Fiberwise two-sided multiplications on homogeneous C*-algebras

Ilja Gogić

Department of Mathematics
University of Zagreb

27th International Conference in Operator Theory
Timișoara, Romania
July 2–6, 2018

joint work with Richard M. Timoney (Trinity College Dublin)

Homogeneous C^*-algebras

In the light of noncommutative topology it is natural to try to view a given unital C^*-algebra A as a set of sections of some sort of the bundle. For example, $C(X)$ is the family of sections of trivial bundle over X.

Homogeneous C^*-algebras

In the light of noncommutative topology it is natural to try to view a given unital C^*-algebra A as a set of sections of some sort of the bundle. For example, $C(X)$ is the family of sections of trivial bundle over X.

This idea in particularly works well for the following class of C^*-algebras:

Definition

*A C^*-algebra is called n-**homogeneous** if all its irreducible representations are of the same finite dimension n.*

Homogeneous C^*-algebras

In the light of noncommutative topology it is natural to try to view a given unital C^*-algebra A as a set of sections of some sort of the bundle. For example, $C(X)$ is the family of sections of trivial bundle over X.

This idea in particularly works well for the following class of C^*-algebras:

Definition

*A C^*-algebra is called n-**homogeneous** if all its irreducible representations are of the same finite dimension n.*

Theorem (Fell & Tomiyama-Takesaki)

If A is a (unital) n-homogeneous C^*-algebra, then its (primitive) spectrum X is a CH space and there is a locally trivial bundle $\mathcal{E}$ over X with fibre $\mathbb{M}_n$ and structure group $\mathrm{Aut}(\mathbb{M}_n) = PU(n) = U(n)/\mathbb{S}^1$ such that A is isomorphic to the algebra $\Gamma(\mathcal{E})$ of sections of $\mathcal{E}$.

Homogeneous C^*-algebras

In the light of noncommutative topology it is natural to try to view a given unital C^*-algebra A as a set of sections of some sort of the bundle. For example, $C(X)$ is the family of sections of trivial bundle over X.

This idea in particularly works well for the following class of C^*-algebras:

Definition

*A C^*-algebra is called n-**homogeneous** if all its irreducible representations are of the same finite dimension n.*

Theorem (Fell & Tomiyama-Takesaki)

If A is a (unital) n-homogeneous C^*-algebra, then its (primitive) spectrum X is a CH space and there is a locally trivial bundle $\mathcal{E}$ over X with fibre $\mathbb{M}_n$ and structure group $\mathrm{Aut}(\mathbb{M}_n) = PU(n) = U(n)/\mathbb{S}^1$ such that A is isomorphic to the algebra $\Gamma(\mathcal{E})$ of sections of $\mathcal{E}$. Moreover, any two such algebras $A_i = \Gamma_0(\mathcal{E}_i)$ with spectra X_i are isomorphic if and only if there is a homeomorphism $f : X_1 \to X_2$ such that $\mathcal{E}_1 \cong f^*(\mathcal{E}_2)$ as bundles over X_1.

In particular, the classification problem of n-homogeneous C^*-algebras over X is equivalent to the classification problem of $PU(n)$-bundles over X.

In particular, the classification problem of n-homogeneous C^*-algebras over X is equivalent to the classification problem of $PU(n)$-bundles over X.

From the general theory of fibre bundles we know that any topological group G admits the **universal G-bundle** $\mathrm{E}G$ over $\mathrm{B}G$ (where $\mathrm{B}G$ is the **classifying space** of G), which has the property that any G-bundle E over a CW-complex X is isomorphic to the induced G-bundle $f^*(\mathrm{E}G)$ for some continuous map $f : X \to \mathrm{B}G$.

In particular, the classification problem of n-homogeneous C^*-algebras over X is equivalent to the classification problem of $PU(n)$-bundles over X.

