ON THE IDEAL STRUCTURE OF THE TENSOR PRODUCT OF NEARLY SIMPLE ALGEBRAS

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Abstract. We define a unital algebra A over a field $\mathbb{F}$ to be nearly simple if A contains a unique non-trivial ideal I_A such that $I_A^2 \neq \{0\}$. If A and B are two nearly simple algebras over $\mathbb{F}$, we consider the ideal structure of their tensor product $A \otimes B$. The obvious non-trivial ideals of $A \otimes B$ are:

$$I_A \otimes I_B, \quad I_A \otimes B, \quad A \otimes I_B, \quad \text{and} \quad I_A \otimes B + A \otimes I_B.$$

The purpose of this paper is to characterize when are all non-trivial ideals of $A \otimes B$ of the above form.

1. Introduction

Let A and B be unital algebras over a field $\mathbb{F}$. If A is a central simple algebra, it is well-known that all ideals of the tensor product $A \otimes B$ are of the form $A \otimes J$, where J is an ideal of B (see e.g. [4, Theorem 4.42] and the comment following its proof).

However, the ideal structure of $A \otimes B$ is generally much more complicated than the one of A and B, even in the simplest cases when A and B are proper field extensions of $\mathbb{F}$. In fact, if $\mathbb{K}$ is any proper field extension of $\mathbb{F}$, then $\mathbb{K} \otimes_{\mathbb{F}} \mathbb{K}$ is never a field, since for any $x \in \mathbb{K} \setminus \mathbb{F}$ the non-zero tensor $1 \otimes x - x \otimes 1$ lies in the kernel of the multiplication $m : \mathbb{K} \otimes_{\mathbb{F}} \mathbb{K} \to \mathbb{K}$, $m(x \otimes y) = xy$. Moreover, the problem of characterizing when is the tensor product of two fields a field (or a domain) is highly non-trivial and for results on this subject we refer to [7] and the references within. We also refer to a survey paper [8] that considers which properties of commutative algebras A and B are conveyed to $A \otimes B$.

In this paper we study the ideal structure of the tensor product of two unital algebras that both contain only one non-trivial ideal. To avoid pathologies, we also add one additional requirement:

Definition 1.1. We define a unital algebra A to be *nearly simple* if A contains a unique non-trivial ideal, denoted by I_A, such that $I_A^2 \neq \{0\}$.

The basic examples of nearly simple algebras are the unitizations of non-unital simple algebras (Example 3.2). Further, if V is a vector space over $\mathbb{F}$ of countably infinite dimension, then the algebra $A = \mathrm{End}_{\mathbb{F}}(V)$ of all linear operators on V is nearly simple, where I_A is the ideal of finite rank operators (Example 3.3).

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If A and B are two nearly simple algebras, then the obvious non-trivial ideals of $A \otimes B$ are:

$$I_A \otimes I_B, \quad I_A \otimes B, \quad A \otimes I_B, \quad \text{and} \quad I_A \otimes B + A \otimes I_B. \tag{1.1}$$

The main result of this paper is Theorem 3.9, in which we characterize when are all non-trivial ideals of $A \otimes B$ of the above form.

2. Preliminaries

Throughout this paper $\mathbb{F}$ denotes a field. Unless specified otherwise, our vector spaces (algebras) will be over $\mathbb{F}$ and all tensor products will be over $\mathbb{F}$. Also, our algebras are assumed to be associative.

Given any algebra A, we write $Z(A)$ for the centre of A. For $x, y \in A$, the commutator $xy - yx$ is denoted by $[x, y]$. By an ideal of A we always mean a two-sided ideal. As usual, we say that an ideal I of A is non-trivial if $I \neq \{0\}$ and $I \neq A$. If A is unital and $Z(A) = \mathbb{F}1$, A is said to be *central*.

For an element $a \in A$ by $\langle a \rangle$ we denote the principle ideal generated by a. Further, for $a, b \in A$ we define a *two-sided multiplication*

$$M_{a,b} : A \to A \qquad \text{by} \qquad M_{a,b} : x \mapsto axb.$$

By an *elementary operator* on A we mean a map $\phi : A \to A$ that can be expressed as a finite sum of two-sided multiplications, that is

$$\phi(x) = \sum_i a_i x b_i$$

for some finite collections of $a_i, b_i \in A$ (the coefficients of ϕ). We denote the set of all elementary operators on A by $\mathcal{E}\ell(A)$.

For a prime algebra A, by $M(A)$ and $Q_s(A)$ we respectively denote the *multiplier algebra* and the *symmetric algebra of quotients* of A (see e.g. [1, 2]). The centre of $Q_s(A)$, denoted by $C(A)$, is called the *extended centroid* and it is a field [1, Corollary 2.1.9]. A unital prime algebra A is said to be *centrally closed* if $C(A) = Z(A) = \mathbb{F}1$. In particular, a unital simple algebra is centrally closed if and only if it is central.

