Let X be a compact metrizable space and let V be a Hilbert $C(X)$-module with its canonical Hilbert bundle $\mathfrak{H}$. The following conditions are equivalent:

(i) *V is topologically finitely generated, i.e. there exists a finite number of elements of V whose $C(X)$-linear span is dense in V.*

(ii) *Fibres $\mathfrak{H}_x$ of $\mathfrak{H}$ have uniformly finite dimensions, and each restriction bundle of $\mathfrak{H}$ over a set where $\dim \mathfrak{H}_x$ is constant is of finite type (as a vector bundle).*

(iii) *there exists $N \in \mathbb{N}$ such that for every Banach $C(X)$-module W, each tensor in the $C(X)$-projective tensor product $V \overset{\pi}{\otimes}_{C(X)} W$ is of (finite) rank at most N.*

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- First suppose that A is subhomogeneous and that $\mathfrak{A}$ is continuous (which is equivalent to the fact that the complete regularization map $\phi_A : \mathrm{Prim}(A) \to \mathrm{Glimm}(A)$ is open).
- In this case we proved that every Glimm ideal of A must be primal and that the dimensions of fibres of $\mathfrak{A}$ are bounded by some finite constant.

- Now, let $X_1, \ldots, X_k$ be a (necessarily finite) partition of X such that the fibers of $\mathfrak{A}|_{X_i}$ are mutually $*$-isomorphic (if $\dim A < \infty$, then A is just a finite direct sum of matrix algebras). If in addition A is separable, then using the fact that the Glimm ideals of A are primal (hence 2-primal) one can show that the condition (5) is equivalent to the fact that each restriction bundle $\mathfrak{A}|_{X_i}$ is of finite type as a vector bundle.

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- If one would know that $\mathfrak{A}|_{X_i}$ are also of finite type as C^*-bundles, then our proof would be more direct (fibres of $\mathfrak{A}|_{X_i}$ are no simple in general, so we cannot use Phillips's result on equivalence of finite type).

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- If one would know that $\mathfrak{A}|_{X_i}$ are also of finite type as C^*-bundles, then our proof would be more direct (fibres of $\mathfrak{A}|_{X_i}$ are no simple in general, so we cannot use Phillips's result on equivalence of finite type).
- Since each $\mathfrak{A}_i$ is locally trivial as a C^*-bundle, on each C^*-algebra $A_i := \Gamma_0(\mathfrak{A}_i)$ one can find a $C_0(X_i)$-valued inner product $\langle\cdot,\cdot\rangle_i$ whose induced norm $a \mapsto \|\langle a, a\rangle_i\|_\infty^{\frac{1}{2}}$ is equivalent to the C^*-norm of A_i (hence $(A_i, \langle\cdot,\cdot\rangle_i)$ is a Hilbert $C_0(X_i)$-module).

Now, using induction on k (=the cardinality of $\{X_i\}$) together with the theorem on topologically finitely generated Hilbert $C(X)$-modules, one obtains the similar result for C^*-algebras:

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Theorem (G. 2011)

Let A be a unital separable C^*-algebra, such that $\mathfrak{A}$ is continuous. The following conditions are equivalent:

(i) *A satisfies (5).*

(ii) *A as a Banach $Z = C(X)$-module is t.f.g.*

(iii) $\sup_{x\in X} \dim \mathfrak{A}_x < \infty$, *and each restriction bundle of $\mathfrak{A}$ over a set where* $\dim \mathfrak{A}_x$ *is constant is of finite type (as a vector bundle).*

(iv) *there exists $N \in \mathbb{N}$ such that for every Banach $C(X)$-module W, each tensor in the $C(X)$-projective tensor product $V \overset{\pi}{\otimes}_{C(X)} W$ is of (finite) rank at most N.*

- Finally, we use a result of Kumar and Sinclair (1998) which says that if A is a subhomogeneous C^*-algebra, then the Haagerup and projective norm on $A \otimes A$ are equivalent. Hence, $A \otimes_{Z,h} A$ and $A \overset{\pi}{\otimes}_{C(X)} A$ are isomorphic as Banach spaces.

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- As we proved, there exists $N \in \mathbb{N}$ such that each tensor $t \in A \overset{\pi}{\otimes}_{C(X)} A$ can be written in a form $t = \sum_{i=1}^{m} a_i \otimes_X b_i$, for some $m \leq N$ and $a_i, b_i \in A$, so the same conclusion holds for tensors in $A \otimes_{Z,h} A$.

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- Finally, since A is subhomogeneous, the cb-norm and the operator norm on $\mathrm{ICB}(A)$ are equivalent, so $\overline{\overline{\mathrm{E}(A)}} = \overline{\overline{\mathrm{E}(A)}}_{cb}$, and since every Glimm ideal of A is primal, Somerset's theorem implies $\overline{\overline{\mathrm{E}(A)}}_{cb} = \mathrm{Im}\,\theta_A^Z = \mathrm{Im}\,\theta_A$. Putting all together, we obtain:

Corollary

Let A be a unital separable C^*-algebra such that $\mathfrak{A}$ is continuous. The following conditions are equivalent:

(i) *A satisfies (5).*

(ii) $\overline{\overline{\mathrm{E}(A)}} = \mathrm{E}(A)$ *or* $\overline{\overline{\mathrm{E}(A)}}_{cb} = \mathrm{E}(A)$ *or* $\mathrm{Im}\,\theta_A = \mathrm{E}(A)$.

(iii) $\sup_{x\in X} \dim \mathfrak{A}_x < \infty$, *and each restriction bundle of $\mathfrak{A}$ over a set where $\dim \mathfrak{A}_x$ is constant is of finite type (as a vector bundle).*

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Problem

What can be said in a more general case, for example in a case when every Glimm ideal of A is 2-primal?

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