MULTIPLICATIVE AND JORDAN MULTIPLICATIVE MAPS ON STRUCTURAL MATRIX ALGEBRAS

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Abstract. Let M_n denote the algebra of $n \times n$ complex matrices and let $\mathcal{A} \subseteq M_n$ be an arbitrary structural matrix algebra, i.e. a subalgebra of M_n that contains all diagonal matrices. We consider injective maps $\phi\colon \mathcal{A} \to M_n$ that satisfy the condition

$$\phi(X \bullet Y) = \phi(X) \bullet \phi(Y), \quad \text{for all } X, Y \in \mathcal{A},$$

where $\bullet$ is either the standard matrix multiplication $(X, Y) \mapsto XY$, or the (normalized) Jordan product $(X, Y) \mapsto \frac{1}{2}(XY + YX)$. We show that all such maps ϕ are automatically additive if and only if $\mathcal{A}$ does not contain a central rank-one idempotent. Moreover, in this case, we fully characterize the form of these maps.

1. Introduction

The interplay between the multiplicative and the additive structure of rings and algebras has been a topic of considerable interest among mathematicians. A classical result by Martindale [21, Corollary] asserts that any bijective multiplicative map from a prime ring containing a nontrivial idempotent onto an arbitrary ring must be additive and, consequently, a ring isomorphism. In the context of matrix rings $M_n(\mathcal{R})$ over a principle ideal domain $\mathcal{R}$, the structure of non-degenerate multiplicative maps $\phi : M_n(\mathcal{R}) \to M_n(\mathcal{R})$ (i.e. maps that are not zero on all zero-determinant matrices) was completely described by Jodeit and Lam in [18]. Specifically, by [18, Corollary], every bijective multiplicative map $\phi : M_n(\mathcal{R}) \to M_n(\mathcal{R})$ has the form

$$\phi(X) = T\omega(X)T^{-1}, \quad \forall X \in M_n(\mathcal{R}),$$

for some invertible matrix $T \in M_n(\mathcal{R})$ and a ring automorphism ω of $\mathcal{R}$, where $\omega(X)$ denotes the matrix in $M_n(\mathcal{R})$ obtained by applying ω entrywise to X. Moreover, in [23], Pierce demonstrated that the Jodeit-Lam characterization does not extend to matrix rings over arbitrary integral domains. More recently, in [25] Šemrl provided a comprehensive description of the (non-degenerate) multiplicative endomorphisms of matrix rings over arbitrary division rings, as well as the structure of multiplicative bijective maps of standard operator algebras (i.e. subalgebras of bounded linear maps on a complex Banach space that contain all finite-rank operators) [24].

In addition to ring homomorphisms, another important class of transformations between rings is that of Jordan homomorphisms. Specifically, a *Jordan homomorphism* between associative rings (algebras) $\mathcal{A}$ and $\mathcal{B}$ is an additive (linear) map $\phi : \mathcal{A} \to \mathcal{B}$ that satisfies the condition

$$\phi(xy + yx) = \phi(x)\phi(y) + \phi(y)\phi(x), \quad \forall x, y \in \mathcal{A}. \tag{1.1}$$

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In the case where the rings (algebras) are 2-torsion-free, this condition is equivalent to the requirement that ϕ preservers squares, meaning that

$$\phi(x^2) = \phi(x)^2, \quad \forall x \in \mathcal{A}.$$

A fundamental problem in Jordan theory, with a rich historical background, is to identify conditions on rings (algebras) $\mathcal{A}$ and $\mathcal{B}$ which ensure that any Jordan homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ (typically under additional assumptions such as surjectivity) is either multiplicative or antimultiplicative, or more generally, can be expressed as a suitable combination of such maps. For foundational results on this subject, we refer to the papers of Herstein, Jacobson-Rickart, and Smiley [14, 15, 27]. The theory of Jordan homomorphisms originates from Jordan algebras, a class of nonassociative algebras that appear in various fields, including functional analysis and the mathematical foundations of quantum mechanics. Most of the practically relevant Jordan algebras naturally arise as subalgebras of an associative real or complex algebra $\mathcal{A}$, equipped with the *(normalized) Jordan product*, given by

$$x \circ y := \frac{1}{2}(xy + yx), \quad \forall x, y \in \mathcal{A}. \tag{1.2}$$

It is clear that an additive map ϕ between algebras $\mathcal{A}$ and $\mathcal{B}$ is a Jordan homomorphism if and only if it preserves the Jordan product, i.e.

$$\phi(x \circ y) = \phi(x) \circ \phi(y), \quad \forall x, y \in \mathcal{A}. \tag{1.3}$$

In [22, Theorem 1], Molnar characterizes the bijective solutions of the functional equation (1.3), when both $\mathcal{A}$ and $\mathcal{B}$ are standard (complex) operator algebras and $\mathcal{A} \ncong \mathbb{C}$. A key consequence of this result is that such maps are automatically additive. The finite-dimensional version of Molnar's theorem (which, along with [18, Corollary], serves as the primary motivation for the present work) asserts that any bijective map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, $n \geq 2$, that satisfies (1.3) is of the form

$$\phi(X) = T\omega(X)T^{-1} \quad \text{or} \quad \phi(X) = T\omega(X)^t T^{-1}, \quad \forall X \in M_n(\mathbb{C}),$$

for some invertible matrix $T \in M_n(\mathbb{C})$ and a ring automorphism ω of $\mathbb{C}$, where $(\cdot)^t$ stands for the transposition. For additional variants and generalizations of Molnar's result, particularly those addressing the automatic additivity of bijective solutions of (1.3), we refer to [16, 17, 19, 20] and the references therein.

The purpose of this paper is to extend both [18, Corollary] and the finite-dimensional variant of [22, Theorem 1] to the setting of injective maps on *structural matrix algebras (SMAs)*. These are subalgebras of the matrix algebra $M_n(\mathbb{F})$ over a field $\mathbb{F}$ spanned by matrix units indexed by a quasi-order on the set $\{1, \ldots, n\}$. For convenience, we focus specifically on the case where $\mathbb{F}$ is the field $\mathbb{C}$ of complex numbers. A simple argument shows that SMAs are precisely subalgebras of $M_n(\mathbb{C})$ that contain all diagonal matrices (see [12, Proposition 3.1]). SMAs were originally introduced by van Wyk in [28] and, since then, they (and the closely related incidence algebras) have been the subject of extensive study, including works such as [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 26, 28]. Let us highlight that the description of the (algebra) automorphisms of SMAs was provided by Coelho in [8, Theorem C], while the description of Jordan embeddings (monomorphisms) between two SMAs in $M_n(\mathbb{C})$ was established in our recent paper [12]. The main result of the current paper, presented in Theorem 3.1, provides a characterization of SMAs $\mathcal{A} \subseteq M_n(\mathbb{C})$ with the property that any injective map $\phi : \mathcal{A} \to M_n(\mathbb{C})$, which either preserves the standard matrix multiplication or the (normalized) Jordan product (1.2), must be additive. This occurs precisely when $\mathcal{A}$ does not contain a central rank-one idempotent. Furthermore, in this case, we describe the exact form of such maps. The same conclusion holds for injective maps $\phi : \mathcal{A} \to M_n(\mathbb{C})$ satisfying (1.1) (see Remark 3.7). We conclude the paper with Example 3.8, which illustrates that Theorem 3.1 cannot be further extended to general subalgebras of $M_n(\mathbb{C})$.