If V is a vector space and L a subspace of V, then a finite subset $\{v_1, \ldots, v_n\}$ of V is said to be *independent over* L if the set $\{v_1 + L, \ldots, v_n + L\}$ is linearly independent in V/L.

We will frequently use the next two well-known facts, but as we have been unable to find a direct reference we include their proofs for completeness.

Lemma 2.1. *Let V and W be vector spaces and let L be a subspace of V. Assume that*

$$t = \sum_{i=1}^{n} v_i \otimes w_i \in V \otimes W$$

is a tensor of rank $n \geq 1$ such that $v_i \notin L$ for some $1 \leq i \leq n$. Then there are $1 \leq k \leq n$, $v'_1, \ldots, v'_n \in V$ and $w'_1, \ldots, w'_n \in W$ such that

$$t = \sum_{i=1}^{n} v'_i \otimes w'_i,$$

where $\{v'_1, \ldots, v'_k\}$ is independent over L and $v'_{k+1}, \ldots, v'_n \in L$.

Proof. Without loss of generality we may assume that $\{v_1,\dots,v_k\}$ is a maximal subset of $\{v_1,\dots,v_n\}$ that is independent over L. If $k=n$ we are done, so assume that $k<n$. Then for each $j=k+1,\dots,n$ there is $f_j\in L$ and scalars $\lambda_{1j},\dots,\lambda_{kj}\in\mathbb{F}$ that are not all zero such that

$$v_j=\sum_{i=1}^{k}\lambda_{ij}v_i+f_j.$$

Then

$$t=\sum_{i=1}^{k}v_i\otimes\left(w_i+\sum_{j=k+1}^{n}\lambda_{ij}w_j\right)+\sum_{j=k+1}^{n}f_j\otimes w_j$$

is a desired decomposition of t. □

Proposition 2.2. *Let A and B be algebras. If I and J are ideals of A and B, respectively, with corresponding canonical maps $q_I:A\to A/I$ and $q_J:B\to B/J$, then the map*

$$q_I\otimes q_J:A\otimes B\to(A/I)\otimes(B/J),\qquad \sum_i a_i\otimes b_i\mapsto\sum_i(a_i+I)\otimes(b_i+J)$$

is an algebra epimorphism with $\ker(q_I\otimes q_J)=I\otimes B+A\otimes J$. In particular,

$$(A\otimes B)/(I\otimes B+A\otimes J)\cong(A/I)\otimes(B/J),$$

as algebras.

Proof. Obviously, $q_I\otimes q_J$ is an algebra epimorphism and $I\otimes B+A\otimes J\subseteq\ker(q_I\otimes q_J)$. For the reverse inclusion, assume that

$$t=\sum_{i=1}^{n}a_i\otimes b_i\in A\otimes B$$

is a tensor of rank $n\geq 1$ such that $(q_I\otimes q_J)(t)=0$. If all a_i belong to I and all b_i belong to J we have nothing to prove. Assume that some $a_i\notin I$. By Lemma 2.1 we may assume that $\{a_1,\dots,a_k\}$ is independent over I and $a_{k+1},\dots,a_n\in I$ for some $1\leq k\leq n$. Then

$$\sum_{i=1}^{k}(a_i+I)\otimes(b_i+J)=(q_I\otimes q_J)(t)=0,$$

which forces $b_1,\dots,b_k\in J$. Hence,

$$t=\sum_{j=k+1}^{n}a_j\otimes b_j+\sum_{i=1}^{k}a_i\otimes b_i\in I\otimes B+A\otimes J.$$

The similar argument also shows that $t\in I\otimes B+A\otimes J$ if some $b_i\notin J$. □

3. Results

We begin this section with the next simple observation.

Proposition 3.1. *Let A be a nearly simple algebra. Then:*

(a) *A is prime.*
(b) *I_A is a simple infinite-dimensional algebra such that $Z(I_A)=\{0\}$.*
(c) *$Z(A)$ is a field and $Q_s(A)=M(I_A)$.*

Proof. (a) Since the only non-trivial ideal I_A of A satisfies $I_A^2 \neq \{0\}$, A is obviously prime.

(b) Assume that J is a non-zero ideal of I_A. Since A is prime, I_A is also prime (see e.g. [1, Lemma 1.1.3]). In particular, $J^3 \neq \{0\}$, so $I_A J I_A$ is a non-zero ideal of A that is contained in J. Then, since A is nearly simple, we have $I_A = I_A J I_A \subseteq J$, and thus $J = I_A$. This shows that I_A is simple.

Assume that I_A is unital with unity e. Then by [4, Lemma 2.54] e is a central idempotent of A such that $I_A = eA$. Then $(1-e)A$ is also a non-trivial ideal of A, so $I_A = (1-e)A$ and thus $I_A = \{0\}$; a contradiction. Hence, I_A is a non-unital simple algebra and consequently $Z(I_A) = \{0\}$.

