Fiberwise two-sided multiplications on homogeneous C*-algebras

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XX Geometrical Seminar
Vrnjačka Banja, Serbia
May 20–23, 2018

joint work with Richard M. Timoney (Trinity College Dublin)

C^*-algebras as noncommutative topology

Definition

A (unital) C^*-**algebra** is a complex Banach $*$-algebra A whose norm $\|\cdot\|$ satisfies the C^*-identity. More precisely:

- A is a Banach algebra with identity over the field $\mathbb{C}$.
- A is equipped with an involution, i.e. a map $* : A \to A$, $a \mapsto a^*$ satisfying the properties:

$$(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*, \quad (ab)^* = b^* a^*, \quad \text{and} \quad (a^*)^* = a,$$

  for all $a, b \in A$ and $\alpha, \beta \in \mathbb{C}$.
- Norm $\|\cdot\|$ satisfies the C^*-**identity**, i.e.

$$\|a^* a\| = \|a\|^2$$

  for all $a \in A$.

Remark

The C^*-identity is a very strong requirement. For instance, together with the spectral radius formula, it implies that the C^*-norm is uniquely determined by the algebraic structure: For all $a \in A$ we have

$$\|a\|^2 = \|a^*a\| = r(a^*a) = \sup\{|\lambda| \ : \ \lambda \in \mathrm{spec}(a^*a)\}.$$

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Example

Let X be a CH (compact Hausdorff) space and let $C(X)$ be the set of all continuous complex-valued functions on X. Then $C(X)$ becomes a commutative C^*-algebra with respect to the pointwise operations, involution $f^*(x) := \overline{f(x)}$, and max-norm $\|f\|_\infty := \sup\{|f(x)| \; : \; x \in X\}$.

In fact, all unital commutative C^*-algebras arise in this fashion:

Theorem (Gelfand-Naimark)

The (contravariant) functor $X \rightsquigarrow C(X)$ defines an equivalence of categories of CH spaces (with continuous maps as morphisms) and commutative C^*-algebras (with $*$-homomorphisms as morphisms).

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In other words: By passing from the space X the function algebra $C(X)$, no information is lost. In fact, X can be recovered from $C(X)$. Thus, topological properties of X can be translated into algebraic properties of $C(X)$, and vice versa, so the theory of C^*-algebras is often thought of as **noncommutative topology**.

Basic examples

- The set $\mathbb{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ becomes a C^*-algebra with respect to the standard operations, usual adjoint and operator norm. In particular, the complex matrix algebras $\mathbb{M}_n = M_n(\mathbb{C})$ are C^*-algebras.
- In fact, every C^*-algebra can be isometrically embedded as a norm-closed self-adjoint subalgebra of $\mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (the noncommutative Gelfand-Naimark theorem).
- To every locally compact group G, one can associate a C^*-algebra $C^*(G)$. Everything about the representation theory of G is encoded in $C^*(G)$.
- The category of C^*-algebras is closed under the formation of direct products, direct sums, extensions, direct limits, pullbacks, pushouts, (some) tensor products, etc.

Homogeneous C^*-algebras

Definition

A **representation** *of a C^*-algebra A is a $*$-homomorphism $\pi : A \to \mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. A representation π is said to be* **irreducible** *if it has no nontrivial closed invariant subspaces (i.e. if $\mathcal{K}$ is a closed subspace of $\mathcal{H}$ such that $\pi(A)\mathcal{K} \subseteq \mathcal{K}$, then $\mathcal{K} = \{0\}$ or $\mathcal{K} = \mathcal{H}$).*

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Noncommutative Gelfand-Naimark theorem

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Noncommutative Gelfand-Naimark theorem

Every C^*-algebra admits an isometric representation on some Hilbert space.

Remark

Because of the previous theorem, C^*-algebras can be concretely defined as norm closed self-adjoint subalgebras of bounded operators on some Hilbert space $\mathcal{H}$.