From the general theory of fibre bundles we know that any topological group G admits the **universal G-bundle** $\mathrm{E}G$ over $\mathrm{B}G$ (where $\mathrm{B}G$ is the **classifying space** of G), which has the property that any G-bundle E over a CW-complex X is isomorphic to the induced G-bundle $f^*(\mathrm{E}G)$ for some continuous map $f : X \to \mathrm{B}G$.
Since any two homotopic maps induce isomorphic bundles, the map $[f] \mapsto [f^*(\mathrm{E}G)]$ defines a bijection between the homotopy classes $[X, \mathrm{B}G]$ onto the isomorphism classes $\mathrm{Bun}(X, G)$ of G-bundles over X .

In particular, the classification problem of n-homogeneous C^*-algebras over X is equivalent to the classification problem of $PU(n)$-bundles over X.

From the general theory of fibre bundles we know that any topological group G admits the **universal G-bundle** $\mathrm{E}G$ over $\mathrm{B}G$ (where $\mathrm{B}G$ is the **classifying space** of G), which has the property that any G-bundle E over a CW-complex X is isomorphic to the induced G-bundle $f^*(\mathrm{E}G)$ for some continuous map $f : X \to \mathrm{B}G$.
Since any two homotopic maps induce isomorphic bundles, the map $[f] \mapsto [f^*(\mathrm{E}G)]$ defines a bijection between the homotopy classes $[X, \mathrm{B}G]$ onto the isomorphism classes $\mathrm{Bun}(X, G)$ of G-bundles over X .

The classifying space of $PU(n)$ is not that easy to describe as it is for the group $U(n)$ (=inductive limits of complex Grassmanians). Hence, the classification problem of $PU(n)$-bundles is more complex than the classification problem of (complex) vector bundles.

Algebraic characterisation of homogeneous C^*-algebras

Standard polynomial of degree k is a polynomial in k non-commuting variables $x_1, \dots, x_k$ defined by

$$s_k(x_1, \dots, x_k) := \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(k)},$$

where S_k is a symmetric group of order k.

Algebraic characterisation of homogeneous C^*-algebras

Standard polynomial of degree k is a polynomial in k non-commuting variables $x_1, \dots, x_k$ defined by

$$s_k(x_1, \dots, x_k) := \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(k)},$$

where S_k is a symmetric group of order k.

Definition

We say that a ring R satisfies the **standard identity** s_k *if for each k-tuple $(r_1, \dots, r_k)$ of elements in R we have $s_k(r_1, \dots, r_k) = 0$.*

Algebraic characterisation of homogeneous C^*-algebras

Standard polynomial of degree k is a polynomial in k non-commuting variables $x_1, \ldots, x_k$ defined by

$$s_k(x_1, \ldots, x_k) := \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(k)},$$

where S_k is a symmetric group of order k.

Definition

We say that a ring R satisfies the **standard identity** s_k *if for each k-tuple $(r_1, \ldots, r_k)$ of elements in R we have $s_k(r_1, \ldots, r_k) = 0$.*

Theorem (Amitsur-Levitzki)

If R is a unital commutative ring, then the ring $M_n(R)$ of $n \times n$ matrices over R satisfies the standard identity s_{2n}.

Definition

*We say that a unital R ring is an A_n-**ring** if:*

(i) *R satisfies the standard identity s_{2n}; and*

(ii) *No non-zero homomorphic image of R satisfies the standard identity $s_{2(n-1)}$.*

Definition

*We say that a unital R ring is an A_n-**ring** if:*

(i) *R satisfies the standard identity s_{2n}; and*

(ii) *No non-zero homomorphic image of R satisfies the standard identity $s_{2(n-1)}$.*

Corollary

A unital C^*-algebra A is an A_n-ring if and only if A is n-homogeneous.

Definition

*We say that a unital R ring is an A_n-**ring** if:*

(i) *R satisfies the standard identity s_{2n}; and*

(ii) *No non-zero homomorphic image of R satisfies the standard identity $s_{2(n-1)}$.*

Corollary

A unital C^*-algebra A is an A_n-ring if and only if A is n-homogeneous.