Now assume that I_A is finite-dimensional. Then, since I_A is simple, Wedderburn's Theorem (see e.g. [4, Theorem 2.61]) implies that there is a natural number n and a division algebra $\mathbb{D}$ over $\mathbb{F}$ such that $I_A \cong \mathrm{M}_n(\mathbb{D})$. In particular, I_A is unital; a contradiction with the preceding paragraph.

(c) Assume that $z \in Z(A)$ is a non-invertible element. Then zA is an ideal of A such that $zA \neq A$ and hence $zA \subseteq I_A$. In particular, by (b),

$$z \in Z(A) \cap I_A \subseteq Z(I_A) = \{0\},$$

that is, $z = 0$. Thus, $Z(A)$ is a field. Finally, the equality $Q_s(A) = M(I_A)$ is a direct consequence of [1, Proposition 2.1.3] and the fact that $I_A^2 = I_A$. □

We now present some examples of nearly simple algebras.

Example 3.2. Let A be a non-unital simple algebra. Then its unitization $A^\sharp = \mathbb{F} \times A$ (see e.g. [4, Section 2.3]) is a nearly simple algebra with $I_{A^\sharp} = A$.

Similarly, set

$$B := C(A) + A \subseteq Q_s(A).$$

As A is simple and non-unital, we have $Q_s(A) = M(A)$ and $C(A) \cap A = Z(A) = \{0\}$. Hence, since A is an ideal of $M(A)$, B is a subalgebra of $M(A)$ such that $B/A \cong C(A)$ (which is a field). Thus, B is a nearly simple algebra with $I_B = A$.

Example 3.3. Let V be a vector space over $\mathbb{F}$ of countably infinite dimension. Consider the algebra $\mathrm{End}_{\mathbb{F}}(V)$ of all linear operators on V. If by $\mathrm{F}(V)$ we denote the ideal of finite rank operators in $\mathrm{End}_{\mathbb{F}}(V)$, it is well-known that $\mathrm{F}(V)$ is the only non-trivial ideal of $\mathrm{End}_{\mathbb{F}}(V)$ and that $\mathrm{End}_{\mathbb{F}}(V) = M(\mathrm{F}(V))$. In particular, by Proposition 3.1 (c), $Q_s(\mathrm{End}_{\mathbb{F}}(V)) = \mathrm{End}_{\mathbb{F}}(V)$, so $\mathrm{End}_{\mathbb{F}}(V)$ is a nearly simple centrally closed algebra (see also [4, Example 7.28]).

Further, if D is any simple subalgebra of $\mathrm{End}_{\mathbb{F}}(V)$ that contains the identity operator 1 (e.g. one of Weyl algebras), define

$$A := D + \mathrm{F}(V) \subseteq \mathrm{End}_{\mathbb{F}}(V).$$

As D is simple, we have $D \cap \mathrm{F}(V) = \{0\}$, so $A/\mathrm{F}(V) \cong D$ is simple and thus $\mathrm{F}(V)$ is the unique non-trivial ideal of A. Hence, A is a nearly simple algebra. Further, since by Proposition 3.1 (c) $Q_s(A) = M(\mathrm{F}(V)) = \mathrm{End}_{\mathbb{F}}(V)$, $C(A) = Z(\mathrm{End}_{\mathbb{F}}(V)) = \mathbb{F}1$, so A is also centrally closed.

Definition 3.4. If A and B are nearly simple algebras, we say that an ideal $\mathcal{J}$ of $A \otimes B$ is *admissible* if $\mathcal{J}$ is either trivial or of the form as in (1.1).

Lemma 3.5. *Assume that A and B are two nearly simple algebras such that all ideals of $A \otimes B$ are admissible. Then the algebra $(A/I_A) \otimes B$ is nearly simple, whose* *only non-trivial ideal is* $(A/I_A) \otimes I_B$*. Similarly,* $A \otimes (B/I_B)$ *is nearly simple with a non-trivial ideal* $I_A \otimes (B/I_B)$*.*

Proof. Let $q_{I_A} : A \to A/I_A$ be the canonical map and consider the algebra epimorphism $q_{I_A} \otimes \mathrm{id}_B : A \otimes B \to (A/I_A) \otimes B$. Let $\mathcal{J}$ be a non-trivial ideal of $(A/I_A) \otimes B$. Then, since all ideals of $A \otimes B$ are admissible, and $\ker(q_{I_A} \otimes \mathrm{id}_B) = I_A \otimes B$ (Proposition 2.2), $(q_{I_A} \otimes \mathrm{id}_B)^{-1}(\mathcal{J})$ is an ideal of $A \otimes B$ that strictly contains $I_A \otimes B$ and so $I_A \otimes B + A \otimes I_B \subseteq (q_{I_A} \otimes \mathrm{id}_B)^{-1}(\mathcal{J})$. Also, since $\mathcal{J}$ is non-trivial, $\mathcal{J} \neq (A/I_A) \otimes B$, so $(q_{I_A} \otimes \mathrm{id}_B)^{-1}(\mathcal{J}) \neq A \otimes B$. Consequently,

$$(q_{I_A} \otimes \mathrm{id}_B)^{-1}(\mathcal{J}) = I_A \otimes B + A \otimes I_B$$

and thus

$$\begin{aligned} (A/I_A) \otimes I_B &= (q_{I_A} \otimes \mathrm{id}_B)(I_A \otimes B + A \otimes I_B) \\ &= (q_{I_A} \otimes \mathrm{id}_B)((q_{I_A} \otimes \mathrm{id}_B)^{-1}(\mathcal{J})) \\ &= \mathcal{J}. \end{aligned}$$

