Elementary Operators and Subhomogeneous $C^*$-algebras

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2 Induced contraction $\theta^Z_A$

3 The surjectivity problem of $\theta_A$

4 On equality $\text{Im} \theta_A = E(A)$
Elementary operators

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- By $\text{Id}(A)$ we denote the set of all ideals of $A$ (by an ideal we mean a closed two-sided ideal) and by $\text{IB}(A)$ (resp. $\text{ICB}(A)$) the set of all bounded (resp. all completely bounded) maps on $A$ that preserve all ideals in $\text{Id}(A)$.
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- Note that every $\phi \in \text{IB}(A)$ is $Z$-(bi)modular and its norm can be computed via the formula

$$\|\phi\| = \sup\{\|\phi_P\| : P \in \text{Prim}(A)\},$$

(1)

where for $J \in \text{Id}(A)$, $\phi_J$ denotes the induced operator $A/J \to A/J$, $\phi_J : a + J \mapsto \phi(a) + J$.
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where for $J \in \text{Id}(A)$, $\phi_J$ denotes the induced operator $A/J \to A/J$, $\phi_J : a + J \mapsto \phi(a) + J$.
- The similar formula is valid for the cb-norm of operators in $\text{ICB}(A)$.
The simplest operators which lie in ICB(A) are the two-sided multiplication operators

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The finite sums of two-sided multiplication operators are known as *elementary operators*.

The set of all elementary operators on $A$ is denoted by $E(A)$. Hence, for each $T \in E(A)$ there exists a finite number of elements $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in A$ such that

$$Tx = \left( \sum_{i=1}^{n} M_{a_i, b_i} \right)(x) = \sum_{i=1}^{n} a_i xb_i \quad (x \in A). \quad (2)$$
Canonical contraction $\theta_A$

If $T \in \mathbb{E}(A)$ has a representation (2), it is easy to see that one has the following estimate for its cb-norm:

$$\| T \|_{cb} \leq \left( \sum_{i=1}^{n} a_i a_i^* \right)^{1/2} \left( \sum_{i=1}^{n} b_i^* b_i \right)^{1/2}.$$
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- Hence, if we endow the algebraic tensor product $A \otimes A$ with the Haagerup norm

$$\|t\|_h := \inf \left\{ \left\| \sum_{i=1}^{n} a_i a_i^* \right\|^{1/2} \left\| \sum_{i=1}^{n} b_i^* b_i \right\|^{1/2} : t = \sum_{i=1}^{n} a_i \otimes b_i \right\},$$

we obtain the well-defined contraction

$$(A \otimes A, \| \cdot \|_h) \to (E(A), \| \cdot \|_{cb}),$$
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- given by

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- Its continuous extension on the completed Haagerup tensor product $A \otimes_h A$ is denoted by $\theta_A$ (and this extension is known as a canonical contraction from $A \otimes_h A$ to $\text{ICB}(A)$).
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- Clearly, if $A$ contains a pair of non-zero orthogonal ideals then $\theta_A$ cannot be injective.
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- The two basic questions concerning the contraction $\theta_A$ are under which conditions on $A$ is $\theta_A$ injective or isometric?

- Clearly, if $A$ contains a pair of non-zero orthogonal ideals then $\theta_A$ cannot be injective.

- Hence, a necessary condition for the injectivity of $\theta_A$ is that $A$ must be a prime $C^*$-algebra.
When is $\theta_A$ injective or isometric?