2. Notation and Preliminaries

Let us now introduce some notation which will be used throughout the paper. First of all, for an arbitrary set S, by $|S|$ we denote its cardinality.

Given a unital associative complex algebra $\mathcal{A}$, by $Z(\mathcal{A})$, $\mathcal{A}^\times$ and $\mathrm{Idem}(\mathcal{A})$ we denote its centre, the group of all invertible elements and the set of all idempotents in $\mathcal{A}$, respectively. By $\circ$ we denote the (normalized) Jordan product, defined by (1.2). Note that $p \in \mathcal{A}$ is an idempotent if and only if it is a Jordan idempotent (i.e. satisfies $p \circ p = p$). For $p \in \mathrm{Idem}(\mathcal{A})$ we denote $p^\perp := 1 - p \in \mathrm{Idem}(\mathcal{A})$. Further, for $p, q \in \mathrm{Idem}(\mathcal{A})$ we write

$$p \leq q \qquad \text{if} \qquad pq = qp = p$$

and

$$p \perp q \qquad \text{if} \qquad pq = qp = 0.$$

Obviously $\leq$ constitutes a partial order on $\mathrm{Idem}(\mathcal{A})$. We have the following straightforward, yet useful lemma.

Lemma 2.1. *For $p, q \in \mathrm{Idem}(\mathcal{A})$ and an arbitrary $a \in \mathcal{A}$ we have:*
(a) $p \circ a = 0$ if and only if $pa = ap = pap = 0$.
(b) $p \circ a = a$ if and only if $pa = ap = pap = a$.
(c) $p \perp q$ if and only if $p \circ q = 0$.
(d) $p \leq q$ if and only if $p \circ q = p$.

Proof. Clearly, (a) $\implies$ (c) and (b) $\implies$ (d), so we prove only (a) and (b).

(a) If $pa = ap = 0$, then trivially $p \circ a = 0$. Conversely, $p \circ a = 0$ is equivalent to $pa + ap = 0$. Multiplying this equality from the left and right by p yields $pa = ap = -pap$. Hence,

$$0 = pa + ap = -pap \implies pa = ap = 0 = pap.$$

(b) If $pa = ap = a$, then obviously $p \circ a = a$. Conversely, $p \circ a = a$ is equivalent to $pa + ap = 2a$. Multiplying this equality from the left and right by p yields $pa = ap = pap$. Therefore,

$$a = \frac{1}{2}(pa + ap) = pap \implies pa = ap = pap = a.$$

□

Given another algebra $\mathcal{B}$, we say that a map $\psi : \mathrm{Idem}(\mathcal{A}) \to \mathrm{Idem}(\mathcal{B})$ is *orthoadditive* if

$$p \perp q \implies \psi(p+q) = \psi(p) + \psi(q), \quad \forall p, q \in \mathrm{Idem}(\mathcal{A}).$$

Let $n \in \mathbb{N}$.

- By $[n]$ we denote the set $\{1, \ldots, n\}$.
- By $M_n = M_n(\mathbb{C})$ we denote the algebra of $n \times n$ complex matrices and by $\mathcal{D}_n$ its subalgebra consisting of all diagonal matrices.
- Given a matrix $X \in M_n$, by $r(X)$ and $\mathrm{Tr}(X)$ we denote the rank and the trace of X, respectively.
- For $X, Y \in M_n$, by $X \propto Y$ we denote the fact that either $X = Y = 0$, or they are both nonzero and collinear.

- For $i, j \in [n]$, by $E_{ij} \in M_n$ we denote the standard matrix unit with 1 at the position (i, j) and 0 elsewhere. As any matrix $X = [X_{ij}]_{i,j=1}^n \in M_n$ can be understood as a map $[n]^2 \to \mathbb{C}, (i, j) \mapsto X_{ij}$, we consider its *support* $\operatorname{supp} X$ as the set of all pairs $(i, j) \in [n]^2$ such that $X_{ij} \neq 0$. Moreover, for a set $S \subseteq [n]^2$ we say that X is *supported in S* if $\operatorname{supp} X \subseteq S$.
- Given a ring endomorphism ω of $\mathbb{C}$, we use the same symbol ω to denote the induced ring endomorphism of M_n, defined by applying the function ω to each entry of the underlying matrix, i.e.
$$\omega(X) = [\omega(X_{ij})]_{i,j=1}^n, \quad \forall X = [X_{ij}]_{i,j=1}^n \in M_n.$$
- Given a binary relation ρ on $[n]$, for a fixed $i \in [n]$ by $\rho(i)$ and $\rho^{-1}(i)$ we denote its image and preimage by ρ, respectively, i.e.
$$\rho(i) = \{j \in [n] : (i, j) \in \rho\}, \qquad \rho^{-1}(i) = \{j \in [n] : (j, i) \in \rho\}.$$
We also write $\rho^\times$ for $\rho \setminus \{(1, 1), \dots, (n, n)\}$.
- By a *quasi-order* on $[n]$ we mean a reflexive and transitive binary relation on $[n]$.

Given a quasi-order ρ on $[n]$ we define the unital subalgebra of M_n by
$$\mathcal{A}_\rho := \{X \in M_n : \operatorname{supp} X \subseteq \rho\} = \operatorname{span}\{E_{ij} : (i, j) \in \rho\},$$
which we call a *structural matrix algebra (SMA) defined by the quasi-order ρ*. As already noted, structural matrix algebras are precisely the subalgebras of M_n that contain $\mathcal{D}_n$ (see [12, Proposition 3.1]). We explicitly state the following result from [12], which will be used in the proof of our main result (Theorem 3.1) on a few occasions.

Theorem 2.2 ([12, Theorem 3.4])**.** *Let $\mathcal{A}_\rho \subseteq M_n$ be an SMA and let $\mathcal{F} \subseteq \mathcal{A}_\rho$ be a commuting family of diagonalizable matrices. Then there exists $S \in \mathcal{A}_\rho^\times$ such that $S\mathcal{F}S^{-1} \subseteq \mathcal{D}_n$.*