Definition

Let A be C^*-algebra.

- *A* **primitive ideal** *of A is an ideal which is the kernel of an irreducible representation of A.*
- *The* **primitive spectrum** *of A is the set $\mathrm{Prim}(A)$ of primitive ideals of A equipped with the* **Jacobson topology***: If S is a set of primitive ideals, its closure is*

$$\overline{S} := \left\{ P \in \mathrm{Prim}(A) \ : \ P \supseteq \bigcap_{Q \in S} Q \right\}.$$

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Example

If $A = C(X)$, let $C_x(X) := \{f \in C(X) \ : \ f(x) = 0\}$ $(x \in X)$. Then $\mathrm{Prim}(C(X)) = \{C_x(X) \ : \ x \in X\}$. Moreover, the correspondence $x \mapsto C_x(X)$ defines a homeomorphism between X and $\mathrm{Prim}(C(X))$.

In the light of noncommutative topology it is natural to try to view a given unital C^*-algebra A as a set of sections of some sort of the bundle. For example, $C(X)$ is the family of sections of trivial bundle over X.

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This idea in particularly works well for the following class of C^*-algebras:

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*A C^*-algebra A is called n-**homogeneous** if $A/P \cong \mathbb{M}_n$ for every $P \in \mathrm{Prim}(A)$.*

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Theorem (Fell & Tomiyama-Takesaki)

If A is a (unital) n-homogeneous C^*-algebra, then its primitive spectrum X is a CH space and there is a locally trivial bundle $\mathcal{E}$ over X with fibre $\mathbb{M}_n$ and structure group $\mathrm{Aut}(\mathbb{M}_n) = PU(n) = U(n)/\mathbb{S}^1$ such that A is isomorphic to the algebra $\Gamma(\mathcal{E})$ of sections of $\mathcal{E}$.

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Moreover, any two such algebras $A_i = \Gamma_0(\mathcal{E}_i)$ with spectra X_i are isomorphic if and only if there is a homeomorphism $f : X_1 \to X_2$ such that $\mathcal{E}_1 \cong f^*(\mathcal{E}_2)$ as bundles over X_1.

In particular, the classification problem of n-homogeneous C^*-algebras over X is equivalent to the classification problem of $PU(n)$-bundles over X.

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From the general theory we know that any topological group G admits the **universal G-bundle** $\mathrm{E}G$ over $\mathrm{B}G$ (where $\mathrm{B}G$ is the **classifying space** of G), which has the property that any G-bundle E over a CW-complex X is isomorphic to the induced G-bundle $f^*(\mathrm{E}G)$ for some continuous map $f : X \to \mathrm{B}G$.

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Since any two homotopic maps induce isomorphic bundles, the map $[f] \mapsto [f^*(\mathrm{E}G)]$ defines a bijection between the homotopy classes $[X, \mathrm{B}G]$ onto the isomorphism classes $\mathrm{Bun}(X, G)$ of G-bundles over X .

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The classifying space of $PU(n)$ is not that easy to describe as it is for the group $U(n)$ (=inductive limits of complex Grassmanians). Hence, the classification problem of $PU(n)$-bundles is more complex than the classification problem of (complex) vector bundles.

However, if our base space X is of the form $\Sigma(Y)$ (suspension of Y) we can use the following result:

Theorem

If the group G is path-connected, then there exists a bijection between the equivalence classes of G-bundles over $X = \Sigma(Y)$ and the homotopy classes $[Y, G]$.

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In particular, since $\Sigma(\mathbb{S}^{k-1}) = \mathbb{S}^k$, we have:

Corollary

If the group G is path-connected, then there is a bijection between the equivalence classes of G-bundles over $\mathbb{S}^k$ and the elements of $(k-1)$th-homotopy group $\pi_{k-1}(G)$.