The converse of the last statement is also true, in fact in the prime case $\theta_A$ is even isometric:
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**Theorem (Mathieu)**

The following conditions are equivalent:

(i) $A$ is prime;

(ii) $\theta_A$ is injective;

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This result was first proved by Haagerup (1980) for the case $A = B(\mathcal{H})$ ($\mathcal{H}$ is a Hilbert space). Chatterjee and Sinclair (1992) showed that $\theta_A$ is isometric if $A$ is a separably-acting von Neumann factor. Finally, Mathieu (2003) proved the result for all prime $C^*$-algebras.
Using Mathieu’s theorem together with the cb-version of formula (1), one obtains the following important formula for the cb-norm of $\theta_A(t)$:

$$\|\theta_A(t)\|_{cb} = \sup \{ \|t^P\|_h : P \in \text{Prim}(A) \},$$

where for $J \in \text{Id}(A)$, $t^J$ denotes the canonical image of $t$ in the quotient algebra $(A \otimes_h A)/(J \otimes_h A + A \otimes_h J)$ (which is isometrically isomorphic to $(A/J) \otimes_h (A/J)$, a result due to Allen, Sinclair and Smith).
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If $A$ has a non-trivial center (so that $A$ is certainly not prime), one can consider the closed ideal $J_A$ of $A \otimes_h A$ generated by the tensors of the form

$$az \otimes b - a \otimes zb \quad (a, b \in A, z \in Z),$$

(note that $J_A \subseteq \ker \theta_A$) and the induced contraction

$$\theta_A^Z : (A \otimes_h A)/J_A \to \text{ICB}(A),$$

and ask whether it is injective or isometric.
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**Definition**

The Banach algebra $(A \otimes_h A)/J_A$ with the quotient norm $\| \cdot \|_{Z,h}$ is known as the central Haagerup tensor product of $A$, and is denoted by $A \otimes_{Z,h} A$. 
Here is a brief historical overview:

- Chatterjee and Smith (1993) first showed that $\theta^Z_A$ is isometric if $A$ is a von Neumann algebra or if $\text{Prim}(A)$ is Hausdorff.
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- Chatterjee and Smith (1993) first showed that $\theta^Z_A$ is isometric if $A$ is a von Neumann algebra or if $\text{Prim}(A)$ is Hausdorff.
- Ara and Mathieu (1994) showed that $\theta^Z_A$ is isometric if $A$ is boundedly centrally closed.

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- Ara and Mathieu (1994) showed that $\theta^Z_A$ is isometric if $A$ is boundedly centrally closed.
- A further generalization was obtained by Somerset (1998):
Theorem (Somerset)

(i) The formula (3) is also valid if we replace $\text{Prim}(A)$ by the larger set $\text{Primal}(A)$. Hence,

$$\|\theta_A(t)\|_{cb} = \sup\{\|t^Q\|_h : Q \in \text{Primal}(A)\}.$$ 

(ii) \( \|t\|_{Z,h} = \sup\{\|t^G\|_h : G \in \text{Glimm}(A)\} \). Hence,

$$J_A = \bigcap\{G \otimes_h A + A \otimes_h G : G \in \text{Glimm}(A)\}.$$ 

(iii) $Q \in \text{Id}(A)$ is 2-primal if and only if $\ker \theta_A \subseteq Q \otimes_h A + A \otimes_h Q$, so

$$\ker \theta_A = \bigcap\{Q \otimes_h A + A \otimes_h Q : Q \in \text{Primal}_2(A)\}. \quad (4)$$

Hence, $\theta_A^Z$ is isometric if every Glimm ideal of $A$ is primal, and $\theta_A^Z$ is injective if and only if every Glimm ideal of $A$ is 2-primal.
After some time, Archbold, Somerset and Timoney (2005) proved that the primality of Glimm ideals of $A$ is also a necessary condition for $\theta_Z^A$ to be isometric, so that the isometry problem of $\theta_Z^A$ was also solved in terms of the ideal structure of $A$: \begin{quote} \text{Theorem (Archbold, Somerset and Timoney)} \text{ } \theta_Z^A \text{ is isometric if and only if every Glimm ideal of } A \text{ is primal.} \end{quote}
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**Theorem (Archbold, Somerset and Timoney)**

$\theta_Z^A$ is isometric if and only if every Glimm ideal of $A$ is primal.
Glimm and primal ideals