Definition

*A unital ring R with centre Z is said to be **Azumaya** over Z if:*

(i) *R is a finitely generated projective Z-module; and*

(ii) *The canonical homomorphism*

$$\theta : A \otimes_Z A^\circ \to \mathrm{End}_Z(R), \quad \theta(a \otimes b)(x) = axb$$

is an isomorphism.

If R iz Azumaya over Z, then R is a finitely generated projective Z-module and hence has a rank function $\mathrm{Spec}(R) \to \mathbb{N}_0$. If this function is constant then R is said to be of **constant rank**. In this case the rank of R is a perfect square.

If R iz Azumaya over Z, then R is a finitely generated projective Z-module and hence has a rank function $\mathrm{Spec}(R) \to \mathbb{N}_0$. If this function is constant then R is said to be of **constant rank**. In this case the rank of R is a perfect square.

Theorem (Artin)

A unital ring R is an A_n-ring if and only if R iz Azumaya of constant rank n^2.

If R iz Azumaya over Z, then R is a finitely generated projective Z-module and hence has a rank function $\mathrm{Spec}(R) \to \mathbb{N}_0$. If this function is constant then R is said to be of **constant rank**. In this case the rank of R is a perfect square.

Theorem (Artin)

A unital ring R is an A_n-ring if and only if R iz Azumaya of constant rank n^2.

Corollary

For a unital C^*-algebra A the following conditions are equivalent:

(i) *A is n-homogeneous.*

(ii) *A is an A_n-ring.*

(iii) *A is Azumaya of constant rank n^2.*

Theorem (G. 2011)

For a C^*-algebra A the following conditions are equivalent:

(i) *A is Azumaya.*

(ii) *A is finitely generated module over the centre of its multiplier algebra.*

(iii) *A is a finite direct sum of unital homogeneous C^*-algebras.*

Fiberwise two-sided multiplications on homogeneous C*-algebras

If A is a C^*-algebra, the important class of bounded linear maps $\phi : A \to A$ are the ones that preserve its (closed two-sided) ideals, i.e. $\phi(I) \subseteq I$ for all ideals I of A. We denote by $\mathrm{IB}(A)$ the set of all such maps.

Fiberwise two-sided multiplications on homogeneous C*-algebras

If A is a C^*-algebra, the important class of bounded linear maps $\phi : A \to A$ are the ones that preserve its (closed two-sided) ideals, i.e. $\phi(I) \subseteq I$ for all ideals I of A. We denote by $\mathrm{IB}(A)$ the set of all such maps.

- Since any ideal in a C^*-algebra is an intersection of all primitive ideals that contain it, a bounded linear map $\phi : A \to A$ lies in $\mathrm{IB}(A)$ if and only if ϕ preserves all primitive ideals of A.
- For any ideal I of A, ϕ induces a map $\phi_I : A/I \to A/I$ which sends $a + I$ to $\phi(a) + I$.

Fiberwise two-sided multiplications on homogeneous C*-algebras

If A is a C^*-algebra, the important class of bounded linear maps $\phi : A \to A$ are the ones that preserve its (closed two-sided) ideals, i.e. $\phi(I) \subseteq I$ for all ideals I of A. We denote by $\mathrm{IB}(A)$ the set of all such maps.

- Since any ideal in a C^*-algebra is an intersection of all primitive ideals that contain it, a bounded linear map $\phi : A \to A$ lies in $\mathrm{IB}(A)$ if and only if ϕ preserves all primitive ideals of A.
- For any ideal I of A, ϕ induces a map $\phi_I : A/I \to A/I$ which sends $a + I$ to $\phi(a) + I$.

The class of maps $\phi \in \mathrm{IB}(A)$ that have the simplest form are the two-sided multiplications $M_{a,b} : x \mapsto axb$, where a and b are elements of A. We denote by $\mathrm{TM}(A)$ the set of all such maps.

