Finitely centrally generated C^*-algebras

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Contents

Definition

A C^*-algebra is a Banach $*$-algebra A which satisfies the C^*-identity

$$\|a^*a\| = \|a\|^2, \quad \forall a \in A.$$

Example

(i) *Let $\mathcal{H}$ be a Hilbert space. The operator algebra $\mathrm{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$ with the operator norm and usual adjoint obeys the C^*-identity. If $\mathcal{H}$ is n-dimensional, we obtain that the $n \times n$ matrices $\mathrm{M}_n(\mathbb{C}) \cong \mathrm{B}(\mathbb{C}^n)$ form a C^*-algebra.*

(ii) *Let X be a locally compact Hausdorff space. The space $C_0(X)$ of complex-valued continuous functions on X that vanish at infinity form a commutative C^*-algebra $C_0(X)$ under pointwise operations, complex conjugation and supremum norm. $C_0(X)$ has a unit if and only if X is compact; in this case we usually write $C(X)$. More generally, if A is a C^*-algebra, then the set $C_0(X, A)$ of norm-continuous functions from X to A vanishing at infinity, with pointwise operations and supremum norm, is a C^*-algebra. In particular, $C_0(X, \mathrm{M}_n(\mathbb{C})) \cong \mathrm{M}_n(C_0(X)) \cong C_0(X) \otimes \mathrm{M}_n(\mathbb{C})$ is C^*-algebra.*

The morphisms in the category of C*-algebras are $*$-*homomorphisms*, that is, linear multiplicative maps which preserves adjoint. It is well known that every $*$-homomorphism $\phi : A \to B$ between C*-algebras A and B is contractive (hence bounded), and that ϕ is isometric if and only if ϕ is injective.

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- *faithful* if π is injective;
- *irreducible* if there is no closed invariant subspace apart from $\{0\}$ and $\mathcal{H}$.

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Two representations $\pi : A \to \mathrm{B}(\mathcal{H})$ and $\rho : A \to \mathrm{B}(\mathcal{H})$ are *(unitarily) equivalent* if there exists a unitary isomorphism $U : \mathcal{K} \to \mathcal{H}$ such that

$$\pi(a) = U\rho(a)U^*, \quad \forall a \in A.$$

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Let A be a C^*-algebra. Then there exists a Hilbert space $\mathcal{H}$ and a faithful representation $\pi : A \to \mathrm{B}(\mathcal{H})$.

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The basic example of an n-homogeneous C^*-algebra is the C^*-algebra $C_0(X, \mathrm{M}_n(\mathbb{C}))$, where X is a LCH space.

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More generally, if E is a locally trivial C^*-bundle over the LCH base space X with fibres $\mathrm{M}_n(\mathbb{C})$ (E is just a usual vector bundle such that the local trivializations, restricted to fibers, are isomorphisms of C^*-algebras) then the C^*-algebra $\Gamma_0(E)$ of all continuous sections vanishing at ∞ of E is n-homogeneous. In the previous example the underlying C^*-bundle E is trivial, that is $E = X \times \mathrm{M}_n(\mathbb{C})$ (with the product topology).

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Theorem (J.M.G. Fell, Acta Math., 1961)

Let A be a n-homogeneous C^*-algebra. Then there exists a locally trivial C^*-bundle E over the locally compact Hausdorff space X whose fibres are isomorphic to $\mathrm{M}_n(\mathbb{C})$ such that $A \cong \Gamma_0(E)$. In this case all irreducible representations of A are (up to a unitary equivalence) evaluations of sections of E at points of X.

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If the base space X of this bundle E admits a finite open covering (U_i) such that each $E|_{U_i}$ is trivial (as a C*-bundle), then E is said to be of *finite type* (and we shall say that in this case A is of finite type).

Each $\mathrm{M}_n(\mathbb{C})$-bundle E is also an n^2-dimensional complex vector bundle (by forgetting the additional structure). If E is of finite type (as a C*-bundle) then of course E is of finite type as a vector bundle. It is interesting (and also non-trivial) that the converse also holds. Moreover, we have the following result:

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Theorem (N.C. Phillips, TAMS, 2007)

Let X be a locally compact Hausdorff space and let E be a locally trivial $\mathrm{M}_n(\mathbb{C})$-bundle over X. Then the following conditions are equivalent:

- (i) *E is of finite type as a C^*-bundle;*
- (ii) *E is of finite type when regarded as a complex vector bundle over X by forgetting the structure;*
- (iii) *E can be extended to a locally trivial $\mathrm{M}_n(\mathbb{C})$-bundle F over the Stone-Čech compactification βX of X.*

Hence, to show that an $\mathrm{M}_n(\mathbb{C})$-bundle E is of finite type as a C*-bundle, it is sufficient to check that the underlying n^2-dimensional vector bundle is of finite type. The next standard fact gives a useful way to do this:

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Lemma

Let E be a locally trivial vector bundle of constant (finite) rank over a paracompact Hausdorff space X. The following conditions are equivalent:

(i) *E is of finite type;*

(ii) *There exists a finite number $a_1, \ldots, a_m$ of continuous bounded sections of E such that*

$$\mathrm{span}\{a_1(x), \ldots, a_m(x)\} = E(x), \quad \forall x \in X.$$

Contents

Let A be a C*-algebra. If A is non-unital, then there are several ways of embedding A in a unital C*-algebra. The *multiplier algebra* of A, denoted by $M(A)$, is a unital C*-algebra which is the largest unital C*-algebra that contains A as an ideal in a "non-degenerate" way. It is the noncommutative generalization of Stone-Čech compactification. Of course, if A is unital then $M(A) = A$.