Additionally, as in [12], given a quasi-order ρ on $[n]$, by $\approx_0$ we denote the associated binary relation on $[n]$, given by
$$i \approx_0 j \overset{\text{def}}{\iff} (i, j) \in \rho \text{ or } (j, i) \in \rho.$$
Its transitive closure is denoted by $\approx$, which forms an equivalence relation. The corresponding quotient set $[n]/\approx$ is denoted by $\mathcal{Q}$. We refer to each element $C \in \mathcal{Q}$ as a *central class* of $\mathcal{A}_\rho$, since by [12, Remark 3.3] we have
$$Z(\mathcal{A}_\rho) = \{\operatorname{diag}(\lambda_1, \dots, \lambda_n) \in \mathcal{D}_n : (\forall i, j \in [n])(i \approx j \implies \lambda_i = \lambda_j)\}.$$
Specifically, $\dim Z(\mathcal{A}_\rho) = |\mathcal{Q}|$. Further, for any subset $S \subseteq [n]$ we define the corresponding diagonal idempotent of $\mathcal{A}_\rho$ by
$$P_S := \sum_{i \in S} E_{ii}. \tag{2.1}$$
In particular, $(P_C)_{C \in \mathcal{Q}}$ is a mutually orthogonal family of idempotents in $Z(\mathcal{A}_\rho)$ such that $\sum_{C \in \mathcal{Q}} P_C = I$ (consequently, $(P_C)_{C \in \mathcal{Q}}$ is a basis for $Z(\mathcal{A}_\rho)$). For each $C \in \mathcal{Q}$ we can identify the ideal $P_C\mathcal{A}_\rho$ of $\mathcal{A}_\rho$ with the subalgebra of $M_{|C|}$ obtained from $\mathcal{A}_\rho$ by deleting all rows and columns not in C. Then $P_C\mathcal{A}_\rho$ becomes a central SMA in $M_{|C|}$ (i.e. $Z(P_C\mathcal{A}_\rho)$ consists only of the scalar multiples of the identity in $M_{|C|}$), so that $\mathcal{A}_\rho$ is isomorphic to the direct sum of central SMAs, i.e.
$$\mathcal{A}_\rho \cong \bigoplus_{C \in \mathcal{Q}} P_C\mathcal{A}_\rho. \tag{2.2}$$

We refer to this fact as the *central decomposition* of $\mathcal{A}_\rho$.

Finally, following [8], given a quasi-order ρ on $[n]$ we say that a map $g : \rho \to \mathbb{C}^\times$ is *transitive* if it satisfies

$$g(i,j)g(j,k) = g(i,k), \quad \forall (i,j),(j,k) \in \rho.$$

Every transitive map g clearly induces an (algebra) automorphism g^* of $\mathcal{A}_\rho$, defined on the basis of matrix units as

$$g^*(E_{ij}) = g(i,j)E_{ij}, \quad \forall (i,j) \in \rho. \tag{2.3}$$

3. Main result

We begin this section by stating our main result.

Theorem 3.1. *Let $\mathcal{A}_\rho \subseteq M_n$ be an SMA and let $\phi : \mathcal{A}_\rho \to M_n$ be an arbitrary injective map which satisfies*

$$\phi(X \bullet Y) = \phi(X) \bullet \phi(Y), \quad \forall X, Y \in \mathcal{A}_\rho, \tag{3.1}$$

where $\bullet$ is either the standard matrix multiplication or the (normalized) Jordan product $\circ$. The following conditions are equivalent:

(i) *$|C| \geq 2$ for all $C \in \mathcal{Q}$ (i.e. $\mathcal{A}_\rho$ does not contain a central rank-one idempotent).*

(ii) *All such maps ϕ are additive.*

(iii) *For any such map ϕ, there exists $T \in M_n^\times$, a transitive map $g : \rho \to \mathbb{C}^\times$, and for each $C \in \mathcal{Q}$ a nonzero ring endomorphism ω_C of $\mathbb{C}$ and an assignment $\dagger_C$ which is either the identity (always the case when ϕ is assumed to be multiplicative) or the transposition such that*

$$\phi(X) = Tg^*\left(\sum_{C \in \mathcal{Q}} \omega_C(P_C X)^{\dagger_C}\right)T^{-1}, \quad \forall X \in \mathcal{A}_\rho,$$

where $(P_C)_{C \in \mathcal{Q}}$ is a basis of mutually orthogonal idempotents of $Z(\mathcal{A}_\rho)$ (defined by (2.1)*).*

In proving Theorem 3.1, we shall utilize the following auxiliary facts.

Lemma 3.2. *Let $\mathcal{A}_\rho \subseteq M_n$ be an SMA. An idempotent $P \in \operatorname{Idem}(\mathcal{A}_\rho)$ is of rank $r \in [n]$ if and only if there exist mutually orthogonal rank-one idempotents $Q_1, \ldots, Q_r \in \operatorname{Idem}(\mathcal{A}_\rho)$ such that $P = Q_1 + \cdots + Q_r$. Moreover, if $S \subseteq [n]$ and $\operatorname{supp} P \subseteq S \times S$, then we can further achieve that $\operatorname{supp} Q_j \subseteq S \times S$ for all $j \in [r]$.*

Proof. We prove only the forward implication as the converse is immediate from the orthoadditivity of the rank. We focus on the second claim, as the first one follows by plugging in $S = [n]$.

Suppose therefore that $P \in \operatorname{Idem}(\mathcal{A}_\rho)$ is an idempotent of rank $r \in [n]$ supported in $S \times S$ for some $S \subseteq [n]$. We have $P \perp P_S^\perp$ (where $P_S \in \operatorname{Idem}(\mathcal{A}_\rho)$ is defined by (2.1)), so by Theorem 2.2 there exists $T \in \mathcal{A}_\rho^\times$ and diagonal idempotents $D, D' \in \operatorname{Idem}(\mathcal{A}_\rho)$ such that

$$P = TDT^{-1}, \qquad P_S^\perp = TD'T^{-1}.$$

Since $P \perp P_S^\perp$, it follows that $D \perp D'$. Set

$$Q_j := TE_{jj}T^{-1}, \quad \text{where } (j,j) \in \operatorname{supp} D$$

and note that $\{Q_j : (j,j) \in \operatorname{supp} D\}$ is a family of r mutually orthogonal rank-one idempotents summing up to P. Further, each Q_j is clearly orthogonal to $P_S^\perp$ and hence supported in $S \times S$. $\square$

Lemma 3.3. *Let ρ be a quasi-order on $[n]$ and let $\mathcal{S} \subseteq \rho^\times$ be a nonempty subset. Suppose that for each $(i,j) \in \mathcal{S}$ we have*

$$(i,k) \in \mathcal{S}, \forall k \in (\rho^\times)(i), \quad (l,j) \in \mathcal{S}, \forall l \in (\rho^\times)^{-1}(j), \quad \text{and} \quad (j,i) \in \rho^\times \implies (j,i) \in \mathcal{S}.$$

If for each $(i,j) \in \mathcal{S}$ we denote by $C \in \mathcal{Q}$ the central class which contains i and j, then we have $\rho^\times \cap (C \times C) \subseteq \mathcal{S}$.

Proof. Fix some $(i,j) \in \mathcal{S}$. We first prove that

$$(j,l) \in \mathcal{S}, \forall l \in (\rho^\times)(j) \setminus \{i\} \quad \text{and} \quad (k,i) \in \mathcal{S}, \forall k \in (\rho^\times)^{-1}(i) \setminus \{j\}.$$

Indeed, from $(j,l) \in \rho^\times$ by transitivity it follows $(i,l) \in \rho^\times$ and hence $(i,l) \in \mathcal{S}$. It follows $(j,l) \in \mathcal{S}$. The same argument works for the other case.