Problem

Suppose that A is a "well-behaved" unital C^*-algebra. If $\phi \in \mathrm{IB}(A)$ has the property that each induced map $\phi_P : A/P \to A/P$ ($P \in \mathrm{Prim}(A)$) is a two-sided multiplication of A/P, does ϕ have to be a two-sided multiplication of A?

Problem

Suppose that A is a "well-behaved" unital C^*-algebra. If $\phi \in \mathrm{IB}(A)$ has the property that each induced map $\phi_P : A/P \to A/P$ ($P \in \mathrm{Prim}(A)$) is a two-sided multiplication of A/P, does ϕ have to be a two-sided multiplication of A?

Remark

In the sequel we consider the above problem when $A = \Gamma(\mathcal{E})$ is a unital n-homogeneous algebra over $X = \mathrm{Prim}(A)$ (which is compact since A is unital), where $\mathcal{E}$ is the canonical $PU(n)$-bundle over X.

Problem

Suppose that A is a "well-behaved" unital C^*-algebra. If $\phi \in \mathrm{IB}(A)$ has the property that each induced map $\phi_P : A/P \to A/P$ ($P \in \mathrm{Prim}(A)$) is a two-sided multiplication of A/P, does ϕ have to be a two-sided multiplication of A?

Remark

In the sequel we consider the above problem when $A = \Gamma(\mathcal{E})$ is a unital n-homogeneous algebra over $X = \mathrm{Prim}(A)$ (which is compact since A is unital), where $\mathcal{E}$ is the canonical $PU(n)$-bundle over X.

Problem

Suppose that A is a "well-behaved" unital C^*-algebra. If $\phi \in \mathrm{IB}(A)$ has the property that each induced map $\phi_P : A/P \to A/P$ ($P \in \mathrm{Prim}(A)$) is a two-sided multiplication of A/P, does ϕ have to be a two-sided multiplication of A?

Remark

In the sequel we consider the above problem when $A = \Gamma(\mathcal{E})$ is a unital n-homogeneous algebra over $X = \mathrm{Prim}(A)$ (which is compact since A is unital), where $\mathcal{E}$ is the canonical $PU(n)$-bundle over X.

Definition

We say that a map $\phi \in \mathrm{IB}(A)$ is a **fiberwise two-sided multiplication** *if $\phi_x \in \mathrm{TM}(A_x)$ for all $x \in X$. The set of all such maps is denoted by $\mathrm{FTM}(A)$.*

Prposition

Let $\phi \in \mathrm{FTM}(A)$ and suppose that $\phi_{x_0} \neq 0$ for some $x_0 \in X$. Then there exists a compact neighborhood N of x_0 and $a, b \in A$ such that $a(x) \neq 0$ and $b(x) \neq 0$ for all $x \in N$ and $\phi = M_{a,b}$ modulo the ideal $I_N = \{a \in A \ : \ a(x) = 0 \text{ for all } \ x \in N\}$.

Prposition

Let $\phi \in \mathrm{FTM}(A)$ *and suppose that* $\phi_{x_0} \neq 0$ *for some* $x_0 \in X$. *Then there exists a compact neighborhood* N *of* x_0 *and* $a, b \in A$ *such that* $a(x) \neq 0$ *and* $b(x) \neq 0$ *for all* $x \in N$ *and* $\phi = M_{a,b}$ *modulo the ideal* $I_N = \{a \in A \ : \ a(x) = 0 \text{ for all } x \in N\}$.

Auxiliary notation

- $\mathrm{TM}_{\mathrm{nv}}(A) = \{\phi \in \mathrm{TM}(A) \ : \ \phi_x \neq 0 \ \forall x \in X\}$;
- $\mathrm{FTM}_{\mathrm{nv}}(A) = \{\phi \in \mathrm{FTM}(A) \ : \ \mathrm{TM}(A_x) \ni \phi_x \neq 0 \ \forall x \in X\}$.