- Informally, they measure the possible topological pathologies on $\operatorname{Prim}(A)$ ($\operatorname{Prim}(A)$ is non-Hausdorff in general).
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- The set of all Glimm ideals of $A$ is denoted by $\text{Glimm}(A)$, and is equipped with the topology from the maximal ideal space of $Z$, such that $\text{Glimm}(A)$ is a compact Hausdorff space homeomorphic to the maximal ideal space of $Z$. 
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- Thus, by the Dauns-Hofmann theorem we can identify \( Z \) with the \( \mathcal{C}^* \)-algebra \( C(\text{Glimm}(A)) \) of continuous complex valued functions on \( \text{Glimm}(A) \).
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- Thus, by the Dauns-Hofmann theorem we can identify \( Z \) with the \( C^* \)-algebra \( C(\text{Glimm}(A)) \) of continuous complex valued functions on \( \text{Glimm}(A) \).
- For \( P \in \text{Prim}(A) \) let \( \phi_A(P) \) be the unique Glimm ideal of \( A \) such that \( \phi_A(P) \subseteq P \). The map \( \phi_A : \text{Prim}(A) \to \text{Glimm}(A) \), \( \phi_A : P \mapsto \phi_A(P) \) is continuous and is known as the complete regularization map.
On Glimm and primal ideals

- An ideal $Q$ of $A$ is said to be $n$-primal ($n \geq 2$) if whenever $J_1, \ldots, J_n$ are ideals of $A$ with $J_1 \cdots J_n = \{0\}$, then at least one $J_i$ is contained in $Q$. 
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- The ideal $Q$ of $A$ is said to be primal if $Q$ is $n$-primal for all $n \geq 2$.
- By $\text{Primal}_n(A)$, resp. $\text{Primal}(A)$, we denote the set of all $n$-primal, resp. all primal ideals of $A$. 
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- Also, one can show that an ideal $Q$ of $A$ is $n$-primal if for all $P_1, \ldots, P_n \in \text{Prim}(A/Q)$ there exists a net $(P_\alpha)$ in $\text{Prim}(A)$ which converges to each element of $\{P_1, \ldots, P_n\}$.
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- Hence, $\text{Prim}(A)$ is Hausdorff if and only if

$$\text{Glimm}(A) = \text{Primal}_2(A) \setminus \{A\} = \text{Prim}(A).$$
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On the other hand, in order to understand the structure of operators lying in $\text{Im} \, \theta_A$, Magajna (2009) considered the problem of when $\text{Im} \, \theta_A$ is as large as possible, hence equal to $\text{ICB}(A)$. He obtained the following result:
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**Theorem (Magajna)**

*Let $A$ be a unital separable C*-algebra. Then $\text{Im} \, \theta_A = \text{ICB}(A)$ if and only if $A$ is a finite sum of (unital separable) homogeneous C*-algebras. Moreover, in this case we have $\text{IB}(A) = \text{ICB}(A) = E(A)$.***
Homogeneous $C^*$-algebras

Recall that (a not necessarily unital) $C^*$-algebra $B$ is said to be \emph{n-homogeneous} if its irreducible representations are of the same finite dimension $n$. In this case $X := \text{Prim}(B)$ is a (locally compact) Hausdorff space, so its canonical $C^*$-bundle $\mathcal{B}$ over $X$, (whose fibres are just matrix algebras $M_n(\mathbb{C})$) is continuous, and moreover locally trivial (a result due to Fell (1961)).
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- If $X$ admits a finite cover $\{U_j\}$ such that each restriction bundle $\mathcal{B}|_{U_j}$ is trivial as a vector (resp. $C^*$-bundle) we say that $\mathcal{B}$ (and hence $B$) is of finite type as a vector bundle (resp. $C^*$-bundle).
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Fortunately, every continuous $M_n(\mathbb{C})$-bundle is of finite type as a vector bundle if and only if it is of finite type as a $C^*$-bundle (a result due to Phillips (2007)).
Two remarks