Now by $C \in \mathcal{Q}$ denote the central class which contains i and j. Denote

$$T := \{k \in [n] : \exists l \in [n] \text{ such that } (k,l) \in \mathcal{S} \text{ or } (l,k) \in \mathcal{S}\}.$$

In view of the assumption and the property just established, we have $k \in T$ if and only if $\mathcal{S}$ contains all pairs $(r,s) \in \rho^\times$ such that $k \in \{r,s\}$. Therefore, to prove the claim, it suffices to show that $C \subseteq T$. Let $k \in C$ be arbitrary. Since $i \in C$, by the definition of $\mathcal{Q}$ we have $i \approx k$ so there exist $m \in \mathbb{N}$ and $i_0, i_1, \ldots, i_m \in [n]$ such that

$$i = i_0 \approx_0 i_1 \approx_0 \cdots \approx_0 i_m = k.$$

Since $i_0 = i \in T$, we clearly have $i_1 \in T$. We continue inductively and conclude $k \in T$. $\square$

Lemma 3.4. *Let $\mathcal{A}_\rho \subseteq M_n$ be an SMA and let $\phi : \mathcal{A}_\rho \to M_n$ be an injective map which satisfies* (3.1)*, where $\bullet$ is either the standard matrix multiplication or the (normalized) Jordan product $\circ$. Then the following holds true.*

(a) ϕ preserves idempotents, i.e. $\phi(\mathrm{Idem}(\mathcal{A}_\rho)) \subseteq \mathrm{Idem}(M_n)$
(b) For $P, Q \in \mathrm{Idem}(\mathcal{A}_\rho)$ we have $P \le Q \implies \phi(P) \le \phi(Q)$.
(c) For each $P \in \mathrm{Idem}(\mathcal{A}_\rho)$ we have $r(\phi(P)) = r(P)$. In particular $\phi(0) = 0$ and $\phi(I) = I$.
(d) For $P, Q \in \mathrm{Idem}(\mathcal{A}_\rho)$ we have $P \perp Q \implies \phi(P) \perp \phi(Q)$.
(e) For each $P \in \mathrm{Idem}(\mathcal{A}_\rho)$ we have $\phi(P^\perp) = \phi(P)^\perp$.
(f) The restriction $\phi|_{\mathrm{Idem}(\mathcal{A}_\rho)} : \mathrm{Idem}(\mathcal{A}_\rho) \to \mathrm{Idem}(M_n)$ is orthoadditive.
(g) Suppose that $P_1, \ldots, P_r \in \mathrm{Idem}(\mathcal{A}_\rho)$ are mutually orthogonal and let $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$. Then

$$\phi\left(\sum_{j=1}^r \lambda_j P_j\right) = \sum_{j=1}^r \phi(\lambda_j P_j).$$

Proof. (a) This is clear.

(b) We have

$$\phi(P) = \phi(P \bullet Q) = \phi(P) \bullet \phi(Q), \qquad \phi(P) = \phi(Q \bullet P) = \phi(Q) \bullet \phi(P),$$

which is (by Lemma 2.1 if necessary) equivalent to $\phi(P) \le \phi(Q)$.

(c) First of all, note that for any $P, Q \in \mathrm{Idem}(M_n)$ we have that $P \le Q$ implies $r(P) \le r(Q)$ with equality if and only if $P = Q$. In view of Theorem 2.2, any $P \in \mathrm{Idem}(\mathcal{A}_\rho)$ is part of a strictly $\le$-increasing chain of idempotents

$$0 = P_0 \lneq P_1 \lneq \cdots \lneq P_n = I$$

in $\mathrm{Idem}(\mathcal{A}_\rho)$. By (b) and the injectivity of ϕ it follows

$$\phi(P_0) \lneq \phi(P_1) \lneq \cdots \lneq \phi(P_n)$$

and hence

$$r(\phi(P_0)) < r(\phi(P_1)) < \cdots < r(\phi(P_n)).$$

Clearly, for each $0 \le j \le n$ we have $r(\phi(P_j)) = j = r(P_j)$.

(d) We have

$$\phi(P) \bullet \phi(Q) = \phi(P \bullet Q) = \phi(0) = 0, \qquad \phi(Q) \bullet \phi(P) = \phi(Q \bullet P) = \phi(0) = 0$$

so (again by Lemma 2.1 if necessary) $\phi(P) \perp \phi(Q)$.

(e) In view of (c) and (d), we have that $\phi(P^\perp)$ is an idempotent orthogonal to $\phi(P)$ of rank $r(P^\perp) = r(\phi(P)^\perp)$. Consequently, $\phi(P^\perp) = \phi(P)^\perp$.

(f) Since $P \perp Q$, we have that $P + Q$ is again an idempotent and $P, Q \le P + Q$. Statements (b) and (d) imply

$$\underbrace{\phi(P), \phi(Q)}_{\text{orthogonal}} \le \phi(P + Q)$$

and hence

$$\phi(P) + \phi(Q) \le \phi(P + Q).$$

Finally, we have

$$\begin{aligned} r(\phi(P) + \phi(Q)) = r(\phi(P)) + r(\phi(Q)) &\overset{(c)}{=} r(P) + r(Q) = r(P + Q) \\ &\overset{(c)}{=} r(\phi(P + Q)), \end{aligned}$$

so equality follows.

(g) We have

$$\begin{aligned} \phi\left(\sum_{j=1}^r \lambda_j P_j\right) &= \phi\left(\left(\sum_{j=1}^r \lambda_j P_j\right) \bullet \left(\sum_{l=1}^r P_l\right)\right) = \phi\left(\sum_{j=1}^r \lambda_j P_j\right) \bullet \phi\left(\sum_{l=1}^r P_l\right) \\ &\overset{(f)}{=} \phi\left(\sum_{j=1}^r \lambda_j P_j\right) \bullet \left(\sum_{l=1}^r \phi(P_l)\right) = \sum_{l=1}^r \left(\phi\left(\sum_{j=1}^r \lambda_j P_j\right) \bullet \phi(P_l)\right) \\ &= \sum_{l=1}^r \phi\left(\left(\sum_{j=1}^r \lambda_j P_j\right) \bullet P_l\right) = \sum_{l=1}^r \phi(\lambda_l P_l). \end{aligned}$$

□

Proof of Theorem 3.1. **(iii) $\implies$ (ii)** This is obvious.

(ii) $\implies$ (i) Suppose that (i) is not true. In the context of the central decomposition (2.2) of SMAs, this precisely means that $\mathcal{A}_\rho$ contains a central summand isomorphic to $\mathbb{C}$. Denote by $\omega : \mathbb{C} \to \mathbb{C}$ some injective multiplicative function which is not additive (see e.g. [22]). Then one can construct a map $\mathcal{A}_\rho \to \mathcal{A}_\rho$ which acts componentwise as the map ω on all one-dimensional central summands of $\mathcal{A}_\rho$, and as the identity map on all other central summands. Such a map is clearly multiplicative and preserves the Jordan product, but is not additive.

