CLASSIFICATION OF JORDAN MULTIPLICATIVE MAPS ON MATRIX ALGEBRAS

ILJA GOGIĆ, MATEO TOMAŠEVIĆ

Abstract. Let $M_n(\mathbb{F})$ be the algebra of $n \times n$ matrices over a field $\mathbb{F}$ of characteristic not equal to 2. If $n \geq 2$, we show that an arbitrary map $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is Jordan multiplicative, i.e. it satisfies the functional equation

$$\phi(XY + YX) = \phi(X)\phi(Y) + \phi(Y)\phi(X), \quad \text{for all } X, Y \in M_n(\mathbb{F})$$

if and only if one of the following holds: either ϕ is constant and equal to a fixed idempotent, or there exists an invertible matrix $T \in M_n(\mathbb{F})$ and a ring monomorphism $\omega : \mathbb{F} \to \mathbb{F}$ such that

$$\phi(X) = T\omega(X)T^{-1} \quad \text{or} \quad \phi(X) = T\omega(X)^t T^{-1}, \quad \text{for all } X \in M_n(\mathbb{F}),$$

where $\omega(X)$ denotes the matrix obtained by applying ω entrywise to X. In particular, any Jordan multiplicative map $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ with $\phi(0) = 0$ is automatically additive. The analogous characterization fails when $\mathbb{F}$ has characteristic 2.

1. Introduction

An interesting class of problems in algebra revolves around exploring the interaction between the multiplicative and additive structures of rings and algebras. A landmark result in this area, due to Martindale [16, Corollary], states that any bijective multiplicative map from a prime ring containing a nontrivial idempotent onto an arbitrary ring is necessarily additive, and thus a ring isomorphism. Another fundamental result by Jodeit and Lam in [13] provides a classification of non-degenerate multiplicative self-maps on the matrix rings $M_n(\mathcal{R})$ over a principle ideal domain $\mathcal{R}$ (i.e. maps that are not identically zero on all zero-determinant matrices). Specifically, they show that for each such map $\phi : M_n(\mathcal{R}) \to M_n(\mathcal{R})$ one of the following holds: either there exists a nonzero idempotent matrix $P \in M_n(\mathcal{R})$ such that the map $\phi - P$ is multiplicative and degenerate, or there exists an invertible matrix $T \in M_n(\mathcal{R})$ and a ring endomorphism ω of $\mathcal{R}$ such that

$$\phi(X) = T\omega(X)T^{-1} \qquad \text{or} \qquad \phi(X) = T\omega(X)^* T^{-1}, \quad \text{for all } X \in M_n(\mathcal{R}),$$

where $\omega(X)$ denotes the matrix obtained by applying ω entrywise to X, and $(\cdot)^*$ represents the corresponding cofactor matrix. In particular, all bijective multiplicative self-maps on $M_n(\mathcal{R})$ are automatically additive and, consequently, ring automorphisms of $M_n(\mathcal{R})$. More recently, Šemrl in [19] provided a extensive classification of the (non-degenerate) multiplicative self-maps on matrix rings over arbitrary division rings. Additionally, in [18], Šemrl described the structure of multiplicative bijective maps on standard operator algebras, which are subalgebras of bounded linear maps on a complex Banach space that contain all finite-rank operators.

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On the other hand, any associative ring (algebra) $\mathcal{A}$ naturally inherits the structure of a Jordan ring (algebra), via the *Jordan product* defined by

$$x \diamond y := xy + yx, \quad \text{for all } x, y \in \mathcal{A}.$$

When working with algebras $\mathcal{A}$ over a filed $\mathbb{F}$ of characteristic not equal to 2, it is often more convenient to use the *normalized Jordan product*, defined by

$$x \circ y := \frac{1}{2}(xy + yx), \quad \text{for all } x, y \in \mathcal{A}. \tag{1.1}$$

The Jordan structure of algebras plays an important role in various areas, especially in the mathematical foundations of quantum mechanics (see e.g. [21]). The corresponding morphisms between rings (algebras) $\mathcal{A}$ and $\mathcal{B}$ are called *Jordan homomorphisms*, which are additive (linear) maps $\psi : \mathcal{A} \to \mathcal{B}$ satisfying

$$\phi(x \diamond y) = \phi(x) \diamond \phi(y), \quad \text{for all } x, y \in \mathcal{A}. \tag{1.2}$$

For 2-torsion-free rings (algebras), condition (1.2) is equivalent to the property that ϕ preservers squares, i.e.

$$\phi(x^2) = \phi(x)^2, \quad \text{for all } x \in \mathcal{A},$$

and, trivially, to the condition

$$\phi(x \circ y) = \phi(x) \circ \phi(y), \quad \text{for all } x, y \in \mathcal{A}, \tag{1.3}$$

when both $\mathcal{A}$ and $\mathcal{B}$ are $\mathbb{F}$-algebras with $\operatorname{char}(\mathbb{F}) \neq 2$. The most notable examples of Jordan homomorphisms are multiplicative and antimultiplicative maps. In fact, one of the central problems in Jordan theory, initially addressed by Jacobson and Rickart in [10] (see also [8, 20]) is to determine the conditions on rings (algebras) that guarantee any (typically surjective) Jordan homomorphism between rings (algebras) is either multiplicative, antimultiplicative, or, more generally, a suitable combination of such maps. For more recent developments on this topic, we refer to Brešar's paper [1] and the references therein.