The similar argument also shows that $I_A \otimes (B/I_B)$ is the only non-trivial ideal of $A \otimes (B/I_B)$. □

We now record some non-examples which helped us to conjecture the main result of this paper, Theorem 3.9.

Example 3.6. Let V be a real vector space of countably infinite dimension. Consider $\mathbb{C}$ as a unital subalgebra of $\mathrm{End}_{\mathbb{R}}(V)$. For example, if $\{e_n : n \in \mathbb{N}\}$ is a basis for V, define a linear operator $T \in \mathrm{End}_{\mathbb{R}}(V)$ by $T(e_{2n-1}) = e_{2n}$ and $T(e_{2n}) = -e_{2n-1}$ for all $n \in \mathbb{N}$. Then obviously $T^2 = -1$, where 1 is the identity operator, so we can identify $\mathbb{C}$ with the subalgebra $\{\alpha 1 + \beta T : \alpha, \beta \in \mathbb{R}\}$ of $\mathrm{End}_{\mathbb{R}}(V)$. Set

$$A := \mathbb{C} + \mathrm{F}(V) \subseteq \mathrm{End}_{\mathbb{R}}(V).$$

By Example 3.3 A is a centrally closed nearly simple (real) algebra with $I_A = \mathrm{F}(V)$. Consider the tensor product $A \otimes A$. Since by Proposition 2.2

$$(A \otimes A)/(\mathrm{F}(V) \otimes A + A \otimes \mathrm{F}(V)) \cong (A/\mathrm{F}(V)) \otimes (A/\mathrm{F}(V)) \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$$

and since $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is not a field, we conclude that $\mathrm{F}(V) \otimes A + A \otimes \mathrm{F}(V)$ is not a maximal ideal of $A \otimes A$. In fact, since $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$ (see e.g. [4, Example 4.45]), $\mathrm{F}(V) \otimes A + A \otimes \mathrm{F}(V)$ is contained in two distinct maximal ideals of $A \otimes A$.

Example 3.7. Let W be a complex vector space of countably infinite dimension. Consider the real algebra

$$B := \mathbb{R}1 + \mathrm{F}(W) \subseteq \mathrm{End}_{\mathbb{C}}(W),$$

where 1 is the identity operator. Then B is a central nearly simple algebra whose only non-trivial ideal is $\mathrm{F}(W)$. Note that B is not centrally closed, since by Proposition 3.1 (c) $Q_s(B) = M(\mathrm{F}(W)) = \mathrm{End}_{\mathbb{C}}(W)$ and thus $C(B) = Z(\mathrm{End}_{\mathbb{C}}(W)) = \mathbb{C}1$ (see also [4, Example 7.37]). Let c_1 and c_2 be elements of $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ defined by $c_1 := 1 \otimes i + i \otimes 1$ and $c_2 := 1 \otimes i - i \otimes 1$. Then

$$\mathcal{J}_1 := c_1(\mathrm{F}(W) \otimes \mathrm{F}(W)) \qquad \text{and} \qquad \mathcal{J}_2 := c_2(\mathrm{F}(W) \otimes \mathrm{F}(W))$$

are two non-zero ideals of $B \otimes B$ such that $\mathcal{J}_1 \mathcal{J}_2 = \{0\}$. In particular, $\mathcal{J}_1$ and $\mathcal{J}_2$ are non-admissible and $B \otimes B$ is not even prime.

Example 3.8. Consider the tensor product $A \otimes B$ of real algebras A and B from Examples 3.6 and 3.7. If $c_1, c_2 \in \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ are as in Example 3.7, then using the isomorphism $A/\mathrm{F}(V) \cong \mathbb{C}$, we see that

$$\mathcal{K}_1 := c_1((A/\mathrm{F}(V)) \otimes \mathrm{F}(W)) \qquad \text{and} \qquad \mathcal{K}_2 := c_2((A/\mathrm{F}(V)) \otimes \mathrm{F}(W))$$

are two non-zero ideals of $(A/\mathrm{F}(V)) \otimes B$ such that $\mathcal{K}_1\mathcal{K}_2 = \{0\}$. In particular, $(A/\mathrm{F}(V)) \otimes B$ is not prime, so by Lemma 3.5 $A \otimes B$ has a non-admissible ideal.

We now state the main result of the paper.