Prposition

Let $\phi \in \mathrm{FTM}(A)$ *and suppose that* $\phi_{x_0} \neq 0$ *for some* $x_0 \in X$. *Then there exists a compact neighborhood* N *of* x_0 *and* $a, b \in A$ *such that* $a(x) \neq 0$ *and* $b(x) \neq 0$ *for all* $x \in N$ *and* $\phi = M_{a,b}$ *modulo the ideal* $I_N = \{a \in A \ : \ a(x) = 0 \text{ for all } x \in N\}$.

Auxiliary notation

- $\mathrm{TM}_{\mathrm{nv}}(A) = \{\phi \in \mathrm{TM}(A) \ : \ \phi_x \neq 0 \ \forall x \in X\}$;
- $\mathrm{FTM}_{\mathrm{nv}}(A) = \{\phi \in \mathrm{FTM}(A) \ : \ \mathrm{TM}(A_x) \ni \phi_x \neq 0 \ \forall x \in X\}$.

Theorem (G.-Timoney 2018)

To each operator $\phi \in \mathrm{FTM}_{\mathrm{nv}}(A)$ *there is a canonically associated complex line subbundle* $\mathcal{L}_\phi$ *of* $\mathcal{E}$ *such that*

$$\phi \in \mathrm{TM}_{\mathrm{nv}}(A) \quad \Longleftrightarrow \quad \mathcal{L}_\phi \text{ is a trivial bundle.}$$

Moreover, for each complex line subbundle $\mathcal{L}$ *of* $\mathcal{E}$ *there is an operator* $\phi_{\mathcal{L}} \in \mathrm{FTM}_{\mathrm{nv}}(A)$ *such that* $\mathcal{L}_{\phi_{\mathcal{L}}} = \mathcal{L}$.

As we know, the principle G-bundles over X are classified by:

- homotopy classes $[X, \mathrm{B}G]$ ($\mathrm{B}G$ is the classifying space of G).
- Čech cohomology $H^1(X; G)$ of equivalent 1-cocycles of a sheaf $\mathcal{S}$ over X, whose local groups are sections $C(U, G)$, $U \subset X$.

When we deal with (principle) complex line bundles, their structure group is $G = U(1) = \mathbb{S}^1$. In this case $\mathrm{B}G = \mathbb{C}P^\infty$ and there exists a natural isomorphisms of groups $H^1(X; G) \to \check{H}^2(X; \mathbb{Z})$.

As we know, the principle G-bundles over X are classified by:

- homotopy classes $[X, \mathrm{B}G]$ ($\mathrm{B}G$ is the classifying space of G).
- Čech cohomology $H^1(X; G)$ of equivalent 1-cocycles of a sheaf $\mathcal{S}$ over X, whose local groups are sections $C(U, G)$, $U \subset X$.

When we deal with (principle) complex line bundles, their structure group is $G = U(1) = \mathbb{S}^1$. In this case $\mathrm{B}G = \mathbb{C}P^\infty$ and there exists a natural isomorphisms of groups $H^1(X; G) \to \check{H}^2(X; \mathbb{Z})$.

In the light of previous theorem, for a homogeneous C^*-algebra $A = \Gamma(\mathcal{E})$ we define a map

$$\theta : \mathrm{FTM}_{\mathrm{nv}}(A) \to \check{H}^2(X; \mathbb{Z})$$

that sends an operator $\phi \in \mathrm{FTM}_{\mathrm{nv}}(A)$ to the corresponding class of the bundle $\mathcal{L}_\phi$ in $\check{H}^2(X; \mathbb{Z})$. Then $\theta^{-1}(0) = \mathrm{TM}_{\mathrm{nv}}(A)$ (by the latter theorem).

Corollary

If $\check{H}^2(X;\mathbb{Z}) = 0$, *then* $\mathrm{FTM}_{\mathrm{nv}}(A) = \mathrm{TM}_{\mathrm{nv}}(A)$.