(i) $\implies$ (iii) This implication is the core of the theorem and its proof will be divided into several steps. First of all, as $E_{11}, \ldots, E_{nn}$ is a mutually orthogonal family of rank-one idempotents in $\mathcal{A}_\rho$, by Lemma 3.4 the same is true for idempotents $\phi(E_{11}), \dots, \phi(E_{nn})$ in M_n. Hence, without loss of generality we can therefore assume that

$$\phi(E_{jj}) = E_{jj}, \quad \forall j \in [n].$$

Then, by the orthoadditivity of ϕ on $\operatorname{Idem}(\mathcal{A}_\rho)$ (Lemma 3.4 (f)), we also have

$$\phi(P) = P, \quad \forall P \in \mathcal{D}_n \cap \operatorname{Idem}(\mathcal{A}_\rho). \tag{3.2}$$

Claim 1. *Let $S \subseteq [n]$ and suppose that a matrix $X \in \mathcal{A}_\rho$ satisfies $\operatorname{supp} X \subseteq S \times S$. Then $\phi(X)$ satisfies the same property.*

Consider the diagonal idempotent $P := P_S^\perp \in \operatorname{Idem}(\mathcal{A}_\rho)$ (where $P_S \in \operatorname{Idem}(\mathcal{A}_\rho)$ is defined by (2.1)). Note that a matrix X is supported in $S \times S$ if and only if $XP = PX = 0$. Then obviously $X \bullet P = P \bullet X = 0$, so

$$0 = \phi(X \bullet P) = \phi(X) \bullet \phi(P) \overset{(3.2)}{=} \phi(X) \bullet P$$

and similarly $0 = P \bullet \phi(X)$ which together (by Lemma 2.1 (a)) imply the claim. $\diamondsuit$

Claim 2. *For each central class $C \in \mathcal{Q}$, there exists a unique injective map $\omega_C : \mathbb{C} \to \mathbb{C}$ such that*

$$\phi(\lambda X) = \omega_C(\lambda)\phi(X), \quad \forall \lambda \in \mathbb{C} \text{ and } X \in \mathcal{A}_\rho \text{ with } \operatorname{supp} X \subseteq C \times C. \tag{3.3}$$

Let $P \in \operatorname{Idem}(\mathcal{A}_\rho)$ be a rank-one idempotent and let $\lambda \in \mathbb{C}^\times$. Then

$$\phi(\lambda P) = \phi((\lambda P) \bullet P) = \phi(\lambda P) \bullet \phi(P), \qquad \phi(\lambda P) = \phi(P \bullet (\lambda P)) = \phi(P) \bullet \phi(\lambda P)$$

In view of Lemma 2.1 (b) we have

$$\phi(\lambda P) = \phi(P)\phi(\lambda P)\phi(P).$$

Since $\phi(\lambda P) \neq 0$ by injectivity, it follows that $\phi(\lambda P)$ has rank one, and shares the same image and kernel as $\phi(P)$ so we conclude $\phi(\lambda P) \propto \phi(P)$. Since $\phi(0) = 0$, it follows that there exists a map $\omega^P : \mathbb{C} \to \mathbb{C}$ such that

$$\phi(\lambda P) = \omega^P(\lambda)\phi(P), \quad \forall \lambda \in \mathbb{C}.$$

Note that if $P, Q \in \operatorname{Idem}(\mathcal{A}_\rho)$ are two rank-one idempotents, we have

$$P \not\perp Q \implies \omega^P = \omega^Q. \tag{3.4}$$

Namely, supposing that $P \bullet Q \neq 0$, we have

$$\phi(P) \bullet \phi(Q) = \phi(P \bullet Q) \neq 0$$

and hence for each $\lambda \in \mathbb{C}$ we have

$$\begin{aligned} \omega^Q(\lambda)(\phi(P) \bullet \phi(Q)) &= \phi(P) \bullet \phi(\lambda Q) = \phi(P \bullet (\lambda Q)) \\ &= \phi((\lambda P) \bullet Q) = \phi(\lambda P) \bullet \phi(Q) \\ &= \omega^P(\lambda)(\phi(P) \bullet \phi(Q)), \end{aligned}$$

so it follows $\omega^Q = \omega^P$. The next objective is to show that for each $C \in \mathcal{Q}$, the map ω^P is in fact the same for all rank-one idempotents $P \in \operatorname{Idem}(\mathcal{A}_\rho)$ supported in $C \times C$. As a first step, note that for any two distinct $i, j \in [n]$ we have

$$i \approx j \implies \omega^{E_{ii}} = \omega^{E_{jj}}.$$

Indeed, by transitivity it suffices to prove this assuming $i \approx_0 j$, i.e. when $(i,j) \in \rho$ or $(j,i) \in \rho$. For concreteness assume the former. Then the rank-one idempotent $E_{ii} + E_{ij} \in \operatorname{Idem}(\mathcal{A}_\rho)$ satisfies

$$E_{ii} \not\perp E_{ii} + E_{ij} \not\perp E_{jj} \overset{(3.4)}{\Longrightarrow} \omega^{E_{ii}} = \omega^{E_{ii}+E_{ij}} = \omega^{E_{jj}}.$$

Fix now some $C \in \mathcal{Q}$ and let $P \in \operatorname{Idem}(\mathcal{A}_\rho)$ be an arbitrary rank-one idempotent such that $\operatorname{supp} P \subseteq C \times C$. As $\operatorname{Tr}(P) = 1$, we can pick some $(i,i) \in \operatorname{supp} P$. We have

$$P \not\perp E_{ii} \overset{(3.4)}{\Longrightarrow} \omega^P = \omega^{E_{ii}}.$$

So indeed we see that the map $P \mapsto \omega^P$ is constant on the set of all rank-one idempotents of $\mathcal{A}_\rho$ supported in $C \times C$, and will henceforth be denoted by ω_C.

Now we prove (3.3). First of all, if X is an idempotent, then (3.3) follows from the fact that X can be decomposed as a sum of mutually orthogonal rank-one idempotents supported in $C \times C$ (Lemma 3.2) and then apply the orthoadditivity of ϕ on $\operatorname{Idem}(\mathcal{A}_\rho)$. For general X, let $P_C \in \operatorname{Idem}(\mathcal{A}_\rho)$ be the corresponding central idempotent (defined by (2.1)). We have

$$\begin{aligned}\phi(\lambda X) &= \phi(X \bullet (\lambda P_C)) = \phi(X) \bullet \phi(\lambda P_C) = \omega_C(\lambda)\phi(X) \bullet \phi(P_C)\\ &= \omega_C(\lambda)\phi(X \bullet P_C) = \omega_C(\lambda)\phi(X),\end{aligned}$$

which implies (3.3). The injectivity of ϕ clearly implies the injectivity of ω_C. $\diamondsuit$

Claim 3. *For each central class $C \in \mathcal{Q}$, the map ω_C is multiplicative.*

Fix $C \in \mathcal{Q}$, an arbitrary $i \in C$ and $\lambda, \mu \in \mathbb{C}$. By Claim 2 we have

$$\begin{aligned}\omega_C(\lambda\mu)E_{ii} &= \phi((\lambda\mu)E_{ii}) = \phi((\lambda E_{ii}) \bullet (\mu E_{ii})) = \phi(\lambda E_{ii}) \bullet \phi(\mu E_{ii})\\ &= \omega_C(\lambda)\omega_C(\mu)E_{ii},\end{aligned}$$

which implies $\omega_C(\lambda\mu) = \omega_C(\lambda)\omega_C(\mu)$, as desired. $\diamondsuit$