Furthermore, when both $\mathcal{A}$ and $\mathcal{B}$ are standard operator algebras, with $\dim \mathcal{A} > 1$, Molnar classifies all bijective maps $\phi : \mathcal{A} \to \mathcal{B}$ satisfying (1.3) in [17, Theorem 1]. An important consequence of Molnar's result is that all such maps are automatically additive. Moreover, the same classification result applies to bijective maps ϕ satisfying (1.2). Indeed, as noted in [7, Remark 3.7], if $\phi : \mathcal{A} \to \mathcal{B}$ is $\diamond$-preserving (where $\mathcal{A}$ and $\mathcal{B}$ are any $\mathbb{F}$-algebras over a field $\mathbb{F}$ with $\operatorname{char}(\mathbb{F}) \neq 2$), then the map $\psi : \mathcal{A} \to \mathcal{B}$ defined by

$$\psi(x) := 2\phi\left(\frac{x}{2}\right), \quad \text{for all } x \in \mathcal{A} \tag{1.4}$$

is evidently $\circ$-preserving. Referring back to Molnar's classification theorem [17, Theorem 1], the finite-dimensional variant asserts that any bijective map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, $n \geq 2$, satisfying (1.3) (or (1.2)) takes the form

$$\phi(X) = T\omega(X)T^{-1} \quad \text{or} \quad \phi(X) = T\omega(X)^t T^{-1}, \quad \text{for all } X \in M_n(\mathbb{C}),$$

where $T \in M_n(\mathbb{C})$ is an invertible matrix and ω is a ring automorphism of $\mathbb{C}$, with $(\cdot)^t$ denoting the matrix transposition. In our recent work [7], the authors extended both [13, Corollary] and the finite-dimensional version of [17, Theorem 1] to the context of injective maps on structural matrix algebras (SMAs), which are subalgebras of $M_n(\mathbb{C})$ containing all diagonal matrices (for a simple characterization of SMAs see [6, Proposition 3.1]). For additional variants and generalizations of Molnar's result, particularly those related to the automatic additivity of bijective maps satisfying (1.2) or (1.3), we refer the reader to [11, 12, 14, 15] and the references therein.

The objective of this paper is to present a complete classification of Jordan multiplicative self-maps on matrix algebras. In contrast to the more intricate Jodeit-Lam's classification of the corresponding multiplicative self-maps, the Jordan multiplicative case exhibits a notably simpler structure:

Theorem 1.1. *Let $\mathbb{F}$ be a field with* $\mathrm{char}(\mathbb{F}) \neq 2$ *and let $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$, $n \geq 2$, be an arbitrary map satisfying either* (1.2) *or* (1.3)*. Then, one of the following holds:*

(a) *ϕ is a constant map, equal to a fixed idempotent, or*
(b) *ϕ is an additive map, and thus a Jordan ring monomorphism of $M_n(\mathbb{F})$. Consequently, there exists an invertible matrix $T \in M_n(\mathbb{F})$ and a ring monomorphism $\omega : \mathbb{F} \to \mathbb{F}$ such that*

$$\phi(X) = T\omega(X)T^{-1} \quad \text{or} \quad \phi(X) = T\omega(X)^t T^{-1}, \quad \text{for all } X \in M_n(\mathbb{F}). \tag{1.5}$$

The proof of Theorem 1.1 will be presented in Section §3. The approach follows a similar strategy to that in [7], relying entirely on elementary linear algebra techniques. Let us also highlight that a related variant of Theorem 1.1, concerning the Jordan multiplicative self-maps on the real subspace of self-adjoint matrices in $M_n(\mathbb{C})$ (with $n \geq 3$), was obtained by Fošner et al. in [4, Proposition 5.2]. We conclude the paper by demonstrating that non-constant Jordan multiplicative maps defined on general central SMAs, the C^*-algebra of bounded linear operators on an infinite-dimensional Hilbert space, or on $M_n(\mathbb{F})$ when $\mathrm{char}(\mathbb{F}) = 2$ (the $\diamond$-multiplicative variant), are no longer automatically additive (Examples 3.6, 3.7 and 3.8).

2. Notation and Preliminaries

We now introduce some notation that will be used throughout the paper. Let $\mathbb{F}$ be a fixed field of characteristic not equal to 2. By $\mathbb{F}^\times$ we denote the group of all nonzero elements in $\mathbb{F}$. Given a unital associative $\mathbb{F}$-algebra $\mathcal{A}$, by $\mathrm{Idem}(\mathcal{A})$ we denote the partially ordered set of all idempotents $\mathcal{A}$, where

$$p \leq q \qquad \text{if} \qquad pq = qp = p.$$

For $p \in \mathrm{Idem}(\mathcal{A})$ we denote $p^\perp := 1 - p \in \mathrm{Idem}(\mathcal{A})$. Further, for $p, q \in \mathrm{Idem}(\mathcal{A})$ we write

$$p \perp q \qquad \text{if} \qquad pq = qp = 0.$$

We use $\circ$ to denote the normalized Jordan product, defined by (1.1). Obviously $p \in \mathcal{A}$ is an idempotent if and only if it is a Jordan idempotent (i.e. satisfies $p \circ p = p$). We explicitly state the following simple lemma from [7], which will be used on several occasions.

