A PROOF OF THE KAPLANSKY–AUPETIT CONJECTURE FOR C^*-ALGEBRAS

ILJA GOGIĆ

ABSTRACT. We prove that a unital surjective linear map between unital C^*-algebras preserves squares and hence is a Jordan homomorphism whenever it sends every exponential to an invertible element. This settles the Kaplansky–Aupetit conjecture for maps between unital C^*-algebras. The proof, found by Anthropic's Claude Opus 5.5, adapts resolvent estimates from a recent preprint by Colbrook and Stepaniants. We also show that a nonscalar unital C^*-algebra A has no tracial states if and only if every bounded unital linear map from A into a unital Banach algebra is a Jordan homomorphism whenever it preserves square-zero elements. The absence of tracial states is also equivalent to the assertion that every unital linear surjection from A onto a unital semisimple Banach algebra is a Jordan homomorphism whenever it is spectrally bounded, meaning that it can increase spectral radii by at most a fixed factor.

1. INTRODUCTION

The Gleason–Kahane–Żelazko theorem [13, 18, 30] states that a linear functional on a unital complex Banach algebra is multiplicative if it takes the identity to 1 and is nonzero on every invertible element. The corresponding question for maps between algebras goes back to Kaplansky [19, p. 12]. The strong form of the Kaplansky–Aupetit conjecture [8, Conjecture 1.1] asks whether every surjective linear map between unital semisimple Banach algebras that preserves the identity and invertibility must also preserve squares. Here *semisimplicity* means that the Jacobson radical is zero. For a commutative semisimple codomain, the map is multiplicative since the characters separate points and composing the map with each character gives a multiplicative functional by the Gleason–Kahane–Żelazko theorem.

Throughout the paper, all algebras are complex and associative, all linear maps are complex-linear, and unital algebras are nonzero. Let $\phi : A \to B$ be a linear map between algebras. If A and B are unital, ϕ is *unital* when $\phi(1) = 1$. The map ϕ is a *Jordan homomorphism* if

$$\phi(x^2) = \phi(x)^2, \qquad x \in A.$$

A bijective Jordan homomorphism is a *Jordan isomorphism*.

For a unital algebra A, write $A^\times$ for its group of invertible elements. If A is a Banach algebra, the spectrum and spectral radius of $x \in A$ are

$$\sigma_A(x) := \{\lambda \in \mathbb{C} : \lambda 1 - x \notin A^\times\}, \qquad r_A(x) := \max_{\lambda \in \sigma_A(x)} |\lambda|.$$

A linear map $\phi : A \to B$ between unital algebras *preserves invertibility* if

$$\phi(A^\times) \subseteq B^\times.$$

For unital Banach algebras A and B, the map is *spectrum-shrinking* if

$$\sigma_B(\phi(x)) \subseteq \sigma_A(x), \qquad x \in A.$$

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If equality holds for every $x \in A$, the map is *spectrum-preserving*. For unital linear maps, spectrum shrinking is equivalent to preservation of invertibility. A unital linear map $\phi : A \to B$ preserves spectra precisely when it *preserves invertibility in both directions*, that is, when

$$x \in A^\times \qquad \Longleftrightarrow \qquad \phi(x) \in B^\times, \qquad x \in A.$$

Early results confirmed the conjecture for codomains with a separating family of finite-dimensional irreducible representations [3, Théorème 2], for positive maps between unital C^*-algebras [10], and for bijections between full operator algebras on Banach spaces [29, Theorem 1.1]. Aupetit [6, Theorem 1.3] proved that every surjective spectrum-preserving linear map between von Neumann algebras is a Jordan isomorphism. Cui and Hou [12, Theorem 2.3] proved the strong form of the conjecture when the domain is a von Neumann algebra. Their argument also applies when the domain is a unital C^*-algebra of *real rank zero*, since the self-adjoint elements with finite spectrum are norm dense in the self-adjoint part [9, Theorem 2.6]. For spectrum-preserving maps, Kovács [20, Theorem 2.5 and Remark 3.4(a)] proved that a linear surjection between unital semisimple Banach algebras is a Jordan isomorphism whenever either algebra is a C^*-algebra of real rank zero.

In his earlier work on spectrum-preserving maps between von Neumann algebras, Aupetit [5] used the Fuglede–Kadison determinant to treat the finite-trace case. Working directly with traces on the codomain, Mathieu and Schulz [28, Section 4, before Corollaries 4.2 and 4.4] showed that a unital linear surjection from a unital Banach algebra onto a unital C^*-algebra with a faithful family of tracial states is a Jordan homomorphism whenever it sends every product of two exponentials to an invertible element.

Here a *tracial state* on a unital C^*-algebra B is a positive linear functional $\tau : B \to \mathbb{C}$ satisfying

$$\tau(1) = 1, \qquad \tau(xy) = \tau(yx), \qquad x, y \in B.$$

A family $\mathcal{F}$ of tracial states on B is *faithful* if, for every $x \in B$,

$$\tau(x^*x) = 0 \quad \text{for all } \tau \in \mathcal{F} \qquad \Longrightarrow \qquad x = 0.$$

Our main result settles the Kaplansky–Aupetit conjecture for maps between unital C^*-algebras. In fact, it suffices to require that the image of each exponential be invertible.

Theorem 1.1. *Let A and B be unital C^*-algebras, and let $\phi : A \to B$ be a unital surjective linear map. Then ϕ is a Jordan homomorphism if and only if*

$$\phi(e^x) \in B^\times, \qquad x \in A.$$

In particular, every unital invertibility-preserving linear surjection between unital C^*-algebras is a Jordan homomorphism.