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By $Z(A)$ we denote the center of A, that is

$$Z(A) := \{z \in A \ : \ zx = xz, \ \forall x \in A\}.$$

We consider A as a $Z(M(A))$-module, under the standard action

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Definition

A C^*-algebra A is said to be finitely centrally generated (shorter FCG) if A as a $Z(M(A))$-module is finitely generated.

Example

Let X be a CH space. Then the C^*-algebra $A := C(X, \mathrm{M}_n(\mathbb{C}))$ is FCG. Indeed, since X is compact A is unital, hence $M(A) = A$. Let $(E_{i,j})$ be the standard matrix units of $\mathrm{M}_n(\mathbb{C})$ considered as constant elements of A. Since the center of $\mathrm{M}_n(\mathbb{C})$ consists only of the scalar multiples of identity, we have (by continuity)

$$Z(A) = \{f1_n : \ f \in C(X)\} \cong C(X).$$

Then for each $a = (a_{i,j}) \in A \cong \mathrm{M}_n((C(X))$ we have $a = \sum_{i,j=1}^n (a_{i,j}1_n)E_{i,j}$, hence

$$A = \mathrm{span}_{Z(A)}\{E_{i,j} : \ 1 \leq i, j \leq n\}.$$

Example

More generally, if E is a locally trivial $\mathrm{M}_n(\mathbb{C})$-bundle over a CH base space X then the (n-homogeneous) C^*-algebra $\Gamma(E)$ is FCG. This can be seen by using the previous example together with the finite partition of unity argument.

Example

More generally, if E is a locally trivial $\mathrm{M}_n(\mathbb{C})$-bundle over a CH base space X then the (n-homogeneous) C*-algebra $\Gamma(E)$ is FCG. This can be seen by using the previous example together with the finite partition of unity argument.

Hence, by Fell's theorem, each unital homogeneous C*-algebra is FCG. Of course, the same conclusion holds for a finite direct sum of unital homogeneous C*-algebras. The converse is also true:

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Theorem (I. Gogić, PEMS, to appear)

Let A be a C^*-algebra. Then A is finitely centrally generated if and only if A is a finite direct sum of unital homogeneous C^*-algebras.

Sketch of the proof

Suppose that A is FCG. The proof of the theorem is divided in several steps:

- Using the functional calculus we first show that A must be unital. The easy consequence of this fact is that if A is FCG so is A/J for each (closed two-sided) ideal J of A.

Sketch of the proof

Suppose that A is FCG. The proof of the theorem is divided in several steps:

- Using the functional calculus we first show that A must be unital. The easy consequence of this fact is that if A is FCG so is A/J for each (closed two-sided) ideal J of A.
- Next, we show that A is subhomogeneous, that is the dimensions of irreducible representations of A are uniformly bounded by some finite constant. This is easy, suppose that

$$A = \mathrm{span}_{Z(A)}\{e_1, \ldots, e_m\}$$

for some $e_1, \ldots, e_m \in A$. Then π maps $Z(A)$ into scalars, so

$$\pi(A) = \mathrm{span}_{\mathbb{C}}\{\pi(e_1), \ldots, \pi(e_m)\} \Rightarrow \dim \pi \leq \sqrt{m} < \infty.$$

- Suppose that A is subhomogeneous of degree n (i.e. the maximal dimension of irreducible representation of A equals n) and let J be the n-homogeneous ideal of A (J is the intersection of the kernels of all irreducible representations of dimension at most $n-1$). To prove that A is a finite direct sum of unital homogeneous C*-algebras, note that it is sufficient to show that J is unital. Indeed, in this case $A \cong J \oplus (A/J)$, where A/J is FCG with the lower degree of subhomogenity.

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- Now, we show that J is of finite type. To see this, let E be a locally trivial $\mathrm{M}_n(\mathbb{C})$-bundle over the LCH base space X such that $J \cong \Gamma_0(E)$. Using the previous lemma, we see that E must be of finite type as a vector bundle, and hence, by Phillips's theorem, E is of finite type as a C*-bundle.

- Next, we reduce the proof to the case when J is essential in A (i.e. if I is any ideal of A such that $IJ = \{0\}$ then $I = \{0\}$). In this case, $A \subseteq M(J)$, and by [3] we have the equalities

$$M(J) = \Gamma_b(E) = \Gamma(F),$$

where $\Gamma_b(E)$ denotes the C*-algebra of all continuous bounded sections of E and F denotes the $\mathrm{M}_n(\mathbb{C})$-bundle over βX which extends E (such F exits by Phillips's theorem).

- Finally, to obtain a contradiction, we assume that J is non-unital so that X is non-compact. In this case it can be seen that there exits a point $s_0 \in \beta X \setminus X$, a compact neighborhood H of s_0 and an ideal I_H of $M(J)$ (which consists of all $a \in M(J)$ such that $a|_H = 0$) such that $A_H := A/(I_H \cap A)$ can be identified with a C*-subalgebra of $C(H, \mathrm{M}_n(\mathbb{C}))$ and

$$a_{1,n}|_{H \setminus U} = 0, \quad \forall a = (a_{i,j})_{1 \le i,j \le n} \in A_H,$$

where $U := X \cap H$. Note that U is a dense open subset of H, and $s_0 \notin U$. Using this fact we then show that the commutative C*-algebra $C_0(U)$ is FCG. By the first part of the proof we conclude that $C_0(U)$ must be unital, so that U is compact, hence equal to H, contradicting the fact that $s_0 \in H \setminus U$.

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