Theorem 3.9. *Let A and B be two nearly simple algebras. Then all ideals of $A\otimes B$ are admissible if and only if all tensor products*

$$Z(A/I_A) \otimes Z(B/I_B), \quad C(A) \otimes Z(B/I_B), \quad Z(A/I_A) \otimes C(B), \quad C(A) \otimes C(B) \tag{3.1}$$

are fields.

Remark 3.10. In Examples 3.6, 3.7 and 3.8 (respectively), when considering $A\otimes A$, $B \otimes B$ and $A \otimes B$ (respectively), all tensor products in (3.1) are fields, except $Z(A/I_A) \otimes Z(A/I_A)$, $C(B) \otimes C(B)$ and $Z(A/I_A) \otimes C(B)$ (respectively). This in particular demonstrates that none of the assumptions of Theorem 3.9 cannot be omitted (if A and B are algebras from Examples 3.6 and 3.7, then by symmetry $B \otimes A$ has a non-admissible ideal, $C(B) \otimes Z(A/I_A)$ is not a field, while $Z(B/I_B) \otimes Z(A/I_A)$, $Z(B/I_B) \otimes C(A)$ and $C(B) \otimes C(A)$ are fields).

The proof of Theorem 3.9 heavily relies on the main result of [10] (see also [9]) and its consequence which we state below.

Theorem 3.11. [10, Theorem] *If A and B are prime algebras, then each non-zero ideal of $A \otimes B$ contains a non-zero elementary tensor if and only if $C(A) \otimes C(B)$ is a field.*

Corollary 3.12. *If A and B are unital simple algebras, then $A \otimes B$ is a simple algebra if and only if $Z(A) \otimes Z(B)$ is a field.*

Proof. If $A\otimes B$ is simple, then $Z(A)\otimes Z(B) \cong Z(A\otimes B)$ (see e.g. [4, Corollary 4.32]) is a field.

Conversely, since both A and B are unital and simple, we have $Q_s(A) = A$ and $Q_s(B) = B$, so in particular $C(A) = Z(A)$ and $C(B) = Z(B)$. Hence if $Z(A) \otimes Z(B)$ is a field, then $A \otimes B$ is simple by [10, Corollary 1]. □

Lemma 3.13. *Let A and B be algebras and let $\mathcal{J}$ be an ideal of $A\otimes B$. If $\phi \in \mathcal{E}\ell(A)$ and $\psi \in \mathcal{E}\ell(B)$ are elementary operators, then $(\phi \otimes \psi)(\mathcal{J}) \subseteq \mathcal{J}$.*

In particular, if both A and B are unital and $a \otimes b \in \mathcal{J}$ for some $a \in A$ and $b \in B$, then $\langle a\rangle \otimes \langle b\rangle \subseteq \mathcal{J}$.

Proof. Assume that $\phi = \sum_i M_{a_i,a_i'}$ and $\psi = \sum_j M_{b_j,b_j'}$ for some finite collections of $a_i, a_i' \in A$ and $b_j, b_j' \in B$. Then

$$\phi \otimes \psi = \sum_j \sum_i M_{a_i\otimes b_j, a_i'\otimes b_j'} \in \mathcal{E}\ell(A \otimes B)$$

and thus $(\phi \otimes \psi)(\mathcal{J}) \subseteq \mathcal{J}$.

Next, assume that both A and B are unital. If $a \otimes b \in \mathcal{J}$ for some $a \in A$ and $b \in B$, then for any $x \in \langle a \rangle$ and $y \in \langle b \rangle$ there are elementary operators $\phi \in \mathcal{E}\ell(A)$ and $\psi \in \mathcal{E}\ell(B)$ such that $\phi(a) = x$ and $\psi(b) = y$. Hence,

$$x \otimes y = (\phi \otimes \psi)(a \otimes b) \in \mathcal{J}$$

and consequently $\langle a \rangle \otimes \langle b \rangle \subseteq \mathcal{J}$. □

Proposition 3.14. *Let A and B be unital prime algebras.*

(a) *If both A and B contain the smallest non-zero ideals I and J, respectively, then $I \otimes J$ is the smallest non-zero ideal of $A \otimes B$ if and only if $C(A) \otimes C(B)$ is a field.*

(b) *If M and N are maximal ideals of A and B, respectively, then $M \otimes B + A \otimes N$ is a maximal ideal of $A \otimes B$ if and only if $Z(A/M) \otimes Z(B/N)$ is a field.*

Proof. (a) Assume that $C(A) \otimes C(B)$ is a field and let $\mathcal{J}$ be a non-zero ideal of $A \otimes B$. By Theorem 3.11, $\mathcal{J}$ contains a non-zero elementary tensor $a \otimes b$. By Lemma 3.13 we have $\langle a \rangle \otimes \langle b \rangle \subseteq \mathcal{J}$. By assumption, $I \subseteq \langle a \rangle$ and $J \subseteq \langle b \rangle$, so $I \otimes J \subseteq \langle a \rangle \otimes \langle b \rangle \subseteq \mathcal{J}$.