A related problem concerns *spectral isometries*, that is, linear maps $\phi : A \to B$ between unital Banach algebras satisfying

$$r_B(\phi(x)) = r_A(x), \qquad x \in A.$$

In their recent preprint, Colbrook and Stepaniants [11, Theorem 1.1] proved that every unital surjective spectral isometry between unital C^*-algebras is a Jordan isomorphism. Theorem 1.1 also applies to maps with nonzero kernel, such as the first-coordinate homomorphism from $\mathbb{C}^2$ onto $\mathbb{C}$.

Both theorems also give the factorization conjectured by Harris and Kadison [15, Conjecture 3.1]. If a linear bijection $\phi : A \to B$ between unital C^*-algebras carries $A^\times$ onto $B^\times$, then $\phi(1)^{-1}\phi$ is a unital linear bijection that preserves invertibility in both directions. It therefore preserves spectra and is a Jordan isomorphism by either theorem.

The proof of Theorem 1.1, given in Section 3, follows Aupetit [6, Theorems 1.1 and 1.2] and Cui and Hou [12, Theorem 2.2] in using local spectral estimates to show that idempotents are preserved (see also [7]). Recall that an idempotent is an element equal to its square, while a projection is a self-adjoint idempotent. A general C^*-algebra may have too few projections to recover square preservation in this way. For instance, the only projections in $C([0,1])$ are 0 and 1. We therefore

work in the bidual, where the spectral theorem allows us to approximate every self-adjoint element in norm by finite real linear combinations of pairwise orthogonal projections. We first use a branch of the logarithm to estimate the resolvent of $\phi(x)$ for normal $x \in A$. For each projection $p \in A^{**}$, we approximate $1 - 2p$ by two nets of unitaries in A whose spectra lie in opposite semicircles. This lets us pass the resolvent estimate to the bidual and show that the second adjoint $\phi^{**} : A^{**} \to B^{**}$ of ϕ sends p to an idempotent. The resolvent, continuity and contour arguments are adapted from Colbrook and Stepaniants [11, Lemmas 2.1, 2.2, 3.2 and 3.3].

We next ask when either preservation of square-zero elements or spectral boundedness suffices for a unital linear map to be a Jordan homomorphism. More precisely, a linear map $\phi : A \to B$ between algebras *preserves square-zero elements* if

$$x^2 = 0 \qquad \Longrightarrow \qquad \phi(x)^2 = 0, \qquad x \in A.$$

When A and B are unital Banach algebras, ϕ is *spectrally bounded* if, for some $\gamma \in [0, \infty)$,

$$r_B(\phi(x)) \le \gamma r_A(x), \qquad x \in A.$$

A unital Jordan homomorphism between unital Banach algebras has both properties, with $\gamma = 1$ in the spectral bound.

Mathieu and Schick [27, Corollary 3.5] proved that every bounded unital square-zero preserver from a properly infinite von Neumann algebra into a unital Banach algebra is a Jordan homomorphism. Here *proper infiniteness* means that the algebra contains two isometries with orthogonal range projections. They obtained the same conclusion for unital spectrally bounded surjections from such an algebra onto unital semisimple Banach algebras [27, Theorem 3.6]. Mathieu also proved the result on spectrally bounded surjections when the domain is a unital simple C^*-algebra that is purely infinite [23, Theorem B] or has real rank zero and no tracial states [24, Theorem 3.1]. For unital C^*-algebras of real rank zero, Lin and Mathieu [21, Theorem 2.4 and Corollary 2.5] proved both conclusions in the purely infinite case, and Mathieu [26, Theorem and Corollary] later obtained them for algebras without tracial states.

To remove the real-rank-zero assumption, we return to the von Neumann algebra setting of Mathieu and Schick. The bidual of a unital C^*-algebra without tracial states is properly infinite [22, p. 172], so their result applies there. The following theorem settles Mathieu's conjecture on square-zero preservers [24, Conjecture 3.2] for nonscalar algebras and gives an affirmative answer to his question about spectrally bounded surjections [25, Problem 2.4]. For $K \subseteq \mathbb{C}$, we write $\operatorname{conv} K$ for its convex hull.

Theorem 1.2. *Let A be a unital C^*-algebra with $\dim A \ge 2$. The following conditions are equivalent.*

(a) *A has no tracial states.*
(b) *Every bounded unital linear map from A into a unital Banach algebra is a Jordan homomorphism whenever it preserves square-zero elements.*
(c) *Every unital surjective spectrally bounded linear map from A onto a unital semisimple Banach algebra is a Jordan homomorphism.*

If A has a tracial state, there is a bounded unital linear bijection $\phi : A \to A$ that preserves square-zero elements and satisfies

$$\operatorname{conv} \sigma_A(\phi(x)) \subseteq \operatorname{conv} \sigma_A(x), \qquad x \in A,$$

but fails to preserve squares.

Note that the scalar case is exceptional because $\mathbb{C}$ has a tracial state and every unital linear map from $\mathbb{C}$ into a unital complex algebra is multiplicative.

The paper is organized as follows. Section 2 recalls the facts about biduals and Jordan homomorphisms used below. Section 3 proves Theorem 1.1 by resolvent estimates. Section 4 proves Theorem 1.2 by passing square-zero preservation to the bidual.

2. Preliminaries

2.1. **Biduals and the strong* topology.** We use the strong* topology to pass estimates from a C^*-algebra to its bidual.

For a C^*-algebra A, its Banach-space bidual A^{**} is its enveloping von Neumann algebra [17, Section 10.1]. We identify A with its canonical image in A^{**}; this embedding is an isometric $*$-homomorphism and is unital when A is unital.