Lemma 2.1 ([7, Lemma 2.1])**.** *Let $\mathcal{A}$ be an $\mathbb{F}$-algebra. For $p, q \in \mathrm{Idem}(\mathcal{A})$ and an arbitrary $a \in \mathcal{A}$ we have:*
(a) $p \circ a = 0$ if and only if $pa = ap = pap = 0$.
(b) $p \circ a = a$ if and only if $pa = ap = pap = a$.
(c) $p \perp q$ if and only if $p \circ q = 0$.
(d) $p \leq q$ if and only if $p \circ q = p$.

Let $n \in \mathbb{N}$ be a fixed positive integer.

– By $[n]$ we denote the set $\{1, \ldots, n\}$ and by Δ_n the diagonal $\{(j, j) : j \in [n]\}$ in $[n]^2$.
– By $M_n = M_n(\mathbb{F})$ we denote the algebra of $n \times n$ matrices over $\mathbb{F}$ and by $\mathcal{D}_n$ its subalgebra consisting of all diagonal matrices.
– The rank of a matrix $X \in M_n$ is denoted by $r(X)$.
– For matrices $X, Y \in M_n$, we write $X \propto Y$ to indicate that either $X = Y = 0$, or they are both nonzero and collinear.

– As usual, for $i, j \in [n]$, by $E_{ij} \in M_n$ we denote the standard matrix unit with 1 at the position (i, j) and 0 elsewhere. For a matrix $X = [X_{ij}]_{i,j=1}^n \in M_n$ we define its *support* as
$$\operatorname{supp} X := \{(i, j) \in [n]^2 : X_{ij} \neq 0\}.$$
– Given a ring endomorphism ω of $\mathbb{F}$, we use the same symbol ω to denote the induced ring endomorphism of M_n, defined by applying ω to the each entry of the corresponding matrix:
$$\omega(X) = [\omega(X_{ij})]_{i,j=1}^n, \quad \text{for all } X = [X_{ij}]_{i,j=1}^n \in M_n.$$

It is well-known that M_n is a simple algebra, and hence a simple Jordan algebra (see e.g. [9, Corollary of Theorem 1.1]). In fact, we have the following simple yet useful observation.

Proposition 2.2. *For an arbitrary matrix $X \in M_n$ define the subset $\mathcal{J}_X \subseteq M_n$ by*
$$\mathcal{J}_X := \{(\cdots(X \circ Y_1) \circ Y_2) \circ \cdots) \circ Y_k : k \in \mathbb{N}, Y_1, \ldots, Y_k \in M_n\}.$$
If $X \neq 0$, then $\mathcal{J}_X = M_n$.

Proof. Fix a nonzero matrix $X \in M_n$. It suffices to show that $I \in \mathcal{J}_X$.

- Suppose that $X_{ij} \neq 0$ for some distinct $i, j \in [n]$. First of all, we have
$$X_{ij}E_{ji} = E_{ji}XE_{ji} = (X \circ E_{ji}) \circ (2E_{ji}) \in \mathcal{J}_X,$$
so that
$$E_{ii} = \left((X_{ij}E_{ji}) \circ \left(\frac{2}{X_{ij}}E_{ij}\right)\right) \circ E_{ii} \in \mathcal{J}_X.$$
- Otherwise, suppose that $X \in \mathcal{D}_n$ and fix some $i \in [n]$ such that $X_{ii} \neq 0$. Then
$$E_{ii} = \left(X \circ \left(\frac{1}{X_{ii}}E_{ii}\right)\right) \circ E_{ii} \in \mathcal{J}_X.$$

In any case, $E_{ii} \in \mathcal{J}_X$ for some $i \in [n]$ also implies $E_{jj} \in \mathcal{J}_X$ for all $j \in [n]$. Indeed, if $j \neq i$:
$$\begin{aligned} E_{ji} = (2E_{ji}) \circ E_{ii} \in \mathcal{J}_X &\implies E_{ii} + E_{jj} = E_{ji} \circ (2E_{ij}) \in \mathcal{J}_X \\ &\implies E_{jj} = (E_{ii} + E_{jj}) \circ E_{jj} \in \mathcal{J}_X. \end{aligned}$$

For $r \in [n]$ denote
$$D_r := \sum_{j \in [r]} E_{jj} \in M_n.$$

We prove that $D_r \in \mathcal{J}_X$ for all $r \in [n]$ by induction on r (then $I = D_n \in \mathcal{J}_X$). To illustrate the process on a concrete example, consider $n = 5$. We have
$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{=A_2 \in \mathcal{J}_X} \circ \underbrace{\begin{bmatrix} 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{=B_2} = \underbrace{\begin{bmatrix} 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\in \mathcal{J}_X},$$
$$\underbrace{\begin{bmatrix} 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\in \mathcal{J}_X} \circ \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{=C_2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = D_2 \implies D_2 \in \mathcal{J}_X.$$

$$\underbrace{\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}}_{\in\mathcal{J}_X}\circ\underbrace{\begin{bmatrix}0&0&4&0&0\\0&-1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}}_{=B_3}=\underbrace{\begin{bmatrix}0&0&2&0&0\\0&-1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}}_{\in\mathcal{J}_X},$$