The lower homotopy groups of $G = PU(n)$ are known. In particular, putting $X = \mathbb{S}^k$, we get:

No. of isomorphism classes of n-homogeneous C^*-algebras over $\mathbb{S}^k$

	$\mathbb{M}_1$	$\mathbb{M}_2$	$\mathbb{M}_3$	$\mathbb{M}_4$	$\mathbb{M}_5$	$\mathbb{M}_6$	$\mathbb{M}_7$	$\mathbb{M}_8$	$\mathbb{M}_9$	$\mathbb{M}_{10}$
$\mathbb{S}^1$	1	1	1	1	1	1	1	1	1	1
$\mathbb{S}^2$	1	2	3	4	5	6	7	8	9	10
$\mathbb{S}^3$	1	1	1	1	1	1	1	1	1	1
$\mathbb{S}^4$	1	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$
$\mathbb{S}^5$	1	2	1	1	1	1	1	1	1	1
$\mathbb{S}^6$	1	2	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$
$\mathbb{S}^7$	1	12	6	1	1	1	1	1	1	1

We end this part of the talk with the following interesting result:

Theorem (Antonevič-Krupnik)

If $\mathcal{E}$ is any $PU(n)$-bundle over $X = \mathbb{S}^k$, then:

(i) *$\mathcal{E}$ is trivial as a vector bundle; and*

(ii) *$\mathcal{E}$ is of the form $\mathcal{E} = \mathrm{End}(\mathcal{V})$ for some n-dimensional vector bundle $\mathcal{V}$ over $\mathbb{S}^k$.*

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Problem

Which manifolds/CW-complexes X satisfy the statements (i) and/or (ii) of the above theorem?

Algebraic characterisation of homogeneous C^*-algebras

Standard polynomial of degree k is a polynomial in k non-commuting variables $x_1, \ldots, x_k$ defined by

$$s_k(x_1, \ldots, x_k) := \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(k)},$$

where S_k is a symmetric group of order k.

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Definition

We say that a ring R satisfies the **standard identity** s_k *if for each k-tuple $(r_1, \ldots, r_k)$ of elements in R we have $s_k(r_1, \ldots, r_k) = 0$.*

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Theorem (Amitsur-Levitzki)

If R is a unital commutative ring, then the ring $M_n(R)$ of $n \times n$ matrices over R satisfies the standard identity s_{2n}.

Definition

*We say that a unital R ring is an A_n-**ring** if:*

(i) *R satisfies the standard identity s_{2n}; and*

(ii) *No non-zero homomorphic image of R satisfies the standard identity $s_{2(n-1)}$.*

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Definition

*A unital ring R with centre Z is said to be **Azumaya** over Z if:*

(i) *R is a finitely generated projective Z-module; and*

(ii) *The canonical homomorphism*

$$\theta : A \otimes_Z A^\circ \to \mathrm{End}_Z(R), \quad \theta(a \otimes b)(x) = axb$$

is an isomorphism.

If R iz Azumaya over Z, then R is a finitely generated projective Z-module and hence has a rank function $\mathrm{Spec}(R) \to \mathbb{N}_0$. If this function is constant then R is said to be of **constant rank**. In this case the rank of R is a perfect square.

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Theorem (Artin)

A unital ring R is an A_n-ring if and only if R iz Azumaya of constant rank n^2.

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Theorem (Artin)

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Corollary

For a unital C^*-algebra A the following conditions are equivalent:

(i) *A is n-homogeneous.*

(ii) *A is an A_n-ring.*

(iii) *A is Azumaya of constant rank n^2.*

Theorem (G. 2011)

For a C^*-algebra A the following conditions are equivalent:

(i) *A is Azumaya.*

(ii) *A is finitely generated module over the centre of its multiplier algebra.*

(iii) *A is a finite direct sum of unital homogeneous C^*-algebras.*

Fiberwise two-sided multiplications on homogeneous C*-algebras

If A is a C^*-algebra, the important class of bounded linear maps $\phi : A \to A$ are the ones that preserve its (closed two-sided) ideals, i.e. $\phi(I) \subseteq I$ for all ideals I of A. We denote by $\mathrm{IB}(A)$ the set of all such maps.