If, on the other hand, $C(A) \otimes C(B)$ is not a field, choose a non-zero non-invertible element $c \in C(A) \otimes C(B)$. Since I is the smallest non-zero ideal of A, we have $C(A)I \subseteq A$. Similarly, $C(B)J \subseteq B$. Then, by the proof of [10, Theorem], $c(I \otimes J)$ defines a non-zero ideal of $A \otimes B$ that does not contain a non-zero elementary tensor. In particular, $c(I \otimes J)$ cannot contain $I \otimes J$.

(b) Obviously $M \otimes B + A \otimes N$ is a maximal ideal of $A \otimes B$ if and only if $(A \otimes B)/(M \otimes B + A \otimes N)$ is a simple algebra. Since by Proposition 2.2

$$(A \otimes B)/(M \otimes B + A \otimes N) \cong (A/M) \otimes (B/N),$$

by Corollary 3.12 $(A \otimes B)/(M \otimes B + A \otimes N)$ is simple if and only if $Z(A/M) \otimes Z(B/N)$ is a field. □

In the proof of Theorem 3.9 we will use the next version of Amitsur's Lemma (see [2, Theorem 4.2.7]) which states that if $T_1, \dots, T_n$ are linear operators between vector spaces V and W such that the vectors $T_1(x), \dots, T_n(x)$ are linearly dependent for every $x \in V$, then a non-trivial linear combination of $T_1, \dots, T_n$ has a finite rank.

We will also use the next fact, which was proved in [6] (see also [3, 5]).

Lemma 3.15. [6, Lemma 3.5] *Let δ be a non-zero derivation of a simple algebra D. If δ has a finite rank, then D is finite-dimensional.*

Proof of Theorem 3.9. First assume that all ideals of $A \otimes B$ are admissible. Proposition 3.14 then implies that $C(A) \otimes C(B)$ and $Z(A/I_A) \otimes Z(B/I_B)$ are fields. Further, by Lemma 3.5, the only non-trivial ideal of $(A/I_A) \otimes B$ is $(A/I_A) \otimes I_B$. In particular, since $(A/I_A) \otimes I_B$ contains non-zero elementary tensors, by Theorem 3.11 $Z(A/I_A) \otimes C(B)$ is a field. A similar argument also shows that $C(A) \otimes Z(B/I_B)$ must be a field.

Now assume that all tensor products in (3.1) are fields and let $\mathcal{J}$ be a non-zero ideal of $A \otimes B$. By Proposition 3.14 (a) we have $I_A \otimes I_B \subseteq \mathcal{J}$. Assume that

$$I_A \otimes I_B \subsetneq \mathcal{J}.$$

If $q_{I_A} : A \to A/I_A$ and $q_{I_B} : B \to B/I_B$ are the canonical maps, then one of the ideals

$$(q_{I_A} \otimes \mathrm{id}_B)(\mathcal{J}) \subseteq (A/I_A) \otimes B \qquad \text{or} \qquad (\mathrm{id}_A \otimes q_{I_B})(\mathcal{J}) \subseteq A \otimes (B/I_B)$$

must be non-zero, since otherwise

$$\begin{aligned}\mathcal{J} &\subseteq \ker(q_{I_A}\otimes \mathrm{id}_B)\cap\ker(\mathrm{id}_A\otimes q_{I_B}) = (I_A\otimes B)\cap(A\otimes I_B)\\ &= I_A\otimes I_B.\end{aligned}$$

Assume that $(q_{I_A}\otimes \mathrm{id}_B)(\mathcal{J})$ is a non-zero ideal of $(A/I_A)\otimes B$. By assumption, $Z(A/I_A)\otimes C(B)$ is a field, so by Theorem 3.11 $(q_{I_A}\otimes \mathrm{id}_B)(\mathcal{J})$ contains a non-zero elementary tensor.

Let $n\geq 1$ be the smallest number with the property that there exists a tensor $t\in\mathcal{J}$ of rank n such that $(q_{I_A}\otimes \mathrm{id}_B)(t)$ is a non-zero elementary tensor in $(A/I_A)\otimes B$. We claim that $n=1$, so that $a\otimes b\in\mathcal{J}$ for some $a\in A\setminus I_A$ and $b\in B\setminus\{0\}$. In this case, $\langle a\rangle = A$ and $I_B\subseteq\langle b\rangle$, so by Lemma 3.13 $\langle a\rangle\otimes\langle b\rangle\subseteq\mathcal{J}$. In particular,

$$A\otimes I_B\subseteq\mathcal{J}. \tag{3.2}$$

In order to obtain a contradiction, assume that $n>1$. Let $t\in\mathcal{J}$ be any tensor of rank n for which there exist $a'\in A\setminus I_A$ and $b'\in B\setminus\{0\}$ such that