Claim 4. *Fix a central class $C \in \mathcal{Q}$. Then*

$$\phi(E_{ij}) \propto E_{ij}, \quad \forall (i,j) \in \rho \cap (C \times C)$$

(always the case when ϕ is assumed to be multiplicative) or

$$\phi(E_{ij}) \propto E_{ji}, \quad \forall (i,j) \in \rho \cap (C \times C).$$

Fix some $(i,j) \in \rho^\times \cap (C \times C)$. By Claim 1, we have $\operatorname{supp} \phi(E_{ij}) \subseteq \{i,j\} \times \{i,j\}$ so denote

$$\phi(E_{ij}) = \sum_{(r,s) \in \{i,j\} \times \{i,j\}} \alpha_{rs} E_{rs}, \quad \alpha_{rs} \in \mathbb{C}$$

On one hand we have

$$0 = \phi(E_{ij} \bullet E_{ij}) = \phi(E_{ij}) \bullet \phi(E_{ij}) = \phi(E_{ij})^2, \tag{3.5}$$

and on the other

$$\begin{aligned}\omega_C\left(\frac{1}{2}\right)\phi(E_{ij}) &\overset{\text{Claim 2}}{=} \phi\left(\frac{1}{2}E_{ij}\right) = \phi(E_{ii} \circ E_{ij}) \overset{(3.2)}{=} E_{ii} \circ \phi(E_{ij})\\ &= \frac{1}{2}\alpha_{ij}E_{ij} + \frac{1}{2}\alpha_{ji}E_{ji} + \alpha_{ii}E_{ii}\end{aligned}$$

(if ϕ preserves $\circ$) or

$$\phi\left(E_{ij}\right) = \phi(E_{ii}E_{ij}) \overset{(3.2)}{=} E_{ii}\phi(E_{ij}) = \alpha_{ij}E_{ij} + \alpha_{ii}E_{ii}$$

(if ϕ is multiplicative). In either case, by (3.5) it ultimately follows $\alpha_{ii} = \alpha_{jj} = 0$ and (by injectivity)

$$\phi(E_{ij}) \propto \begin{cases} E_{ij} \text{ or } E_{ji}, & \text{if } \bullet = \circ \\ E_{ij}, & \text{if } \bullet = \cdot. \end{cases}$$

It remains to prove that in the case $\bullet = \circ$, the same option happens for each pair in $\rho \cap (C \times C)$. For the sake of concreteness, assume that $\phi(E_{ij}) \propto E_{ij}$. If $(j,i) \in \rho$, then clearly $\phi(E_{ji}) \propto E_{ji}$ as otherwise

$$\phi\left(\frac{1}{2}(E_{ii} + E_{jj})\right) = \phi(E_{ij} \circ E_{ji}) = \phi(E_{ij}) \circ \phi(E_{ji}) \propto E_{ij} \circ E_{ij} = 0$$

is a contradiction. Next we show that $\phi(E_{ik}) \propto E_{ik}$ for any $k \in (\rho^\times)(i)$, and $\phi(E_{lj}) \propto E_{lj}$ for any $l \in (\rho^\times)^{-1}(j)$.

– Suppose that $k \in (\rho^\times)(i) \setminus \{j\}$ and that $\phi(E_{ik}) \propto E_{ki}$. Then

$$0 = \phi(E_{ij} \circ E_{ik}) = \phi(E_{ij}) \circ \phi(E_{ik}) \propto E_{ij} \circ E_{ki} = \frac{1}{2}E_{kj}$$

is a contradiction so it must be $\phi(E_{ik}) \propto E_{ik}$.

– Suppose that $l \in (\rho^\times)^{-1}(j) \setminus \{i\}$ and that $\phi(E_{lj}) \propto E_{jl}$. Then

$$0 = \phi(E_{ij} \circ E_{lj}) = \phi(E_{ij}) \circ \phi(E_{lj}) \propto E_{ij} \circ E_{jl} = \frac{1}{2}E_{il}$$

is a contradiction so it must be $\phi(E_{lj}) \propto E_{lj}$.

Now we can apply Lemma 3.3 to the set

$$\mathcal{S} := \{(r,s) \in \rho^\times : \phi(E_{rs}) \propto E_{rs}\}.$$

Since $(i,j) \in \mathcal{S}$, the set $\mathcal{S}$ contains all $(r,s) \in \rho^\times \cap (C \times C)$, which proves the claim. ♢

In view of Claim 4, we can define a map $g : \rho \to \mathbb{C}^\times$ which to a pair $(i,j) \in \rho$ assigns a scalar $g(i,j)$ such that

$$\phi(E_{ij}) = g(i,j)E_{ij} \qquad \text{or} \qquad \phi(E_{ij}) = g(i,j)E_{ji}$$

(we know that $g(i,j) = 1$ if $i = j$, and otherwise exactly one option is true).

Claim 5. *For each central class $C \in \mathcal{Q}$, the map ω_C is additive.*

Let $C \in \mathcal{Q}$. By invoking Claim 4, without losing generality we can assume that

$$\phi(E_{ij}) = g(i,j)E_{ij}, \quad \forall (i,j) \in \rho \cap (C \times C).$$

Since $|C| \geq 2$, there exists some $(i,j) \in \rho^\times \cap (C \times C)$. For fixed $x, y \in \mathbb{C}$ consider the idempotents

$$E_{ii} + xE_{ij}, E_{jj} + yE_{ij} \in \operatorname{Idem}(\mathcal{A}_\rho).$$

By Claim 1 we have

$$\operatorname{supp} \phi(E_{ii} + xE_{ij}) \subseteq \{i,j\} \times \{i,j\}.$$

Denote

$$\phi(E_{ii} + xE_{ij}) = \sum_{(r,s) \in \{i,j\} \times \{i,j\}} \alpha_{rs} E_{rs}, \quad \alpha_{rs} \in \mathbb{C}.$$

Suppose that ϕ preserves $\circ$. We have

$$\begin{aligned}\omega_C\left(\frac{1}{2}x\right)g(i,j)E_{ij} &\overset{\text{Claim 2}}{=} \phi\left(\frac{1}{2}xE_{ij}\right) = \phi((E_{ii}+xE_{ij})\circ E_{jj}) \overset{(3.2)}{=} \phi(E_{ii}+xE_{ij})\circ E_{jj} \\ &= \frac{1}{2}\alpha_{ij}E_{ij} + \frac{1}{2}\alpha_{ji}E_{ji} + \alpha_{jj}E_{jj}.\end{aligned}$$

Since $\phi(E_{ii}+xE_{ij})$ is an idempotent and $\omega_C^{-1}(\{0\}) = \{0\}$, we conclude

$$\alpha_{ij} = 2\omega_C\left(\frac{1}{2}x\right)g(i,j), \qquad \alpha_{ji} = \alpha_{jj} = 0, \qquad \alpha_{ii} = 1.$$

Hence

$$\phi(E_{ii}+xE_{ij}) = E_{ii} + 2\omega_C\left(\frac{1}{2}x\right)g(i,j)E_{ij}.$$