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- Since any ideal in a C^*-algebra is an intersection of all primitive ideals that contain it, a bounded linear map $\phi : A \to A$ lies in $\mathrm{IB}(A)$ if and only if ϕ preserves all primitive ideals of A.
- For any ideal I of A, ϕ induces a map $\phi_I : A/I \to A/I$ which sends $a + I$ to $\phi(a) + I$.

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The class of maps $\phi \in \mathrm{IB}(A)$ that have the simplest form are the two-sided multiplications $M_{a,b} : x \mapsto axb$, where a and b are elements of A. We denote by $\mathrm{TM}(A)$ the set of all such maps.

Problem

Suppose that A is a "well-behaved" unital C^*-algebra. If $\phi \in \mathrm{IB}(A)$ has the property that each induced map $\phi_P : A/P \to A/P$ ($P \in \mathrm{Prim}(A)$) is a two-sided multiplication of A/P, does ϕ have to be a two-sided multiplication of A?

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Remark

In the sequel we consider the above problem when $A = \Gamma(\mathcal{E})$ is a unital n-homogeneous algebra over $X = \mathrm{Prim}(A)$ (which is compact since A is unital), where $\mathcal{E}$ is the canonical $PU(n)$-bundle over X.

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Definition

We say that a map $\phi \in \mathrm{IB}(A)$ *is a* **fiberwise two-sided multiplication** *if* $\phi_x \in \mathrm{TM}(A_x)$ *for all* $x \in X$*. The set of all such maps is denoted by* $\mathrm{FTM}(A)$*.*

Prposition

Let $\phi \in \mathrm{FTM}(A)$ *and suppose that* $\phi_{x_0} \neq 0$ *for some* $x_0 \in X$*. Then there exists a compact neighborhood* N *of* x_0 *and* $a, b \in A$ *such that* $a(x) \neq 0$ *and* $b(x) \neq 0$ *for all* $x \in N$ *and* $\phi = M_{a,b}$ *modulo the ideal* $I_N = \{a \in A \ : \ a(x) = 0 \text{ for all } x \in N\}$.

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Auxiliary notation

- $\mathrm{TM}_{\mathrm{nv}}(A) = \{\phi \in \mathrm{TM}(A) \ : \ \phi_x \neq 0 \ \forall x \in X\}$;
- $\mathrm{FTM}_{\mathrm{nv}}(A) = \{\phi \in \mathrm{FTM}(A) \ : \ \mathrm{TM}(A_x) \ni \phi_x \neq 0 \ \forall x \in X\}$.

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Theorem (G.-Timoney 2018)

To each operator $\phi \in \mathrm{FTM}_{\mathrm{nv}}(A)$ there is a canonically associated complex line subbundle $\mathcal{L}_\phi$ of $\mathcal{E}$ such that

$$\phi \in \mathrm{TM}_{\mathrm{nv}}(A) \quad \Longleftrightarrow \quad \mathcal{L}_\phi \text{ is a trivial bundle.}$$

Moreover, for each complex line subbundle $\mathcal{L}$ of $\mathcal{E}$ there is an operator $\phi_{\mathcal{L}} \in \mathrm{FTM}_{\mathrm{nv}}(A)$ such that $\mathcal{L}_{\phi_{\mathcal{L}}} = \mathcal{L}$.

As we know, the principle G-bundles over X are classified by:

- homotopy classes $[X, \mathrm{B}G]$ ($\mathrm{B}G$ is the classifying space of G).
- Čech cohomology $H^1(X; G)$ of equivalent 1-cocycles of a sheaf $\mathcal{S}$ over X, whose local groups are sections $C(U, G)$, $U \subset X$.