Corollary

If $\check{H}^2(X;\mathbb{Z}) = 0$*, then* $\mathrm{FTM}_{\mathrm{nv}}(A) = \mathrm{TM}_{\mathrm{nv}}(A)$.

Theorem (G.-Timoney 2018)

Suppose that $\dim X \le d < \infty$*. For each* $n \ge 1$ *let* $A_n = C(X, \mathbb{M}_n)$*. If* $p := \left\lceil \sqrt{(d+1)/2} \right\rceil$*, then for every* $n \ge p$ *the mapping* $\theta : \mathrm{FTM}_{\mathrm{nv}}(A) \to \check{H}^2(X;\mathbb{Z})$ *is surjective. In particular, if* $\check{H}^2(X;\mathbb{Z}) \neq 0$*, then* $\mathrm{TM}_{\mathrm{nv}}(A_n) \subsetneq \mathrm{FTM}_{\mathrm{nv}}(A_n)$ *for all* $n \ge p$.

Corollary

If $\check{H}^2(X;\mathbb{Z}) = 0$*, then* $\mathrm{FTM}_{\mathrm{nv}}(A) = \mathrm{TM}_{\mathrm{nv}}(A)$.

Theorem (G.-Timoney 2018)

Suppose that $\dim X \le d < \infty$*. For each* $n \ge 1$ *let* $A_n = C(X, \mathbb{M}_n)$*. If* $p := \left\lceil \sqrt{(d+1)/2} \right\rceil$*, then for every* $n \ge p$ *the mapping* $\theta : \mathrm{FTM}_{\mathrm{nv}}(A) \to \check{H}^2(X;\mathbb{Z})$ *is surjective. In particular, if* $\check{H}^2(X;\mathbb{Z}) \neq 0$*, then* $\mathrm{TM}_{\mathrm{nv}}(A_n) \subsetneq \mathrm{FTM}_{\mathrm{nv}}(A_n)$ *for all* $n \ge p$*.*

Corollary

If $X = \mathbb{S}^2$ *or* $X = \mathbb{S}^1 \times \mathbb{S}^1$*, then for* $A = C(X, \mathbb{M}_n)$ *we have* $\mathrm{TM}_{\mathrm{nv}}(A) \subsetneq \mathrm{FTM}_{\mathrm{nv}}(A)$ *for all* $n \ge 2$*.*

Theorem (G.-Timoney 2108)

Let $A = \Gamma(\mathcal{E})$ be a unital homogeneous C^*-algebra with $X = \mathrm{Prim}(A)$. Consider the following two conditions:

(a) $\forall U \subset X$ *open, each complex line subbundle of* $\mathcal{E}|_U$ *is trivial.*

(b) $\mathrm{FTM}(A) = \mathrm{TM}(A)$.

Then (a) $\Rightarrow$ (b). If A is separable, then (a) and (b) are equivalent.

Theorem (G.-Timoney 2108)

Let $A = \Gamma(\mathcal{E})$ be a unital homogeneous C^*-algebra with $X = \mathrm{Prim}(A)$. Consider the following two conditions:

(a) *$\forall U \subset X$ open, each complex line subbundle of $\mathcal{E}|_U$ is trivial.*

(b) $\mathrm{FTM}(A) = \mathrm{TM}(A)$.

Then (a) $\Rightarrow$ (b). If A is separable, then (a) and (b) are equivalent.

Corollary

Suppose that $n \geq 2$.

(a) *If X is second-countable with $\dim X < 2$, or if X is (homeomorphic to) a subset of a non-compact connected 2-manifold, then $\mathrm{FTM}(A) = \mathrm{TM}(A)$.*

(b) *If X contains a nonempty open subset homeomorphic to (an open subset of) $\mathbb{R}^d$ for some $d \geq 3$, then $\mathrm{FTM}(A) \setminus \mathrm{TM}(A) \neq \emptyset$.*