$$\underbrace{\begin{bmatrix}0&0&2&0&0\\0&-1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}}_{=A_3\in\mathcal{J}_X}\circ\underbrace{\begin{bmatrix}0&0&0&0&0\\0&-1&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}}_{=C_3}=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}=D_3\implies D_3\in\mathcal{J}_X.$$

$$\underbrace{\begin{bmatrix}1&0&0&0&0\\-2&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}}_{=A_4=A_4\circ D_2\in\mathcal{J}_X}\circ\underbrace{\begin{bmatrix}0&0&0&4&0\\0&0&-4&8&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}}_{=B_4}=\underbrace{\begin{bmatrix}0&0&0&2&0\\0&0&-2&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}}_{\in\mathcal{J}_X},$$

$$\underbrace{\begin{bmatrix}0&0&0&2&0\\0&0&-2&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}}_{\in\mathcal{J}_X}\circ\underbrace{\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&-1&0&0&0\\1&0&0&0&0\\0&0&0&0&0\end{bmatrix}}_{=C_4}=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\end{bmatrix}=D_4\implies D_4\in\mathcal{J}_X,$$

$$\underbrace{\begin{bmatrix}1&0&0&0&0\\-2&1&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}}_{=A_5=A_5\circ D_3\in\mathcal{J}_X}\circ\underbrace{\begin{bmatrix}0&0&0&0&4\\0&0&0&4&8\\0&0&-1&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}}_{=B_5}=\underbrace{\begin{bmatrix}0&0&0&0&2\\0&0&0&2&0\\0&0&-1&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}}_{\in\mathcal{J}_X},$$

$$\underbrace{\begin{bmatrix}0&0&0&0&2\\0&0&0&2&0\\0&0&-1&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}}_{\in\mathcal{J}_X}\circ\underbrace{\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&-1&0&0\\0&1&0&0&0\\1&0&0&0&0\end{bmatrix}}_{=C_5}=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}=D_5=I\implies I\in\mathcal{J}_X.$$

Of course, if $p:=\operatorname{char}(\mathbb{F})\neq 0$, all operations are performed modulo p.

We continue with the proof for general n. For $r=1$ we have $D_1=E_{11}\in\mathcal{J}_X$. Suppose that $D_{r-1}\in\mathcal{J}_X$ for some $2\leq r\leq n$. We prove that $D_r\in\mathcal{J}_X$. Let $(p_j)_{j\in\mathbb{N}}$ be a sequence in $\mathbb{F}$ defined as

$$p_j:=\begin{cases}1, & \text{if } j=1,\\ 2\cdot 3^{j-2}, & \text{if } j\geq 2,\end{cases}$$

and

$$A_r := \begin{cases} \left(\sum_{1\le j\le k} E_{jj}\right) - 2\left(\sum_{1\le j<i\le k} E_{ij}\right), & \text{if } r = 2k, \\ \left(\sum_{1\le j\le k+1} E_{jj}\right) - 2\left(\sum_{1\le j<i\le k} E_{ij}\right), & \text{if } r = 2k+1, \end{cases}$$

$$B_r := \begin{cases} -8E_{k,k+1} + 4\left(\sum_{1\le i\le j\le k} p_i E_{j,2k+i-j}\right), & \text{if } r = 2k, \\ -E_{k+1,k+1} + 4\left(\sum_{1\le i\le j\le k} p_i E_{j,2k+1+i-j}\right), & \text{if } r = 2k+1, \end{cases}$$

$$C_r := \begin{cases} -E_{k+1,k} + \sum_{1\le j\le k} E_{2k+1-j,j}, & \text{if } r = 2k, \\ -E_{k+1,k+1} + \sum_{1\le j\le k} E_{2k+2-j,j}, & \text{if } r = 2k+1. \end{cases}$$

We have that $\operatorname{supp} A_r \subseteq [r-1]\times[r-1]$, so

$$A_r = A_r \circ D_{r-1} \in \mathcal{J}_X.$$

Hence, using the observation $p_j = 2(p_1 + \cdots + p_{j-1})$, for $j \ge 2$, a straightforward calculation shows that

$$D_r = (A_r \circ B_r) \circ C_r \in \mathcal{J}_X.$$

□

We shall also require the following elementary fact, which is a simplified version of [7, Lemma 3.3] (applicable to general SMAs).

Lemma 2.3. *Let $\mathcal{S} \subseteq [n]^2 \setminus \Delta_n, n \ge 2$, be a nonempty subset. Suppose that for each $(i,j) \in \mathcal{S}$ we have:*
(a) $(i,k) \in \mathcal{S},$ for all $k \in [n]\setminus\{i\}$,
(b) $(l,j) \in \mathcal{S},$ for all $l \in [n]\setminus\{j\}$,
(c) $(j,i) \in \mathcal{S}$.
Then $\mathcal{S} = [n]^2 \setminus \Delta_n$.

Proof. Fix some $(i,j) \in \mathcal{S}$ and let $(k,l) \in [n]^2 \setminus \Delta_n$ be arbitrary. If $k \ne j$, then

$$(i,j) \in \mathcal{S} \overset{(b)}{\Longrightarrow} (k,j) \in \mathcal{S} \overset{(a)}{\Longrightarrow} (k,l) \in \mathcal{S}.$$

If $l \ne i$, then

$$(i,j) \in \mathcal{S} \overset{(a)}{\Longrightarrow} (i,l) \in \mathcal{S} \overset{(b)}{\Longrightarrow} (k,l) \in \mathcal{S}.$$

Finally, if $(k,l) = (j,i)$, then the claim follows directly from (c). □

3. Proof of Theorem 1.1

Let $m,n \in \mathbb{N}$ be fixed throughout the proof. Before proving our main result, we first establish some preliminary results, starting with the following straightforward consequence of Proposition 2.2.