When we deal with (principle) complex line bundles, their structure group is $G = U(1) = \mathbb{S}^1$. In this case $\mathrm{B}G = \mathbb{C}P^\infty$ and there exists a natural isomorphisms of groups $H^1(X; G) \to \check{H}^2(X; \mathbb{Z})$.

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- homotopy classes $[X, \mathrm{B}G]$ ($\mathrm{B}G$ is the classifying space of G).
- Čech cohomology $H^1(X; G)$ of equivalent 1-cocycles of a sheaf $\mathcal{S}$ over X, whose local groups are sections $C(U, G)$, $U \subset X$.

When we deal with (principle) complex line bundles, their structure group is $G = U(1) = \mathbb{S}^1$. In this case $\mathrm{B}G = \mathbb{C}P^\infty$ and there exists a natural isomorphisms of groups $H^1(X; G) \to \check{H}^2(X; \mathbb{Z})$.

In the light of previous theorem, for a homogeneous C^*-algebra $A = \Gamma(\mathcal{E})$ we define a map

$$\theta : \mathrm{FTM}_{\mathrm{nv}}(A) \to \check{H}^2(X; \mathbb{Z})$$

that sends an operator $\phi \in \mathrm{FTM}_{\mathrm{nv}}(A)$ to the corresponding class of the bundle $\mathcal{L}_\phi$ in $\check{H}^2(X; \mathbb{Z})$. Then $\theta^{-1}(0) = \mathrm{TM}_{\mathrm{nv}}(A)$ (by the latter theorem).

Corollary

If $\check{H}^2(X;\mathbb{Z}) = 0$*, then* $\mathrm{FTM}_{\mathrm{nv}}(A) = \mathrm{TM}_{\mathrm{nv}}(A)$.

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Theorem (G.-Timoney 2018)

Suppose that $\dim X \leq d < \infty$. *For each* $n \geq 1$ *let* $A_n = C(X, \mathbb{M}_n)$. *If* $p := \left\lceil \sqrt{(d+1)/2} \right\rceil$, *then for every* $n \geq p$ *the mapping* $\theta : \mathrm{FTM}_{\mathrm{nv}}(A) \to \check{H}^2(X;\mathbb{Z})$ *is surjective. In particular, if* $\check{H}^2(X;\mathbb{Z}) \neq 0$, *then* $\mathrm{TM}_{\mathrm{nv}}(A_n) \subsetneq \mathrm{FTM}_{\mathrm{nv}}(A_n)$ *for all* $n \geq p$.

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Corollary

If $X = \mathbb{S}^2$ *or* $X = \mathbb{S}^1 \times \mathbb{S}^1$*, then for* $A = C(X, \mathbb{M}_n)$ *we have* $\mathrm{TM}_{\mathrm{nv}}(A) \subsetneq \mathrm{FTM}_{\mathrm{nv}}(A)$ *for all* $n \ge 2$*.*

Theorem (G.-Timoney 2018)

Let $A = \Gamma(\mathcal{E})$ be a unital homogeneous C^*-algebra with $X = \mathrm{Prim}(A)$. Consider the following two conditions:

(a) *$\forall U \subset X$ open, each complex line subbundle of $\mathcal{E}|_U$ is trivial.*

(b) $\mathrm{FTM}(A) = \mathrm{TM}(A)$.

Then (a) $\Rightarrow$ (b). If A is separable, then (a) and (b) are equivalent.

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Corollary

Suppose that $n \geq 2$.

(a) *If X is second-countable with $\dim X < 2$, or if X is (homeomorphic to) a subset of a non-compact connected 2-manifold, then $\mathrm{FTM}(A) = \mathrm{TM}(A)$.*

(b) *If X contains a nonempty open subset homeomorphic to (an open subset of) $\mathbb{R}^d$ for some $d \geq 3$, then $\mathrm{FTM}(A) \setminus \mathrm{TM}(A) \neq \emptyset$.*