- Magajna’s theorem is also valid in a non-unital case, but then $\theta_A$ is defined on $M(A) \otimes_h M(A)$, and theorem then says that $\text{Im} \, \theta_A = \text{ICB}(A)$ if and only if $A$ is a finite direct sum of homogeneous C*-algebras of finite type.
Two remarks

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- We note that in the inseparable case the problem remains open.
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*Characterize all (unital) $C^*$-algebras $A$ for which $\text{Im} \, \theta_A$ is as small as possible, hence equal $E(A)$.***
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**Problem**

Characterize all (unital) C*-algebras $A$ for which $\text{Im} \theta_A$ is as small as possible, hence equal $E(A)$.

Using Somerset’s description (4) of $\ker \theta_A$ and some additional calculations inside $A \otimes_h A$, we obtained the following result:
Theorem (G. 2011)

Suppose that $A$ satisfies the equality $\text{Im} \, \theta_A = E(A)$. Then $A$ is necessarily subhomogeneous. Moreover, if $A$ is separable then there exists a finite number of elements $a_1, \ldots, a_n \in A$ whose canonical images linearly generate every two-primal quotient of $A$, i.e.

$$\text{span}\{a_1 + Q, \ldots, a_n + Q\} = A/Q \quad \text{for all } Q \in \text{Primal}_2(A).$$

- Recall, $A$ is said to be subhomogeneous if the dimensions of its irreducible representations are bounded by some finite constant.
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$$\text{span}\{a_1 + Q, \ldots, a_n + Q\} = A/Q \quad \text{for all } Q \in \text{Primal}_2(A).$$ (5)

- Recall, $A$ is said to be subhomogeneous if the dimensions of its irreducible representations are bounded by some finite constant.
- The condition (5) seems to be rather technical, but it has a nice interpretation in some cases.
For example, Phillips (2007) introduced the class of *recursively subhomogeneous* $C^*$-algebras, which play an important role in K-theory. In separable case, those are just subhomogeneous $C^*$-algebras satisfying the following condition: If

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is a standard composition series for $A$, then each homogeneous quotient $J_i/J_{i-1}$ is of finite type.

We proved that a unital separable $C^*$-algebra $A$ is recursively subhomogeneous if and only if there exists a finite number of elements $a_1, \ldots, a_n \in A$ whose canonical images linearly generate every primitive quotient of $A$. 

Since $\text{Primal}(A)$ contains $\text{Prim}(A)$, (5) implies that every unital separable $C^*$-algebras satisfying $\text{Im} \theta_A = E(A)$ must be recursively subhomogeneous (the converse is not true in general).
• For example, Phillips (2007) introduced the class of *recursively subhomogeneous C*-algebras*, which play an important role in K-theory. In separable case, those are just subhomogeneous C*-algebras satisfying the following condition: If

$$0 = J_0 \trianglelefteq J_1 \trianglelefteq \cdots \trianglelefteq J_n = A$$

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• We proved that a unital separable C*-algebra $A$ is recursively subhomogeneous if and only if there exists a finite number of elements $a_1, \ldots, a_n \in A$ whose canonical images linearly generate every primitive quotient of $A$.

• Since Primal$_2(A)$ contains Prim$(A)$, (5) implies that every unital separable C*-algebras satisfying Im $\theta_A = E(A)$ must be recursively subhomogeneous (the converse is not true in general).
Bundles

In order to prove the partial converse, recall that to every unital (or more generally quasi-central) $C^*$-algebra $A$ one can associate the canonical upper semicontinuous $C^*$-bundle $\mathcal{A}$ over $X := \text{Max}(Z)$, such that $A \cong \Gamma(\mathcal{A})$, where $\Gamma(\mathcal{A})$ denotes the algebra of all continuous sections of $\mathcal{A}$ (fibres of $\mathcal{A}$ are just the Glimm quotients).
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- The similar statement is true for Hilbert $C(X)$-modules, but it is an important fact that their canonical Hilbert bundles are automatically continuous.