Let $(x_j)_{j\in J}$ be a norm-bounded net in a von Neumann algebra M, indexed by a directed set J, and let $x \in M$. The net converges to x in the strong* topology if $x_j \longrightarrow x$ and $x_j^* \longrightarrow x^*$ in the strong operator topology of a faithful normal representation of M. On norm-bounded sets, the strong* topology is independent of the choice of representation [1, Remark I.2]. On such sets, strong* convergence implies weak* convergence, and multiplication is jointly strong*-continuous. By Kaplansky's density theorem [16, Theorem 5.3.5], each self-adjoint contraction in A^{**} is the strong operator limit of a net of self-adjoint contractions in A. Since the terms and the limit are self-adjoint, the convergence is strong*.

We also record the following standard continuity property of the continuous functional calculus.

Lemma 2.1. *Let $(x_j)_{j\in J}$ be a norm-bounded net of self-adjoint elements of a von Neumann algebra M converging strong* to a self-adjoint element $x \in M$, where J is a directed set. Let $I \subseteq \mathbb{R}$ be a compact interval containing the spectra of x and of every x_j. For every continuous function $f : I \to \mathbb{C}$, we have*

$$f(x_j) \longrightarrow f(x) \qquad \text{in the strong}^* \text{ topology.}$$

Proof. For polynomials this follows from joint strong* continuity of multiplication on bounded sets. Uniform polynomial approximation on I, together with the norm estimate in continuous functional calculus, gives the assertion for every continuous $f : I \to \mathbb{C}$. □

The second adjoint $\phi^{**} : A^{**} \to B^{**}$ of a bounded linear map $\phi : A \to B$ between Banach spaces is a bounded weak*-continuous extension of ϕ. When A and B are C^*-algebras, Akemann's continuity theorem [1, Corollary III.10] shows that ϕ^{**} is strong*-continuous on norm-bounded subsets of A^{**}.

2.2. **Jordan homomorphisms.** We show that unital Jordan homomorphisms preserve invertibility and that, on a von Neumann algebra, sending projections to idempotents suffices for square preservation.

For an algebra A, define the *Jordan product* by

$$x \circ y := \frac{1}{2}(xy + yx), \qquad x, y \in A.$$

The identities

$$2x \circ y = (x + y)^2 - x^2 - y^2, \qquad x \circ x = x^2, \qquad x, y \in A,$$

show that a linear map between complex algebras preserves squares if and only if it preserves the Jordan product.

Lemma 2.2. (a) *A unital Jordan homomorphism between unital algebras preserves invertibility.*
(b) *A bounded linear map from a von Neumann algebra into a Banach algebra is a Jordan homomorphism if it sends projections to idempotents.*

Proof. (a) Let $\phi : A \to B$ be a unital Jordan homomorphism between unital algebras. For $x, y \in A$, expanding the Jordan products gives

$$xyx = 2x \circ (x \circ y) - x^2 \circ y.$$

Applying ϕ and using the same identity in B, we obtain

$$\begin{aligned} \phi(xyx) &= 2\phi(x) \circ (\phi(x) \circ \phi(y)) - \phi(x)^2 \circ \phi(y) \\ &= \phi(x)\phi(y)\phi(x). \end{aligned}$$

For $x \in A^\times$, substituting x^{-2} for y gives

$$\phi(x)\phi(x^{-2})\phi(x) = 1.$$

Thus $\phi(x)\phi(x^{-2})$ is a left inverse of $\phi(x)$ and $\phi(x^{-2})\phi(x)$ is a right inverse. Hence $\phi(x) \in B^\times$.

(b) We include the argument of Mathieu and Schick [27, Lemma 2.1]. Let M be a von Neumann algebra, let B be a Banach algebra, and let $\psi : M \to B$ be a bounded linear map sending projections to idempotents. For orthogonal projections $p, q \in M$, the images of p, q and $p+q$ are idempotents. Expanding the square of $\psi(p+q)$ gives

$$\psi(p)\psi(q) + \psi(q)\psi(p) = 0.$$

Thus the mixed terms cancel when we square the image of a finite real linear combination of pairwise orthogonal projections, and the diagonal terms agree because the images are idempotent. Hence ψ preserves squares of these combinations.

By the spectral theorem [16, Theorem 5.2.2], every self-adjoint element of M is a norm limit of finite real linear combinations of pairwise orthogonal projections in M. Since ψ is bounded, it preserves squares of all self-adjoint elements. Every element of M has the form $x+iy$ with $x, y \in M$ self-adjoint. Applying the square identity to $x+y$ gives $\psi(x \circ y) = \psi(x) \circ \psi(y)$, and hence

$$\begin{aligned}\psi\left((x+iy)^2\right) &= \psi(x)^2 + 2i\psi(x) \circ \psi(y) - \psi(y)^2 \\ &= \psi(x+iy)^2.\end{aligned}$$

□

3. Proof of Theorem 1.1

We prove the nontrivial implication by showing that the second adjoint sends projections to idempotents. Until the final proof of the theorem, let A and B be unital C^*-algebras and let $\phi : A \to B$ be a unital surjective linear map that sends exponentials to invertible elements.

3.1. **A resolvent estimate.**

The principal logarithm gives an invertibility criterion along rays. We use this criterion to establish boundedness and then estimate the resolvents of images of normal elements.