Lemma 3.1. *Let $\mathcal{A}$ be an arbitrary $\mathbb{F}$-algebra and let $\phi : M_n \to \mathcal{A}$ be a $\circ$-multiplicative map such that $\phi(X) = 0$ for some nonzero matrix $X \in M_n$. Then ϕ is the zero map.*

Proof. By Proposition 2.2 we have $\mathcal{J}_X = M_n$, and therefore $\phi(X) = 0$ implies that ϕ is the zero map. □

The following lemma, which is a variant of [7, Lemma 3.4] (originally for injective $\circ$-multiplicative maps on SMAs), outlines the general properties of (not necessarily injective) $\circ$-multiplicative maps between matrix algebras.

Lemma 3.2. *Let $\phi : M_n \to M_m$, be a $\circ$-multiplicative map. Then the following holds true:*

(a) ϕ preserves idempotents, i.e. $\phi(\mathrm{Idem}(M_n)) \subseteq \mathrm{Idem}(M_m)$.
(b) For $P, Q \in \mathrm{Idem}(M_n)$ we have $P \le Q \implies \phi(P) \le \phi(Q)$.

Suppose now that ϕ is nonzero but $\phi(0) = 0$. Then:

(c) For $P, Q \in \mathrm{Idem}(M_n)$ we have $P \perp Q \implies \phi(P) \perp \phi(Q)$.
(d) For each nonzero $P \in \mathrm{Idem}(M_n)$ we have $r(\phi(P)) \ge r(P)$ (in particular, $m \ge n$).

Further, if $m = n$, then:

(e) For each $P \in \mathrm{Idem}(M_n)$ we have $r(\phi(P)) = r(P)$.
(f) For each $P \in \mathrm{Idem}(M_n)$ we have $\phi(P^\perp) = \phi(P)^\perp$.
(g) The restriction $\phi|_{\mathrm{Idem}(M_n)} : \mathrm{Idem}(M_n) \to \mathrm{Idem}(M_n)$ is orthoadditive, i.e.

$$P \perp Q \implies \phi(P+Q) = \phi(P) + \phi(Q), \quad \text{for all } P, Q \in \mathrm{Idem}(M_n).$$

(h) Suppose that $P_1, \ldots, P_r \in \mathrm{Idem}(M_n)$ are mutually orthogonal and let $\lambda_1, \ldots, \lambda_r \in \mathbb{F}$. Then

$$\phi\left(\sum_{j=1}^r \lambda_j P_j\right) = \sum_{j=1}^r \phi(\lambda_j P_j).$$

Proof. (a) This is clear.

(b) We have

$$\phi(P) = \phi(P \circ Q) = \phi(P) \circ \phi(Q)$$

which is by Lemma 2.1 equivalent to $\phi(P) \le \phi(Q)$.

(c) We have

$$\phi(P) \circ \phi(Q) = \phi(P \circ Q) = \phi(0) = 0,$$

so again by Lemma 2.1, $\phi(P) \perp \phi(Q)$.

(d) Let $P \in \mathrm{Idem}(M_n)$ be an arbitrary idempotent of rank $r \ge 1$. There exist mutually orthogonal rank-one idempotents $P_1, \ldots, P_r \in \mathrm{Idem}(M_n)$ such that $P = P_1 + \cdots + P_r$. Since ϕ is not the zero map, by Lemma 3.1 ϕ cannot annihilate any nonzero matrix, so in particular $\phi(P_j) \ne 0$ for all $j \in [r]$. Therefore,

$$P_1, \ldots, P_r \le P \overset{(b)}{\implies} \underbrace{\phi(P_1), \ldots, \phi(P_r)}_{\text{mutually orthogonal by (c)}} \le \phi(P).$$

Consequently, $r(\phi(P)) \ge r$.

(e) Let $P \in \mathrm{Idem}(M_n)$ be an arbitrary idempotent. By (c) we have $\phi(P) \perp \phi(P^\perp)$ and hence,

$$n = r(P) + r(P^\perp) \overset{(d)}{\le} r(\phi(P)) + r(\phi(P^\perp)) \le n$$

and thus $r(\phi(P)) = r(P)$.

(f) In view of (c) and (e), we have that $\phi(P^\perp)$ is an idempotent orthogonal to $\phi(P)$ of rank $r(P^\perp) = r(\phi(P)^\perp)$. Consequently, $\phi(P^\perp) = \phi(P)^\perp$.

(g) Since $P \perp Q$, we have that $P + Q$ is again an idempotent and $P, Q \le P + Q$. Statements (b) and (c) imply

$$\underbrace{\phi(P), \phi(Q)}_{\text{orthogonal}} \le \phi(P+Q)$$

and hence

$$\phi(P) + \phi(Q) \le \phi(P+Q).$$

Finally, we have

$$r(\phi(P)+\phi(Q)) = r(\phi(P)) + r(\phi(Q)) \overset{(e)}{=} r(P) + r(Q) = r(P+Q) \overset{(e)}{=} r(\phi(P+Q)),$$

so equality follows.