$$(q_{I_A}\otimes \mathrm{id}_B)(t) = (a'+I_A)\otimes b'. \tag{3.3}$$

If t is represented as

$$t=\sum_{i=1}^n a_i\otimes b_i, \tag{3.4}$$

then obviously not all a_i belong to I_A. By Lemma 2.1 we may assume that the set $\{a_1,\ldots,a_k\}$ is independent over I_A and that $a_{k+1},\ldots,a_n\in I_A$ for some $1\leq k\leq n$. Also, since t is of rank n, the set $\{b_1,\ldots,b_n\}$ is linearly independent. We first show that $k=1$. Indeed, by (3.3) and (3.4) we have

$$\sum_{i=1}^k (a_i+I_A)\otimes b_i = (a'+I_A)\otimes b' \tag{3.5}$$

in $(A/I_A)\otimes B$. Clearly, $b'\in\mathrm{span}\{b_1,\ldots,b_k\}$, since otherwise the set $\{b',b_1,\ldots,b_k\}$ would be linearly independent and consequently $a_1,\ldots,a_k\in I_A$; a contradiction. Hence, there are scalars $\lambda_1,\ldots,\lambda_k\in\mathbb{F}$ such that

$$b'=\sum_{i=1}^k\lambda_i b_i.$$

Then, by (3.5),

$$\sum_{i=1}^k (a_i-\lambda_i a'+I_A)\otimes b_i = 0,$$

which forces $a_i-\lambda_i a'\in I_A$ for all $i=1,\ldots,k$. Since the set $\{a_1,\ldots,a_k\}$ is independent over I_A, this is only possible if $k=1$, as claimed. Hence $a_1\notin I_A$, while $a_2,\ldots,a_n\in I_A$.

Further, without loss of generality we may assume that $a_1=1_A$. Indeed, since $a_1\notin I_A$, we have $\langle a_1\rangle = A$, so there is an elementary operator $\phi\in\mathcal{E}\ell(A)$ such that $\phi(a_1)=1_A$. For $2\leq i\leq n$ set $a_i':=\phi(a_i)$. Then, $a_2',\ldots,a_n'\in I_A$ and by Lemma 3.13 we have

$$t' := 1_A\otimes b_1+\sum_{i=2}^n a_i'\otimes b_i = (\phi\otimes\mathrm{id}_B)(t)\in\mathcal{J}.$$

Obviously, $(q_{I_A} \otimes \mathrm{id}_B)(t') = (1_A + I_A) \otimes b_1 \neq 0$ in $(A/I_A) \otimes B$, so by minimality of n, t' is also a tensor of rank n. So if $a_1 \neq 1_A$, we may substitute t by t'. Hence, in the sequel we assume that

$$t = 1_A \otimes b_1 + \sum_{i=2}^{n} a_i \otimes b_i \in \mathcal{J}, \tag{3.6}$$

where $a_2, \dots, a_n \in I_A$.

Set $\mathbb{K} := C(A)$. By Proposition 3.1 (c), we have $C(A) = Z(M(I_A))$, so $\mathbb{K} I_A \subseteq I_A$. Hence, we may consider I_A as an algebra over $\mathbb{K}$. Without loss of generality we may assume that $\{a_2, \dots, a_l\}$ is a maximal $\mathbb{K}$-independent subset of $\{a_2, \dots, a_n\}$. Then for each $j = l+1, \dots, n$ there are $\alpha_{ij} \in \mathbb{K}$ such that

$$a_j = \sum_{i=2}^{l} \alpha_{ij} a_i,$$

so by (3.6)

$$t = 1_A \otimes b_1 + \sum_{i=2}^{l} a_i \otimes b_i + \sum_{j=l+1}^{n} \left(\sum_{i=2}^{l} \alpha_{ij} a_i \right) \otimes b_j. \tag{3.7}$$

We claim there is an element $x_0 \in I_A$ such that the set $\{[a_2, x_0], \dots, [a_l, x_0]\}$ is $\mathbb{K}$-independent. Indeed, if this set would be $\mathbb{K}$-dependent for all $x_0 \in I_A$, then by Amitsur's Lemma there are $\beta_2, \dots, \beta_l \in \mathbb{K}$ which are not all zero such that the inner derivation $\delta : I_A \to I_A$ defined by

$$\delta(x) := [\beta_2 a_2 + \ldots + \beta_l a_l, x]$$

has a finite rank. Since by Proposition 3.1 (b) I_A is simple and infinite-dimensional, Lemma 3.15 implies that δ is zero. As $Z(I_A) = \{0\}$ (again by Proposition 3.1 (b)), we conclude that

$$\beta_2 a_2 + \ldots + \beta_l a_l = 0.$$

Since the set $\{a_2, \dots, a_l\}$ is $\mathbb{K}$-independent, this yields $\beta_2 = \ldots = \beta_l = 0$; a contradiction.