For $\lambda, \eta \in \mathbb{C}$ with $|\eta| = 1$, put

$$L(\lambda, \eta) := \{\lambda + \alpha\eta : \alpha \in [0, \infty)\}.$$

Lemma 3.1. *Let $x \in A$ and let $\lambda, \eta \in \mathbb{C}$ with $|\eta| = 1$. Then*

$$L(\lambda, \eta) \cap \sigma_A(x) = \emptyset \qquad \Longrightarrow \qquad \lambda 1 - \phi(x) \in B^\times. \tag{3.1}$$

Proof. Suppose that $L(\lambda, \eta) \cap \sigma_A(x) = \emptyset$. By the spectral mapping theorem and the assumption on the ray,

$$\sigma_A\left(\overline{\eta}(\lambda 1 - x)\right) = \{\overline{\eta}(\lambda - \zeta) : \zeta \in \sigma_A(x)\} \subseteq \mathbb{C} \setminus (-\infty, 0].$$

Indeed, if $\overline{\eta}(\lambda - \zeta) = -\alpha$ for some $\zeta \in \sigma_A(x)$ and $\alpha \in [0, \infty)$, then $\zeta = \lambda + \alpha\eta$ lies on the ray. The principal logarithm and the holomorphic functional calculus show that $\overline{\eta}(\lambda 1 - x)$ is an exponential in A. By linearity and unitality, its image under ϕ is $\overline{\eta}(\lambda 1 - \phi(x))$, which is invertible by hypothesis. Since $|\eta| = 1$, the element $\lambda 1 - \phi(x)$ is invertible. □

Let $x \in A$. For every $\lambda \in \mathbb{C}$ with $|\lambda| > r_A(x)$, the ray $L(\lambda, \lambda/|\lambda|)$ is disjoint from $\sigma_A(x)$. It follows that

$$r_B(\phi(x)) \overset{(3.1)}{\leq} r_A(x), \qquad x \in A.$$

Thus ϕ is spectrally bounded. Since ϕ is surjective and the C^*-algebra B is semisimple, Aupetit's automatic continuity theorem [4, Theorem 5.5.2] shows that ϕ is bounded. By the open mapping theorem, we may fix $\kappa \in (0,\infty)$ such that, for every $y \in B$, there is $x \in A$ satisfying

$$\phi(x) = y, \qquad \|x\| \le \kappa \|y\|.$$

We use this constant throughout the rest of the section.

For nonempty sets $K_1, K_2 \subseteq \mathbb{C}$, put

$$\operatorname{dist}(K_1,K_2) := \inf_{\mu\in K_1,\,\zeta\in K_2} |\mu-\zeta|, \qquad \operatorname{dist}(\lambda,K_2) := \operatorname{dist}(\{\lambda\},K_2), \qquad \lambda\in\mathbb{C}.$$

Lemma 3.2. *Let $x_0 \in A$ be normal and let $\lambda,\eta \in \mathbb{C}$ with $|\eta| = 1$. If $\delta \in (0,\infty)$ satisfies*

$$\operatorname{dist}(L(\lambda,\eta),\sigma_A(x_0)) \ge \delta,$$

then $\lambda 1 - \phi(x_0) \in B^\times$ and

$$\|(\lambda 1 - \phi(x_0))^{-1}\| \le \frac{\kappa}{\delta}. \tag{3.2}$$

Proof. Let x_0, λ, η and δ satisfy the stated conditions. The element $\lambda 1 - \phi(x_0)$ is invertible by Lemma 3.1. Take $y \in B$ with $\|y\| < \delta/\kappa$ and choose $x \in A$ with $\phi(x) = y$ and $\|x\| \le \kappa\|y\| < \delta$. For every $\mu \in L(\lambda,\eta)$, the continuous functional calculus for x_0 gives

$$\|(\mu 1 - x_0)^{-1}\| = \operatorname{dist}(\mu,\sigma_A(x_0))^{-1} \le \delta^{-1}.$$

In particular, $\|(\mu 1 - x_0)^{-1}x\| < 1$. The factorization

$$\mu 1 - (x_0 + x) = (\mu 1 - x_0)\left(1 - (\mu 1 - x_0)^{-1}x\right)$$

and the Neumann series show that $\mu 1 - (x_0 + x)$ is invertible. Thus the ray misses $\sigma_A(x_0 + x)$, and (3.1) gives $\lambda 1 - \phi(x_0) - y \in B^\times$.

Put $c := \lambda 1 - \phi(x_0)$. The open ball of radius δ/κ about c is contained in $B^\times$. Following the proof of [11, Lemma 2.2], put

$$|c| := (c^*c)^{1/2}, \qquad u := c|c|^{-1} \in B.$$

Since c is invertible, u is unitary and $c = u|c|$. Continuous functional calculus gives

$$\nu := \min \sigma_B(|c|) = \|c^{-1}\|^{-1}.$$

Since $\nu \in \sigma_B(|c|)$, the element $c - \nu u = u(|c| - \nu 1)$ is noninvertible. Its distance from c is ν, so $\nu \ge \delta/\kappa$. This proves (3.2). □

3.2. **Images of projections in the bidual.** To use the resolvent estimate at a projection in the bidual, we approximate the associated self-adjoint unitary by unitaries whose spectra lie in the upper and lower semicircles.