(h) We have

$$\begin{aligned}\phi\left(\sum_{j=1}^{r}\lambda_j P_j\right) &= \phi\left(\left(\sum_{j=1}^{r}\lambda_j P_j\right)\circ\left(\sum_{l=1}^{r}P_l\right)\right) = \phi\left(\sum_{j=1}^{r}\lambda_j P_j\right)\circ\phi\left(\sum_{l=1}^{r}P_l\right)\\ &\overset{(g)}{=} \phi\left(\sum_{j=1}^{r}\lambda_j P_j\right)\circ\left(\sum_{l=1}^{r}\phi(P_l)\right) = \sum_{l=1}^{r}\left(\phi\left(\sum_{j=1}^{r}\lambda_j P_j\right)\circ\phi(P_l)\right)\\ &= \sum_{l=1}^{r}\phi\left(\left(\sum_{j=1}^{r}\lambda_j P_j\right)\circ P_l\right) = \sum_{l=1}^{r}\phi(\lambda_l P_l).\end{aligned}$$

□

In the sequel, $\mathbb{K}$ will denote the prime subfield of $\mathbb{F}$, i.e. $\mathbb{K}$ is generated by the multiplicative identity of $\mathbb{F}$ (see e.g. [5]). Note that $\mathbb{K}\cong\mathbb{Q}$ if $\operatorname{char}(\mathbb{F})=0$, or $\mathbb{K}\cong\mathbb{Z}/p\mathbb{Z}$ if $p=\operatorname{char}(\mathbb{F})>0$.

Lemma 3.3. *Let $\phi: M_n\to M_n$ be a nonzero $\circ$-multiplicative map such that $\phi(0)=0$. There exists a unique multiplicative map $\omega:\mathbb{F}\to\mathbb{F}$ such that*

$$\phi(\lambda X) = \omega(\lambda)\phi(X), \quad \text{for all } \lambda\in\mathbb{F} \text{ and } X\in M_n. \tag{3.1}$$

Further, if $n\geq 2$, the map $\omega:\mathbb{F}\to\mathbb{F}$ is a ring monomorphism. In particular, ϕ is $\mathbb{K}$-homogeneous.

Proof. In view of Lemmas 3.1 and 3.2 (c) and (e), $\phi(E_{11}),\dots,\phi(E_{nn})$ are mutually orthogonal rank-one idempotents and therefore can be simultaneously diagonalized. Hence, by passing to map $T^{-1}\phi(\cdot)T$, for a suitable invertible matrix $T\in M_n$, without loss of generality we can assume that

$$\phi(E_{jj}) = E_{jj} \quad \text{for all } j\in[n]. \tag{3.2}$$

Obviously $\phi(\mathbb{F}^\times E_{jj})\neq\{0\}$ (again by Lemma 3.1), for all $j\in[n]$. Note that for each $X\in M_n$ and $S\subseteq[n]$ we have

$$\operatorname{supp}X\subseteq S\times S \implies \operatorname{supp}\phi(X)\subseteq S\times S. \tag{3.3}$$

Indeed, denote the diagonal idempotent $P:=\sum_{j\in[n]\setminus S}E_{jj}$ and note that a matrix $X\in M_n$ is supported in $S\times S$ if and only if $XP=PX=0$. In that case, obviously $X\circ P=0$, so

$$0=\phi(X\circ P)=\phi(X)\circ\phi(P)\overset{\text{Lemma 3.2}(g),\,(3.2)}{=}\phi(X)\circ P$$

and hence Lemma 2.1 (a) implies the claim.

Let $j\in[n]$ and $\lambda\in\mathbb{F}^\times$. Then

$$\phi(\lambda E_{jj}) = \phi((\lambda E_{jj})\circ E_{jj}) = \phi(\lambda E_{jj})\circ E_{jj}.$$

In view of Lemma 2.1 (b) we have

$$\phi(\lambda E_{jj}) = E_{jj}\phi(\lambda E_{jj})E_{jj} = \phi(\lambda E_{jj})_{jj}E_{jj}.$$

Since $\phi(0) = 0$, it follows that there exists a unique map $\omega_j : \mathbb{F} \to \mathbb{F}$ such that

$$\phi(\lambda E_{jj}) = \omega_j(\lambda)\phi(E_{jj}), \qquad \text{for all } \lambda \in \mathbb{F}.$$

Fix distinct $i, j \in [n]$. For $\lambda \in \mathbb{F}^\times$ by (3.2) we have

$$\begin{aligned}\omega_i(2\lambda)\phi(E_{ij}) \circ E_{ii} &= \phi(E_{ij} \circ (2\lambda E_{ii})) = \phi(\lambda E_{ij}) = \phi(E_{ij} \circ (2\lambda E_{jj})) \\ &= \omega_j(2\lambda)\phi(E_{ij}) \circ E_{jj}.\end{aligned}$$