- Using this canonical duality between Hilbert $C(X)$-modules and continuous Hilbert bundles over $X$, we obtained the following result:
Theorem (G. 2011)

Let $X$ be a compact metrizable space and let $V$ be a Hilbert $C(X)$-module with its canonical Hilbert bundle $\mathcal{H}$. The following conditions are equivalent:

(i) $V$ is topologically finitely generated, i.e. there exists a finite number of elements of $V$ whose $C(X)$-linear span is dense in $V$.

(ii) Fibres $\mathcal{H}_x$ of $\mathcal{H}$ have uniformly finite dimensions, and each restriction bundle of $\mathcal{H}$ over a set where $\dim \mathcal{H}_x$ is constant is of finite type (as a vector bundle).

(iii) there exists $N \in \mathbb{N}$ such that for every Banach $C(X)$-module $W$, each tensor in the $C(X)$-projective tensor product $V \overset{\pi}{\otimes}_{C(X)} W$ is of (finite) rank at most $N$. 
Partial converse

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- First suppose that $A$ is subhomogeneous and that $A$ is continuous (which is equivalent to the fact that the complete regularization map $\phi_A : \text{Prim}(A) \to \text{Glimm}(A)$ is open).
Partial converse

- We shall use the latter theorem in order to prove the partial converse of our theorem on $\text{Im} \, \theta_A = E(A)$.

- First suppose that $A$ is subhomogeneous and that $\mathcal{A}$ is continuous (which is equivalent to the fact that the complete regularization map $\phi_A : \text{Prim}(A) \to \text{Glimm}(A)$ is open).

- In this case we proved that every Glimm ideal of $A$ must be primal and that the dimensions of fibres of $\mathcal{A}$ are bounded by some finite constant.
Now, let $X_1, \ldots, X_k$ be a (necessarily finite) partition of $X$ such that the fibers of $\mathcal{A}|_{X_i}$ are mutually $\ast$-isomorphic (if $\dim A < \infty$, then $A$ is just a finite direct sum of matrix algebras). If in addition $A$ is separable, then using the fact that the Glimm ideals of $A$ are primal (hence 2-primal) one can show that the condition (5) is equivalent to the fact that each restriction bundle $\mathcal{A}|_{X_i}$ is of finite type as a vector bundle.
Now, let $X_1, \ldots, X_k$ be a (necessarily finite) partition of $X$ such that the fibers of $\mathcal{A}|_{X_i}$ are mutually $*$-isomorphic (if $\dim A < \infty$, then $A$ is just a finite direct sum of matrix algebras). If in addition $A$ is separable, then using the fact that the Glimm ideals of $A$ are primal (hence 2-primal) one can show that the condition (5) is equivalent to the fact that each restriction bundle $\mathcal{A}|_{X_i}$ is of finite type as a vector bundle.

If one would know that $\mathcal{A}|_{X_i}$ are also of finite type as $C^*$-bundles, then our proof would be more direct (fibres of $\mathcal{A}|_{X_i}$ are no simple in general, so we cannot use Phillips’s result on equivalence of finite type).
Now, let $X_1, \ldots, X_k$ be a (necessarily finite) partition of $X$ such that the fibers of $\mathcal{A}|_{X_i}$ are mutually $*$-isomorphic (if $\dim A < \infty$, then $A$ is just a finite direct sum of matrix algebras). If in addition $A$ is separable, then using the fact that the Glimm ideals of $A$ are primal (hence 2-primal) one can show that the condition (5) is equivalent to the fact that each restriction bundle $\mathcal{A}|_{X_i}$ is of finite type as a vector bundle.

If one would know that $\mathcal{A}|_{X_i}$ are also of finite type as $C^*$-bundles, then our proof would be more direct (fibres of $\mathcal{A}|_{X_i}$ are no simple in general, so we cannot use Phillips’s result on equivalence of finite type).