Proposition 3.3. *The second adjoint $\phi^{**} : A^{**} \to B^{**}$ sends projections to idempotents.*

Proof. Let $p \in A^{**}$ be a projection and put

$$\psi := \phi^{**} : A^{**} \to B^{**}, \qquad w := \psi(1 - 2p).$$

Since ψ extends ϕ, it is unital. We first construct the approximating unitaries. Applying Kaplansky's density theorem to $2p - 1$ and rescaling by the affine function $\beta \mapsto (1+\beta)/2$ from $[-1,1]$ to $[0,1]$, we obtain a net $(h_j)_{j\in J}$ of positive contractions in A converging strong* to p, where J is a directed set. Define

$$E := \{-1,1\}, \qquad \Gamma_\varepsilon := \{\zeta \in \mathbb{C} : |\zeta| = 1,\ \varepsilon \operatorname{Im}\zeta \ge 0\}, \qquad \varepsilon \in E.$$

For $\varepsilon \in E$ and $j \in J$, put

$$u_{j,\varepsilon} := e^{i\varepsilon\pi h_j}.$$

Since $0 \le h_j \le 1$, continuous functional calculus shows that $u_{j,\varepsilon}$ is unitary and

$$\sigma_A(u_{j,\varepsilon}) \subseteq \Gamma_\varepsilon, \qquad j \in J,\ \varepsilon \in E.$$

Lemma 2.1 gives

$$u_{j,\varepsilon} \longrightarrow e^{i\varepsilon\pi p} = (1-p) + e^{i\varepsilon\pi}p = 1 - 2p$$

in the strong* topology, for each $\varepsilon \in E$.

Fix $\lambda \in \mathbb{C} \setminus E$ and choose $\varepsilon \in E$ with $\varepsilon \operatorname{Im}\lambda \le 0$. For $\mu \in L(\lambda, -i\varepsilon)$ and $\zeta \in \Gamma_\varepsilon$, we have $\operatorname{Re}\mu = \operatorname{Re}\lambda$, $|\operatorname{Im}\mu| \ge |\operatorname{Im}\lambda|$, and $\operatorname{Im}\mu \operatorname{Im}\zeta \le 0$. Since $|\operatorname{Re}\zeta| \le 1$, it follows that $\operatorname{Re}(\mu\overline{\zeta}) \le |\operatorname{Re}\lambda|$. Hence

$$\begin{aligned} |\mu - \zeta|^2 &= |\mu|^2 + 1 - 2\operatorname{Re}(\mu\overline{\zeta}) \\ &\ge |\lambda|^2 + 1 - 2|\operatorname{Re}\lambda| = \operatorname{dist}(\lambda, E)^2. \end{aligned}$$

Put $\delta := \operatorname{dist}(\lambda, E) > 0$. Since $\sigma_A(u_{j,\varepsilon}) \subseteq \Gamma_\varepsilon$ for every $j \in J$, we obtain

$$\operatorname{dist}(L(\lambda, -i\varepsilon), \sigma_A(u_{j,\varepsilon})) \ge \delta, \qquad j \in J.$$

For $j \in J$, put $c_j := \lambda 1 - \phi(u_{j,\varepsilon})$. Each $u_{j,\varepsilon}$ is normal, so Lemma 3.2 gives $c_j \in B^\times$ and $\|c_j^{-1}\| \le \kappa/\delta$. By the C^*-identity and continuous functional calculus,

$$(\delta/\kappa)^2 1 \overset{(3.2)}{\le} c_j^* c_j \qquad \text{and} \qquad (\delta/\kappa)^2 1 \overset{(3.2)}{\le} c_j c_j^*, \qquad j \in J.$$

We now pass these bounds to the bidual. The net $(c_j)_{j\in J}$ is bounded because ϕ is bounded and the $u_{j,\varepsilon}$ are unitary. By the strong* continuity of ψ on bounded sets recalled in Section 2,

$$c_j \longrightarrow c := \lambda 1 - w \qquad \text{in the strong}^* \text{ topology.}$$

Joint strong* continuity of multiplication gives convergence of $c_j^* c_j$ and $c_j c_j^*$ to c^*c and cc^*, respectively. Since the positive cone is weak* closed, it follows that

$$(\delta/\kappa)^2 1 \le c^*c, \qquad (\delta/\kappa)^2 1 \le cc^*.$$

Thus $(c^*c)^{-1}c^*$ and $c^*(cc^*)^{-1}$ are a left and a right inverse of c in B^{**}, respectively. They coincide, so c is invertible. By continuous functional calculus, the lower bound on c^*c gives

$$\|c^{-1}\|^2 = \|(c^*c)^{-1}\| \le (\kappa/\delta)^2.$$

Since λ was arbitrary, we have shown that

$$\begin{aligned} &\sigma_{B^{**}}(w) \subseteq E, \\ &\|(\lambda 1 - w)^{-1}\| \le \frac{\kappa}{\operatorname{dist}(\lambda, E)}, \qquad \lambda \in \mathbb{C} \setminus E. \end{aligned} \tag{3.3}$$

It remains to show that $w^2 = 1$. Let $\rho \in (0,1)$ and, for each $\varepsilon \in E$, let γ_ε be the positively oriented circle of radius ρ about ε. The circles have disjoint interiors and together enclose the spectrum of w by (3.3). Holomorphic functional calculus gives

$$w^2 - 1 = \frac{1}{2\pi i} \sum_{\varepsilon \in E} \int_{\gamma_\varepsilon} (\lambda^2 - 1)(\lambda 1 - w)^{-1}\, d\lambda.$$

For $\varepsilon \in E$ and $\lambda \in \gamma_\varepsilon$, the other point of E is at distance at least $2 - \rho > \rho$. Consequently,

$$\operatorname{dist}(\lambda, E) = \rho, \qquad |\lambda^2 - 1| = |\lambda - \varepsilon|\,|\lambda + \varepsilon| \le \rho(2 + \rho).$$

The integrand therefore has norm at most $\kappa(2+\rho)$ on both circles, whose total length is $4\pi\rho$. Hence

$$\|w^2 - 1\| \overset{(3.3)}{\le} 2\kappa\rho(2+\rho).$$

Letting ρ decrease to zero gives $w^2 = 1$. Since ψ is unital,

$$\psi(p)^2 = \left(\frac{1-w}{2}\right)^2 = \frac{1-w}{2} = \psi(p).$$

□

Proof of Theorem 1.1. Suppose that ϕ sends exponentials to invertible elements. As established above, ϕ is bounded, and Proposition 3.3 shows that ϕ^{**} sends projections to idempotents. Lemma 2.2(b) therefore implies that ϕ^{**} and its restriction ϕ are Jordan homomorphisms.