Note that (3.3) implies that $\operatorname{supp} \phi(\lambda E_{ij}) \subseteq \{i, j\} \times \{i, j\}$. By $\phi(\lambda E_{ij})^2 = 0$ and Lemma 3.1 it follows that

$$\phi(\lambda E_{ij}) \propto E_{ij} \text{ or } E_{ji}. \tag{3.4}$$

Returning to the previous equation, it follows $\omega_i(2\lambda) = \omega_j(2\lambda)$. We conclude $\omega_i = \omega_j$ so there exists a unique globally defined map $\omega : \mathbb{F} \to \mathbb{F}$ such that

$$\phi(\lambda E_{jj}) = \omega(\lambda)E_{jj}, \qquad \text{for all } \lambda \in \mathbb{F}, j \in [n].$$

Now we prove (3.1). For $\lambda \in \mathbb{F}$ we have

$$\begin{aligned}\phi(\lambda I) = \phi\left(\sum_{j\in[n]} \lambda E_{jj}\right) &\overset{\text{Lemma 3.2 (h)}}{=} \sum_{j\in[n]} \phi(\lambda E_{jj}) = \sum_{j\in[n]} \omega(\lambda)\phi(E_{jj}) \\ &\overset{(3.2)}{=} \omega(\lambda)I.\end{aligned}$$

Now, for arbitrary $X \in M_n$ and $\lambda \in \mathbb{F}$ we have

$$\phi(\lambda X) = \phi(X \circ (\lambda I)) = \phi(X) \circ \phi(\lambda I) = \lambda\phi(X).$$

For some $i \in [n]$ and $\lambda, \mu \in \mathbb{F}$ (again using (3.2)) we have

$$\begin{aligned}\omega(\lambda\mu)E_{ii} &= \phi((\lambda\mu)E_{ii}) = \phi((\lambda E_{ii}) \circ (\mu E_{ii})) = \phi(\lambda E_{ii}) \circ \phi(\mu E_{ii}) \\ &= \omega(\lambda)\omega(\mu)E_{ii},\end{aligned}$$

which implies $\omega(\lambda\mu) = \omega(\lambda)\omega(\mu)$, so ω is a multiplicative map.

Assume now that $n \geq 2$. The argument that ω is additive is similar to the proof of [7, Theorem 3.1, Claim 5]. For completeness, we include the details. Let $i, j \in [n]$ be distinct. For fixed $x, y \in \mathbb{F}$ consider the idempotents

$$E_{ii} + xE_{ij}, E_{jj} + yE_{ij} \in \operatorname{Idem}(M_n).$$

By (3.3) we see that

$$\operatorname{supp} \phi(E_{ii} + xE_{ij}) \subseteq \{i, j\} \times \{i, j\}.$$

Denote

$$\phi(E_{ii} + xE_{ij}) = \sum_{(r,s)\in\{i,j\}\times\{i,j\}} \alpha_{rs}E_{rs}, \quad \alpha_{rs} \in \mathbb{F}.$$

From now on, in view of (3.4) assume that $\phi(E_{ij}) = \beta E_{ij}$ for some $\beta \in \mathbb{F}^\times$ as the other case (i.e. $\phi(E_{ij}) = \beta E_{ji}$) is similar. We have

$$\begin{aligned}\omega\left(\frac{1}{2}x\right)\beta E_{ij} &= \phi\left(\frac{1}{2}xE_{ij}\right) = \phi((E_{ii} + xE_{ij}) \circ E_{jj}) \overset{(3.2)}{=} \phi(E_{ii} + xE_{ij}) \circ E_{jj} \\ &= \frac{1}{2}\alpha_{ij}E_{ij} + \frac{1}{2}\alpha_{ji}E_{ji} + \alpha_{jj}E_{jj}.\end{aligned}$$

Since $\phi(E_{ii}+xE_{ij})$ is an idempotent and $\omega^{-1}(\{0\})=\{0\}$, we conclude

$$\alpha_{ij}=2\omega\left(\frac{1}{2}x\right)\beta, \qquad \alpha_{ji}=\alpha_{jj}=0, \qquad \alpha_{ii}=1.$$

Hence

$$\phi(E_{ii}+xE_{ij})=E_{ii}+2\omega\left(\frac{1}{2}x\right)\beta E_{ij}.$$

In an analogous way we arrive at the equality

$$\phi(E_{jj}+yE_{ij})=E_{jj}+2\omega\left(\frac{1}{2}y\right)\beta E_{ij}.$$

We have

$$\begin{aligned}\omega\left(\frac{x+y}{2}\right)\beta E_{ij} &= \phi\left(\frac{x+y}{2}E_{ij}\right)=\phi((E_{ii}+xE_{ij})\circ(E_{jj}+yE_{ij}))\\ &=\phi(E_{ii}+xE_{ij})\circ\phi(E_{jj}+yE_{ij})\\ &=\left(E_{ii}+2\omega\left(\frac{1}{2}x\right)\beta E_{ij}\right)\circ\left(E_{jj}+2\omega\left(\frac{1}{2}y\right)\beta E_{ij}\right)\\ &=\left(\omega\left(\frac{1}{2}x\right)+\omega\left(\frac{1}{2}y\right)\right)\beta E_{ij}\end{aligned}$$

and hence

$$\omega\left(\frac{x+y}{2}\right)=\omega\left(\frac{1}{2}x\right)+\omega\left(\frac{1}{2}y\right).$$

Since $x,y\in\mathbb{F}$ were arbitrary, this concludes the proof. □

The proof of the next lemma follows exactly the same lines as the proof of [7, Theorem 3.1, Claim 8], so we omit it.