In an analogous way we arrive at the equality

$$\phi(E_{jj}+yE_{ij}) = E_{jj} + 2\omega_C\left(\frac{1}{2}y\right)g(i,j)E_{ij}.$$

We have

$$\begin{aligned}\omega_C\left(\frac{x+y}{2}\right)g(i,j)E_{ij} &\overset{\text{Claim 2}}{=} \phi\left(\frac{x+y}{2}E_{ij}\right) = \phi((E_{ii}+xE_{ij})\circ(E_{jj}+yE_{ij})) \\ &= \phi(E_{ii}+xE_{ij})\circ\phi(E_{jj}+yE_{ij}) \\ &= \left(E_{ii}+2\omega_C\left(\frac{1}{2}x\right)g(i,j)E_{ij}\right)\circ\left(E_{jj}+2\omega_C\left(\frac{1}{2}y\right)g(i,j)E_{ij}\right) \\ &= \left(\omega_C\left(\frac{1}{2}x\right)+\omega_C\left(\frac{1}{2}y\right)\right)g(i,j)E_{ij}\end{aligned}$$

and hence

$$\omega_C\left(\frac{x+y}{2}\right) = \omega_C\left(\frac{1}{2}x\right)+\omega_C\left(\frac{1}{2}y\right).$$

As $x, y \in \mathbb{C}$ were arbitrarily chosen, this closes the proof for $\circ$.

If ϕ is a multiplicative map, a similar calculation implies

$$\phi(E_{ii}+xE_{ij}) = E_{ii}+\omega_C(x)\,g(i,j)E_{ij}, \qquad \phi(E_{jj}+yE_{ij}) = E_{jj}+\omega_C(y)\,g(i,j)E_{ij}$$

and hence

$$\begin{aligned}\omega_C(x+y)g(i,j)E_{ij} &= \phi((x+y)E_{ij}) = \phi((E_{ii}+xE_{ij})(E_{jj}+yE_{ij})) \\ &= (\omega_C(x)+\omega_C(y))g(i,j)E_{ij}\end{aligned}$$

which likewise implies the desired claim. ♢

It follows that each map $\omega_C : \mathbb{C} \to \mathbb{C}$ is an injective ring endomorphism of $\mathbb{C}$ (i.e. a monomorphism), and hence acts as the identity on the subfield $\mathbb{Q}$ of rational numbers.

Claim 6. *ϕ is a $\mathbb{Q}$-homogeneous map.*

First of all, for $\lambda \in \mathbb{Q}$ we have

$$\begin{aligned}\phi(\lambda I) = \phi\left(\sum_{C\in\mathcal{Q}} \lambda P_C\right) &\overset{\text{Lemma 3.4 (g)}}{=} \sum_{C\in\mathcal{Q}} \phi(\lambda P_C) \overset{\text{Claim 2}}{=} \sum_{C\in\mathcal{Q}} \omega_C(\lambda)\phi(P_C) \\ &\overset{(3.2)}{=} \sum_{C\in\mathcal{Q}} \lambda P_C = \lambda I.\end{aligned}$$

Now, for arbitrary $X \in \mathcal{A}_\rho$ and $\lambda \in \mathbb{Q}$ we have

$$\phi(\lambda X) = \phi(X \bullet (\lambda I)) = \phi(X) \bullet \phi(\lambda I) = \lambda\phi(X).$$

$\diamondsuit$

Claim 7. *The map g is transitive.*

Fix $(i,j),(j,k) \in \rho$. Then $(i,k) \in \rho$ as well. Since $i,j,k \in C$ for some central class $C \in \mathcal{Q}$, for concreteness assume that

$$\phi(E_{ij}) = g(i,j)E_{ij}, \qquad \phi(E_{jk}) = g(j,k)E_{jk}, \qquad \phi(E_{ik}) = g(i,k)E_{ik}.$$

First assume that ϕ preserves $\circ$. If $i \neq k$, then

$$\begin{aligned}\frac{1}{2}g(i,k)E_{ik} &\overset{\text{Claim 6}}{=} \phi\left(\frac{1}{2}E_{ik}\right) = \phi(E_{ij}\circ E_{jk}) = \phi(E_{ij})\circ\phi(E_{jk}) \\ &= \frac{1}{2}g(i,j)g(j,k)E_{ik},\end{aligned}$$

which implies $g(i,k) = g(i,j)g(j,k)$. Similarly, if $i = k$, then

$$\begin{aligned}\frac{1}{2}(E_{ii}+E_{jj}) &\overset{\text{Claim 6,(3.2)}}{=} \phi\left(\frac{1}{2}(E_{ii}+E_{jj})\right) = \phi(E_{ij}\circ E_{ji}) = \phi(E_{ij})\circ\phi(E_{ji}) \\ &= \frac{1}{2}g(i,j)g(j,i)(E_{ii}+E_{jj})\end{aligned}$$

which implies $g(i,i) = 1 = g(i,j)g(j,i)$. The proof is even shorter for multiplicative maps. $\diamondsuit$

Therefore, by passing to the map $(g^*)^{-1}\circ\phi$, without loss of generality we can assume that for each $C \in \mathcal{Q}$ there exists an assignment $\dagger_C \in \{\text{identity}, \text{transposition}\}$ (always the identity when ϕ is multiplicative) so that

$$\phi(E_{ij}) = E_{ij}^{\dagger_C}, \quad \forall (i,j) \in \rho \cap (C\times C).$$

Claim 8. *Let $X \in \mathcal{A}_\rho$ and $P \in \mathrm{Idem}(\mathcal{A}_\rho)$. Then*

$$\phi(PXP) = \phi(P)\phi(X)\phi(P).$$

This is clearly true for multiplicative maps, so assume that ϕ is $\circ$-preserving. One easily verifies the equality

$$(P - P^\perp)\circ(X\circ P) = PXP. \tag{3.6}$$

We also have

$$\phi(P-P^\perp) \overset{\text{Lemma 3.4}}{=} \phi(P) + \phi(-P^\perp) \overset{\text{Claim 6 and Lemma 3.4}}{=} \phi(P) - \phi(P)^\perp.$$

Hence

$$\begin{aligned}\phi(PXP) &\overset{(3.6)}{=} \phi((P-P^\perp)\circ(X\circ P)) = (\phi(P)-\phi(P)^\perp)\circ(\phi(X)\circ\phi(P)) \\ &\overset{(3.6)}{=} \phi(P)\phi(X)\phi(P).\end{aligned}$$

$\diamondsuit$

Claim 9. *For all $C \in \mathcal{Q}$ and $X \in \mathcal{A}_\rho$ with* $\operatorname{supp} X \subseteq C \times C$ *we have*

$$\phi(X) = \sum_{(i,j)\in\rho\cap(C\times C)} \omega_C(X_{ij})E_{ij}^{\dagger_C} = \omega_C(X)^{\dagger_C}.$$