Conversely, a unital Jordan homomorphism preserves invertibility by Lemma 2.2(a). Since e^x has inverse e^{-x} for every $x \in A$, the conclusion follows. □

4. Square-zero preservers and tracial states

For a unital C^*-algebra without tracial states, the bidual is properly infinite, so Mathieu and Schick's theorem applies once square-zero preservation passes to the second adjoint. For Banach algebra codomains, Haagerup's extension theorem supplies the required continuity.

Recall that a von Neumann algebra is *finite* if every isometry in it is unitary. By the central decomposition and the halving lemma, it is properly infinite precisely when it has no nonzero finite central summand [17, Proposition 6.3.7 and Lemma 6.3.3].

Mathieu [22, p. 172] observed the following equivalence. We include a proof for completeness.

Lemma 4.1. *A unital C^*-algebra A has no tracial states if and only if A^{**} is properly infinite.*

Proof. Suppose that A has no tracial states and that $z \in A^{**}$ is a nonzero central projection for which zA^{**} is finite. Composing its centre-valued trace with a state on its centre gives a tracial state on zA^{**} [17, Theorem 8.2.8]. Since z is central and zA^{**} has identity z, the map $x \mapsto zx$ is a unital $*$-homomorphism from A to zA^{**}. Pulling back the tracial state gives a tracial state on A, a contradiction.

Conversely, suppose that A^{**} is properly infinite and that τ is a tracial state on A. Let $\widetilde{\tau}$ be the canonical weak*-continuous extension of τ to A^{**}. Since $\|\widetilde{\tau}\| = \widetilde{\tau}(1) = 1$, this extension is a state. By weak* density of A and separate weak* continuity of multiplication, the trace identity extends to A^{**} one variable at a time. Choose isometries $v_1, v_2 \in A^{**}$ with orthogonal ranges. Since $v_1v_1^* + v_2v_2^* \leq 1$, we obtain

$$2 = \widetilde{\tau}(v_1^*v_1) + \widetilde{\tau}(v_2^*v_2) = \widetilde{\tau}(v_1v_1^* + v_2v_2^*) \leq 1,$$

a contradiction. □

For a unital Banach algebra B, we write $\square$ for the first Arens product on B^{**} [2, pp. 839–840]. For $f \in B^*$ and $x \in B$, define $f_x \in B^*$ by

$$f_x(y) := f(xy), \qquad y \in B.$$

For $F, G \in B^{**}$, the product is given by

$$(F \,\square\, G)(f) := F\left(x \mapsto G(f_x)\right), \qquad f \in B^*.$$

The map $x \mapsto G(f_x)$ belongs to B^* because $\|f_x\| \leq \|f\|\|x\|$. With this product, B^{**} is a unital Banach algebra, and the canonical embedding of B into B^{**} is a unital homomorphism [2]. We identify B with its canonical image in B^{**}. The product is weak* continuous in the first variable and, when the first factor belongs to B, also in the second.

Lemma 4.2. *Let A be a unital C^*-algebra and B a unital Banach algebra. If a bounded linear map $\phi : A \to B$ preserves square-zero elements, then so does $\phi^{**} : A^{**} \to B^{**}$, where B^{**} carries the first Arens product.*

Proof. Let $w \in A^{**}$ satisfy $w^2 = 0$. We adapt the approximation in [11, proof of Lemma 3.4]. In a faithful normal representation of A^{**}, let $p \in A^{**}$ be the spectral projection of ww^* for $(0, \infty)$. Since $\ker(ww^*) = \ker w^*$, this is the projection onto the closure of the range of w. Hence $pw = w$. Since $w^2 = 0$, the range of w lies in its closed kernel, so $wp = 0$ and $pw(1-p) = w$.

Apply Kaplansky's density theorem to $2p-1$ and, after rescaling as needed, to the real and imaginary parts of w. On the product J of the three index sets, this gives a bounded net $(x_j)_{j\in J}$ in A converging strong* to w and a net $(h_j)_{j\in J}$ of self-adjoint contractions in A converging strong* to $2p-1$. For $j\in J$, put

$$h_j^{\pm} := \frac{|h_j| \pm h_j}{2}, \qquad w_j := h_j^+ x_j h_j^-.$$

The elements h_j^+ and h_j^- are positive contractions with $h_j^- h_j^+ = 0$. Thus $(w_j)_{j\in J}$ is bounded and

$$w_j^2 = h_j^+ x_j (h_j^- h_j^+) x_j h_j^- = 0, \qquad j \in J.$$

By Lemma 2.1, $h_j^+ \longrightarrow p$ and $h_j^- \longrightarrow 1-p$ strong*. Joint strong* continuity of multiplication gives $w_j \longrightarrow pw(1-p) = w$ strong*, hence weak*. Since ϕ^{**} is weak*-continuous, $\phi(w_j) \longrightarrow \phi^{**}(w)$ weak* in B^{**}.