Since each $\mathcal{A}_i$ is locally trivial as a $C^*$-bundle, on each $C^*$-algebra $A_i := \Gamma_0(\mathcal{A}_i)$ one can find a $C_0(X_i)$-valued inner product $\langle \cdot, \cdot \rangle_i$ whose induced norm $a \mapsto \|\langle a, a \rangle_i\|_2$ is equivalent to the $C^*$-norm of $A_i$ (hence $(A_i, \langle \cdot, \cdot \rangle_i)$ is a Hilbert $C_0(X_i)$-module).
Now, using induction on $k (=\text{the cardinality of } \{X_i\})$ together with the theorem on topologically finitely generated Hilbert $C(X)$-modules, one obtains the similar result for $C^*$-algebras:
Now, using induction on \( k \) (=the cardinality of \( \{X_i\} \)) together with the theorem on topologically finitely generated Hilbert \( C(X) \)-modules, one obtains the similar result for \( C^* \)-algebras:

**Theorem (G. 2011)**

Let \( A \) be a unital separable \( C^* \)-algebra, such that \( \mathfrak{A} \) is continuous. The following conditions are equivalent:

(i) \( A \) satisfies (5).

(ii) \( A \) as a Banach \( Z = C(X) \)-module is t.f.g.

(iii) \( \sup_{x \in X} \dim \mathfrak{A}_x < \infty \), and each restriction bundle of \( \mathfrak{A} \) over a set where \( \dim \mathfrak{A}_x \) is constant is of finite type (as a vector bundle).

(iv) there exists \( N \in \mathbb{N} \) such that for every Banach \( C(X) \)-module \( W \), each tensor in the \( C(X) \)-projective tensor product \( \pi \otimes_{C(X)} W \) is of (finite) rank at most \( N \).
Finally, we use a result of Kumar and Sinclair (1998) which says that if $A$ is a subhomogeneous $C^*$-algebra, then the Haagerup and projective norm on $A \otimes A$ are equivalent. Hence, $A \otimes Z, h A$ and $A \otimes_{C(X)} A$ are isomorphic as Banach spaces.
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As we proved, there exists $N \in \mathbb{N}$ such that each tensor $t \in A \otimes_{C(X)} A$ can be written in a form $t = \sum_{i=1}^{m} a_i \otimes_X b_i$, for some $m \leq N$ and $a_i, b_i \in A$, so the same conclusion holds for tensors in $A \otimes Z, h A$. 

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As we proved, there exists $N \in \mathbb{N}$ such that each tensor $t \in A \pi \otimes C(X) A$ can be written in a form $t = \sum_{i=1}^{m} a_i \otimes X b_i$, for some $m \leq N$ and $a_i, b_i \in A$, so the same conclusion holds for tensors in $A \otimes \mathcal{Z}, h A$.

Finally, since $A$ is subhomogeneous, the cb-norm and the operator norm on $\text{ICB}(A)$ are equivalent, so $\overline{E(A)} = \overline{E(A)}_{cb}$, and since every Glimm ideal of $A$ is primal, Somerset’s theorem implies $\overline{E(A)}_{cb} = \text{Im} \theta_{A}^{Z} = \text{Im} \theta_{A}$. Putting all together, we obtain:
Corollary

Let $A$ be a unital separable $C^*$-algebra such that $\mathcal{A}$ is continuous. The following conditions are equivalent:

(i) $A$ satisfies (5).

(ii) $\overline{E(A)} = E(A)$ or $\overline{E(A)}_{cb} = E(A)$ or $\text{Im} \, \theta_A = E(A)$.

(iii) $\sup_{x \in X} \dim \mathcal{A}_x < \infty$, and each restriction bundle of $\mathcal{A}$ over a set where $\dim \mathcal{A}_x$ is constant is of finite type (as a vector bundle).

(iv) $A$ as a Banach $Z = C(X)$-module is t.f.g.
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(iv) $A$ as a Banach $Z = C(X)$-module is t.f.g.

Problem

What can be said in a more general case, for example in a case when every Glimm ideal of $A$ is 2-primal?
References

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References