We prove the claim for $\bullet = \circ$, as the multiplicative case is similar, only simpler. Fix $C \in \mathcal{Q}$. For concreteness, assume that $\dagger_C = \mathrm{id}$ and fix some $X \in \mathcal{A}_\rho$ such that $\operatorname{supp} X \subseteq C \times C$. Clearly, by Claim 1, $\phi(X)$ is also supported in $C \times C$. Let $(i,j) \in C \times C$. If $i = j$, we have

$$\omega_C(X_{ii})E_{ii} = \phi(X_{ii}E_{ii}) = \phi(E_{ii}XE_{ii}) \overset{\text{Claim 8}}{=} E_{ii}\phi(X)E_{ii} = \phi(X)_{ii}E_{ii},$$

so $\phi(X)_{ii} = \omega_C(X_{ii})$. Now assume $i \neq j$. Assume first that $(i,j),(j,i) \in \rho$. As ω_C is multiplicative and acts as the identity on $\mathbb{Q}$, it follows

$$\begin{aligned}\frac{1}{2}\omega_C(X_{ij})E_{ji} &= \phi\left(\frac{1}{2}X_{ij}E_{ji}\right) = \phi\left(\frac{1}{2}E_{ji}XE_{ji}\right) = \phi((E_{ji}\circ X)\circ E_{ji})\\ &= (\phi(E_{ji})\circ\phi(X))\circ\phi(E_{ji}) = (E_{ji}\circ\phi(X))\circ E_{ji}\\ &= \frac{1}{2}\phi(X)_{ij}E_{ji},\end{aligned}$$

which implies $\phi(X)_{ij} = \omega_C(X_{ij})$. Suppose now that $(i,j) \in \rho$ but $(j,i) \notin \rho$ (so that $X_{ji} = 0$). We have

$$\begin{aligned}\frac{1}{4}\omega_C(X_{ij})E_{ij} &= \phi\left(\frac{1}{4}X_{ij}E_{ij}\right) = \phi\left(\frac{1}{4}X_{ij}E_{ij} + \frac{1}{4}X_{ji}E_{ji}\right)\\ &= \phi\left(\frac{1}{4}(E_{ii}XE_{jj} + E_{jj}XE_{ii})\right) = \phi((E_{ii}\circ X)\circ E_{jj})\\ &= (E_{ii}\circ\phi(X))\circ E_{jj} = \frac{1}{4}\phi(X)_{ij}E_{ij} + \frac{1}{4}\phi(X)_{ji}E_{ji},\end{aligned}$$

so $\phi(X)_{ij} = \omega_C(X_{ij})$ and $\phi(X)_{ji} = 0$. Finally, the same calculation also shows that $\phi(X)_{ij} = \phi(X)_{ji} = 0$ for each $i,j \in C$ such that $(i,j),(j,i) \notin \rho$. This proves the claim. $\diamondsuit$

We are now in the position to finish the proof of the theorem.

Claim 10. *For each $X \in \mathcal{A}_\rho$, we have*

$$\phi(X) = \sum_{C\in\mathcal{Q}} \omega_C(P_CX)^{\dagger_C}.$$

Indeed,

$$\begin{aligned}\phi(X) &= \phi(X)\bullet I = \phi(X)\bullet\left(\sum_{C\in\mathcal{Q}} P_C\right) = \sum_{C\in\mathcal{Q}}(\phi(X)\bullet P_C)\\ &\overset{(3.2)}{=} \sum_{C\in\mathcal{Q}}(\phi(X)\bullet\phi(P_C)) = \sum_{C\in\mathcal{Q}}\phi(X\bullet P_C) = \sum_{C\in\mathcal{Q}}\phi(\underbrace{P_CX}_{\text{supported in } C\times C})\\ &\overset{\text{Claim 9}}{=} \sum_{C\in\mathcal{Q}}\omega_C(P_CX)^{\dagger_C}.\end{aligned}$$

$\diamondsuit$

$\square$

Remark 3.5. Using the setting of Theorem 3.1, adding the additional assumption of the continuity at a single point for the map ϕ ensures that all monomorphisms $\omega_C : \mathbb{C} \to \mathbb{C}$ are either the identity or the complex conjugation.

Remark 3.6. In contrast to [12, Theorem 4.9], the injectivity assumption of the map ϕ in Theorem 3.1 cannot be relaxed to the condition that only $\phi(E_{ij}) \neq 0$ for all $(i,j) \in \rho$. This is illustrated by a constant map that assigns each matrix to a fixed nonzero idempotent.

Remark 3.7. Given a unital algebra $\mathcal{A}$ over a field $\mathbb{F}$, it is also common to consider the (non-normalized) Jordan product given by

$$x \triangle y := xy + yx, \quad \forall x, y \in \mathcal{A}.$$

If $\mathrm{char}(\mathbb{F}) \neq 2$, note that all $\triangle$-preserving injective, surjective, or bijective maps $\mathcal{A} \to \mathcal{B}$ (where $\mathcal{B}$ is another unital algebra over $\mathbb{F}$) are automatically additive if and only if the same holds true for the corresponding $\circ$-preserving maps. Indeed, if for example $\phi : \mathcal{A} \to \mathcal{B}$ is $\triangle$-preserving, then the map $\psi : \mathcal{A} \to \mathcal{B}$ defined by

$$\psi(x) := 2\phi\left(\frac{x}{2}\right), \quad \forall x \in \mathcal{A}$$

is clearly $\circ$-preserving. In particular, when $\mathcal{A} \subseteq M_n$ is an SMA, Theorem 3 also applies to all $\triangle$-preserving maps $\phi : \mathcal{A} \to M_n$.

We close the paper with an example which demonstrates that Theorem 3.1 cannot be generalized to arbitrary unital subalgebras of M_n.

Example 3.8. Consider $\mathcal{A} \subseteq M_5$ defined by

$$\mathcal{A} := \left\{ \begin{bmatrix} x_{11} & 0 & 0 & 0 & 0 \\ x_{21} & y & z & 0 & 0 \\ 0 & 0 & x_{33} & 0 & 0 \\ 0 & 0 & z & y & x_{45} \\ 0 & 0 & 0 & 0 & x_{55} \end{bmatrix} : x_{ij}, y, z \in \mathbb{C} \right\}.$$

One can easily check that $\mathcal{A}$ is a central subalgebra of M_5. On the other hand (as in the proof of (ii) $\implies$ (i) of Theorem 3.1), choose any injective multiplicative non-additive function $\omega : \mathbb{C} \to \mathbb{C}$ and define a map $\phi : \mathcal{A} \to M_5$ by

$$\phi\left(\begin{bmatrix} x_{11} & 0 & 0 & 0 & 0 \\ x_{21} & y & z & 0 & 0 \\ 0 & 0 & x_{33} & 0 & 0 \\ 0 & 0 & z & y & x_{45} \\ 0 & 0 & 0 & 0 & x_{55} \end{bmatrix} \right) := \begin{bmatrix} x_{11} & 0 & 0 & 0 & 0 \\ x_{21} & y & z & x_{45} & 0 \\ 0 & 0 & x_{33} & 0 & 0 \\ 0 & 0 & 0 & x_{55} & 0 \\ 0 & 0 & 0 & 0 & \omega(x_{11}) \end{bmatrix}.$$

It is then straightforward to verify that ϕ is an injective non-additive map that is both multiplicative and Jordan multiplicative.

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