To pass to the limit in products, we use Haagerup's extension theorem for scalar-valued bilinear forms. Fix $f \in B^*$ and define $V : A \times A \to \mathbb{C}$ by

$$V(x,y) := f\left(\phi(x)\phi(y)\right), \qquad x, y \in A.$$

This form is bounded because ϕ is bounded. By Haagerup's theorem [14, Corollary 2.4], it has an extension $\widetilde{V} : A^{**} \times A^{**} \to \mathbb{C}$ that is separately weak*-continuous and jointly strong*-continuous on bounded sets. The continuity properties of the Arens product and separate weak* continuity of $\widetilde{V}$ give

$$\begin{aligned}\left(\phi^{**}(w) \,\square\, \phi^{**}(w)\right)(f) &= \lim_{j\in J}\lim_{k\in J} V(w_j, w_k) = \lim_{j\in J} \widetilde{V}(w_j, w) \\ &= \widetilde{V}(w,w).\end{aligned}$$

Since $\phi(w_j)^2 = 0$ for every $j \in J$, joint strong* continuity gives

$$\widetilde{V}(w,w) = \lim_{j\in J} V(w_j, w_j) = 0.$$

As f was arbitrary, $\phi^{**}(w) \,\square\, \phi^{**}(w) = 0$. □

Proof of Theorem 1.2. (a) $\implies$ (b). Suppose that A has no tracial states. By Lemma 4.1, A^{**} is properly infinite. Let B be a unital Banach algebra and let $\phi : A \to B$ be a bounded unital linear map that preserves square-zero elements. Put $\psi := \phi^{**} : A^{**} \to B^{**}$. The map ψ is bounded and unital, and it preserves square-zero elements by Lemma 4.2. Mathieu and Schick's result [27, Corollary 3.5] therefore shows that ψ is a Jordan homomorphism. Thus

$$\psi(x^2) = \psi(x) \,\square\, \psi(x), \qquad x \in A^{**}.$$

The canonical embeddings of A and B into their biduals are homomorphisms, so restricting this identity to A shows that ϕ is a Jordan homomorphism.

(b) $\implies$ (c). Let B be a unital semisimple Banach algebra and let $\phi : A \to B$ be a unital surjective spectrally bounded linear map. Aupetit's automatic continuity theorem [4, Theorem 5.5.2] gives boundedness of ϕ. Mathieu and Schick's result [27, Corollary 3.2] shows that ϕ preserves square-zero elements. Condition (b) therefore implies that ϕ is a Jordan homomorphism.

(c) $\implies$ (a). Suppose that A has a tracial state τ. Fix $\alpha \in (0,1)$ and define $\phi : A \to A$ by

$$\phi(x) := \alpha x + (1-\alpha)\tau(x)1, \qquad x \in A.$$

This is the identity-map case of Mathieu's construction [24, formula (1)]. Since $\tau(1) = 1$, we have $A = \mathbb{C}1 \oplus \ker\tau$. The map acts as the identity on $\mathbb{C}1$ and as multiplication by α on $\ker\tau$. Since τ is bounded, ϕ is a bounded unital linear bijection.

Fix $x \in A$. Since τ is a tracial state, its value $\tau(x)$ lies in $\operatorname{conv}\sigma_A(x)$ [28, Section 2]. The affine spectral mapping theorem then gives

$$\sigma_A(\phi(x)) = \{\alpha\zeta + (1-\alpha)\tau(x) : \zeta \in \sigma_A(x)\} \subseteq \operatorname{conv}\sigma_A(x).$$

Taking convex hulls gives the asserted inclusion. The closed disc of radius $r_A(x)$ about 0 contains $\operatorname{conv}\sigma_A(x)$, so $r_A(\phi(x)) \le r_A(x)$.

If $x^2 = 0$, then $\sigma_A(x) = \{0\}$, whence $\tau(x) = 0$ and $\phi(x)^2 = \alpha^2 x^2 = 0$.

Since $\dim A \ge 2$ and τ is positive, $\ker\tau$ is nonzero and invariant under the involution. Taking a nonzero real or imaginary part of an element of $\ker\tau$, we obtain a nonzero self-adjoint $h \in \ker\tau$. Since $\phi(h) = \alpha h$,

$$\begin{aligned}\phi(h^2) - \phi(h)^2 &= (1-\alpha)\left(\alpha h^2 + \tau(h^2)1\right)\\ &\ge \alpha(1-\alpha)h^2 \ne 0.\end{aligned}$$

Here h^2 is positive and nonzero, and $\tau(h^2) \ge 0$. Thus ϕ fails to preserve squares. Since A is semisimple as a C^*-algebra, ϕ is a counterexample to condition (c). □

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References

[1] C. A. Akemann, *The dual space of an operator algebra*, Trans. Amer. Math. Soc. 126 (1967), no. 2, 286–302, doi: 10.1090/S0002-9947-1967-0206732-8.

[2] R. Arens, *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc. 2 (1951), no. 6, 839–848, doi: 10.1090/S0002-9939-1951-0045941-1.

[3] B. Aupetit, *Une généralisation du théorème de Gleason–Kahane–Żelazko pour les algèbres de Banach*, Pacific J. Math. 85 (1979), no. 1, 11–17, doi: 10.2140/pjm.1979.85.11.

[4] B. Aupetit, *A primer on spectral theory*, Universitext, Springer-Verlag, New York, 1991, doi: 10.1007/978-1-4612-3048-9.

[5] B. Aupetit, *Sur les transformations qui conservent le spectre*, in *Banach Algebras '97* (E. Albrecht and M. Mathieu, eds.), Walter de Gruyter, Berlin, 1998, pp. 55–78, doi: 10.1515/9783110802009.55.