If $x_0 \in I_A$ is an element from the preceding paragraph, by (3.7) we have

$$\mathcal{J} \ni [t, x_0 \otimes 1_B] = \sum_{i=2}^{l} [a_i, x_0] \otimes b_i + \sum_{j=l+1}^{n} \left(\sum_{i=2}^{l} \alpha_{ij} [a_i, x_0] \right) \otimes b_j. \tag{3.8}$$

Since $\{[a_2, x_0], \dots, [a_l, x_0]\}$ is $\mathbb{K}$-independent, by [2, Theorem 2.3.3] for each $i = 2, \dots, l$ there is an elementary operator $\phi_i \in \mathcal{E}\ell(I_A)$ such that

$$\phi_i([a_i, x_0]) \neq 0 \qquad \text{and} \qquad \phi_i([a_j, x_0]) = 0 \qquad \forall j \in \{2, \dots, l\} \setminus \{i\}.$$

Further, since $0 \neq \phi_i([a_i, x_0]) \in I_A$ and since I_A is simple (Proposition 3.1 (b)), $I_A \phi_i([a_i, x_0]) I_A$ is a non-zero ideal of I_A and thus $I_A \phi_i([a_i, x_0]) I_A = I_A$. Hence, for each $i = 2, \dots, l$ there is another elementary operator $\psi_i \in \mathcal{E}\ell(I_A)$ such that

$$\psi_i(\phi_i([a_i, x_0])) = a_i.$$

Set $\theta_i := \psi_i \circ \phi_i$. Then θ_i is an elementary operator on I_A such that

$$\theta_i([a_i, x_0]) = a_i \qquad \text{and} \qquad \theta_i([a_j, x_0]) = 0 \qquad \forall j \in \{2, \dots, l\} \setminus \{i\}.$$

By extending θ_i to an elementary operator on A (with the same coefficients), (3.8) and Lemma 3.13 imply

$$\mathcal{J} \ni (\theta_i \otimes \mathrm{id}_B)([t, x_0 \otimes 1_B]) = a_i \otimes b_i + \sum_{j=l+1}^{n} (\alpha_{ij} a_i) \otimes b_j,$$

for each $i = 2, \ldots, l$, so by (3.7)

$$\mathcal{J} \ni \sum_{i=2}^{l} \left(a_i \otimes b_i + \sum_{j=l+1}^{n} (\alpha_{ij} a_i) \otimes b_j \right) = t - 1_A \otimes b_1.$$

Since $t \in \mathcal{J}$, we conclude that $1_A \otimes b_1 \in \mathcal{J}$, which contradicts the assumption that $n > 1$. In particular, (3.2) holds.

The similar arguments would also show that if $(\mathrm{id}_A \otimes q_{I_B})(\mathcal{J})$ is a non-zero ideal of $A \otimes (B/I_B)$, then $I_A \otimes B \subseteq \mathcal{J}$.

Finally, if $\mathcal{J}$ strictly contains $I_A \otimes B$ or $A \otimes I_B$ (respectively), then the above proof shows that $\mathcal{J}$ also contains $A \otimes I_B$ or $I_A \otimes B$ (respectively). In any of those two cases we clearly have $I_A \otimes B + A \otimes I_B \subseteq \mathcal{J}$. Since, by assumption, $Z(A/I_A) \otimes Z(B/I_B)$ is a field, Proposition 3.14 (b) implies that $I_A \otimes B + A \otimes I_B$ is a maximal ideal of $A \otimes B$, which finishes the proof. □

Corollary 3.16. *Let A be a nearly simple algebra. The following conditions are equivalent:*

(i) *For any nearly simple algebra B, all ideals of $A \otimes B$ are admissible.*
(ii) *All ideals of $A \otimes A$ are admissible.*
(iii) *A is centrally closed and A/I_A is central.*

Proof. (i) $\Longrightarrow$ (ii). This is trivial.

(ii) $\Longrightarrow$ (iii). If A would not be centrally closed, $C(A)$ would be a proper field extension of $\mathbb{F}$. But then $C(A) \otimes C(A)$ is not a field, so by Proposition 3.14 (a) $A \otimes A$ has a non-admissible ideal. Similarly, if A/I_A is not central, then $Z(A/I_A) \otimes Z(A/I_A)$ is not a field, so by Proposition 3.14 (b) $A \otimes A$ has a non-admissible ideal.

(iii) $\Longrightarrow$ (i). By assumption $C(A) \cong Z(A/I_A) \cong \mathbb{F}$, so the assertion follows directly from Theorem 3.9. □

Example 3.17. In view of Example 3.3, if V is a vector space over $\mathbb{F}$ of countably infinite dimension, Corollary 3.16 in particular applies to algebras $\mathrm{End}_{\mathbb{F}}(V)$ and $D + \mathrm{F}(V)$, where D is any central simple subalgebra of $\mathrm{End}_{\mathbb{F}}(V)$ (that $\mathrm{End}_{\mathbb{F}}(V)/\mathrm{F}(V)$ is central follows from [6, Proposition 2.9]).

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