[6] B. Aupetit, *Spectrum-preserving linear mappings between Banach algebras or Jordan–Banach algebras*, J. London Math. Soc. (2) 62 (2000), no. 3, 917–924, doi: 10.1112/S0024610700001514.

[7] M. Brešar and P. Šemrl, *Invertibility preserving maps preserve idempotents*, Michigan Math. J. 45 (1998), no. 3, 483–488, doi: 10.1307/mmj/1030132294.

[8] M. Brešar and P. Šemrl, *An extension of the Gleason–Kahane–Żelazko theorem: A possible approach to Kaplansky's problem*, Expo. Math. 26 (2008), no. 3, 269–277, doi: 10.1016/j.exmath.2007.11.004.

[9] L. G. Brown and G. K. Pedersen, *C^*-algebras of real rank zero*, J. Funct. Anal. 99 (1991), no. 1, 131–149, doi: 10.1016/0022-1236(91)90056-B.

[10] M.-D. Choi, D. Hadwin, E. Nordgren, H. Radjavi and P. Rosenthal, *On positive linear maps preserving invertibility*, J. Funct. Anal. 59 (1984), no. 3, 462–469, doi: 10.1016/0022-1236(84)90060-0.

[11] M. J. Colbrook and G. Stepaniants, *Spectral isometries of unital C^*-algebras*, preprint (2026), arXiv:2610.40470v1, doi: 10.48550/arXiv.2610.40470.

[12] J. L. Cui and J. C. Hou, *A characterization of homomorphisms between Banach algebras*, Acta Math. Sin. (Engl. Ser.) 20 (2004), no. 4, 761–768, doi: 10.1007/s10114-004-0312-8.

[13] A. M. Gleason, *A characterization of maximal ideals*, J. Analyse Math. 19 (1967), 171–172, doi: 10.1007/BF02788714.

[14] U. Haagerup, *The Grothendieck inequality for bilinear forms on C^*-algebras*, Adv. Math. 56 (1985), no. 2, 93–116, doi: 10.1016/0001-8708(85)90026-X.

[15] L. A. Harris and R. V. Kadison, *Affine mappings of invertible operators*, Proc. Amer. Math. Soc. 124 (1996), no. 8, 2415–2422, doi: 10.1090/S0002-9939-96-03445-4.

[16] R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras. Vol. I. Elementary theory*, Graduate Studies in Mathematics, Vol. 15, American Mathematical Society, Providence, RI, 1997, doi: 10.1090/gsm/015.

[17] R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras. Vol. II. Advanced theory*, Graduate Studies in Mathematics, Vol. 16, American Mathematical Society, Providence, RI, 1997, doi: 10.1090/gsm/016.
[18] J.-P. Kahane and W. Żelazko, *A characterization of maximal ideals in commutative Banach algebras*, Studia Math. 29 (1968), no. 3, 339–343, doi: 10.4064/sm-29-3-339-343.
[19] I. Kaplansky, *Algebraic and analytic aspects of operator algebras*, CBMS Regional Conference Series in Mathematics, Vol. 1, American Mathematical Society, Providence, RI, 1970, doi: 10.1090/cbms/001.
[20] I. Kovács, *Invertibility-preserving maps of C^*-algebras with real rank zero*, Abstr. Appl. Anal. 2005 (2005), no. 6, 685–689, doi: 10.1155/AAA.2005.685.
[21] Y.-F. Lin and M. Mathieu, *Jordan isomorphism of purely infinite C^*-algebras*, Q. J. Math. 58 (2007), no. 2, 249–253, doi: 10.1093/qmath/hal024.
[22] M. Mathieu, *Spectrally bounded traces on C^*-algebras*, Bull. Austral. Math. Soc. 68 (2003), no. 1, 169–173, doi: 10.1017/S0004972700037527.
[23] M. Mathieu, *Spectrally bounded operators on simple C^*-algebras*, Proc. Amer. Math. Soc. 132 (2004), no. 2, 443–446, doi: 10.1090/S0002-9939-03-07215-0.
[24] M. Mathieu, *Spectrally bounded operators on simple C^*-algebras, II*, Irish Math. Soc. Bull. 54 (2004), 33–40, doi: 10.33232/BIMS.0054.33.40.
[25] M. Mathieu, *Elementary operators—still not elementary?*, Opuscula Math. 36 (2016), no. 6, 787–797, doi: 10.7494/OpMath.2016.36.6.787.
[26] M. Mathieu, *Characterizing Jordan homomorphisms*, Acta Sci. Math. (Szeged) 86 (2020), no. 3–4, 697–701, doi: 10.14232/actasm-020-067-7.
[27] M. Mathieu and G. J. Schick, *Spectrally bounded operators from von Neumann algebras*, J. Operator Theory 49 (2003), no. 2, 285–293, Journal archive.
[28] M. Mathieu and F. Schulz, *Invertibility preserving mappings onto finite C^*-algebras*, Studia Math. 272 (2023), no. 1, 113–120, doi: 10.4064/sm230101-27-3.
[29] A. R. Sourour, *Invertibility preserving linear maps on $\mathcal{L}(X)$*, Trans. Amer. Math. Soc. 348 (1996), no. 1, 13–30, doi: 10.1090/S0002-9947-96-01428-6.
[30] W. Żelazko, *A characterization of multiplicative linear functionals in complex Banach algebras*, Studia Math. 30 (1968), no. 1, 83–85, doi: 10.4064/sm-30-1-83-85.

I. Gogić, University of Zagreb Faculty of Science, Department of Mathematics, Bijenička 30, 10000 Zagreb, Croatia
Email address: ilja@math.hr, ilja.gogic@math.pmf.unizg.hr