Lemma 3.4. *Let $\phi: M_n\to M_n, n\geq 2$, be a $\circ$-multiplicative map such that $\phi(0)=0$. Then*

$$\phi(PXP)=\phi(P)\phi(X)\phi(P), \qquad \text{for all } X\in M_n, P\in \mathrm{Idem}(M_n).$$

Proof of Theorem 1.1. First, as noted in the introduction (and following [7, Remark 3.7]), it suffices to prove Theorem 1.1 for $\circ$-preserving maps, since the transformation (1.4) allows us to extend the result to $\diamond$-preserving maps. Therefore, assume that $\phi: M_n\to M_n$, $n\geq 2$, is $\circ$-multiplicative.

Suppose that ϕ is not the zero map. Since $\phi(0)$ is an idempotent, without loss of generality we can assume that

$$\phi(0)=\begin{bmatrix} I_r & 0\\ 0 & 0\end{bmatrix}, \qquad \text{for some } 0\leq r\leq n.$$

Assume that $r\geq 1$. Then we claim that ϕ is the constant map equal to $\phi(0)$. Indeed, first note that for all $X\in M_n$ we have

$$\phi(0)=\phi(X\circ 0)=\phi(X)\circ\phi(0)$$

which easily implies that

$$\phi(X)=\begin{bmatrix} I_r & 0\\ 0 & \psi(X)\end{bmatrix}, \tag{3.5}$$

for some uniquely determined matrix $\psi(X)\in M_{n-r}$. In particular, if $r=n$, it follows that ϕ is the constant map globally equal to I. Otherwise, it makes sense to consider the map $\psi: M_n\to M_{n-r}$ defined by (3.5), which is again $\circ$-multiplicative, and satisfies $\psi(0)=0$. Since

$n-r<n$, Lemma 3.2 (d) implies that ψ must be the zero map, and therefore $\phi(X)=\phi(0)$ for all $X\in M_n$.

Suppose now that $r=0$, i.e. that $\phi(0)=0$. We claim that ϕ takes the form given in (1.5), and as a result, it is a nonzero additive map. As in Lemma 3.3 and its proof, without loss of generality we can assume that

$$\phi(E_{jj})=E_{jj}, \quad \text{for all } j\in[n]$$

and consequently, by Lemma 3.2 (g),

$$\phi(P)=P, \quad \text{for all } P\in \mathrm{Idem}(M_n)\cap\mathcal{D}_n. \tag{3.6}$$

As in (3.4) for $\lambda=1$, under these assumptions we also obtain

$$\phi(E_{ij})\propto E_{ij} \text{ or } E_{ji}.$$

We claim that the same option holds throughout. We follow a similar approach as outlined in the proof of [7, Theorem 3.1, Claim 4]. For completeness we include details. Consider the set

$$\mathcal{S}:=\{(r,s)\in[n]^2\setminus\Delta_n : \phi(E_{rs})\propto E_{rs}\}.$$

For the sake of concreteness, assume that $(i,j)\in\mathcal{S}$. In that case clearly $\phi(E_{ji})\propto E_{ji}$, as otherwise

$$\phi\left(\frac{1}{2}(E_{ii}+E_{jj})\right)=\phi(E_{ij}\circ E_{ji})=\phi(E_{ij})\circ\phi(E_{ji})\propto E_{ij}\circ E_{ij}=0$$

is a contradiction with Lemma 3.1, so $(j,i)\in\mathcal{S}$. The next objective is to show that $(i,k)\in\mathcal{S}$ for any $k\in[n]\setminus\{i\}$, and $(l,j)\in\mathcal{S}$ for any $l\in[n]\setminus\{j\}$.

- Assume that $k\in[n]\setminus\{i,j\}$ and that $\phi(E_{ik})\propto E_{ki}$. Then

$$0=\phi(E_{ij}\circ E_{ik})=\phi(E_{ij})\circ\phi(E_{ik})\propto E_{ij}\circ E_{ki}=\frac{1}{2}E_{kj}$$

  is a contradiction, so it must be $\phi(E_{ik})\propto E_{ik}$.
- Assume that $l\in[n]\setminus\{i,j\}$ and that $\phi(E_{lj})\propto E_{jl}$. Then

$$0=\phi(E_{ij}\circ E_{lj})=\phi(E_{ij})\circ\phi(E_{lj})\propto E_{ij}\circ E_{jl}=\frac{1}{2}E_{il}$$

  is a contradiction, so that $\phi(E_{lj})\propto E_{lj}$.

By Lemma 2.3 it follows that $\mathcal{S}=[n]^2\setminus\Delta_n$, so there exists a map $g:[n]^2\to\mathbb{F}^\times$ such that

$$\phi(E_{ij})=g(i,j)E_{ij}, \quad \text{for all } i,j\in[n].$$

We claim that the map g is transitive in the sense of [3], i.e. it satisfies

$$g(i,j)g(j,k)=g(i,k), \quad \text{for all } (i,j),(j,k)\in[n]^2.$$

Fix $(i,j),(j,k)\in[n]^2$. If $i\neq k$, then

$$\begin{aligned}\frac{1}{2}g(i,k)E_{ik} &\overset{\text{Lemma 3.3}}{=} \phi\left(\frac{1}{2}E_{ik}\right)=\phi(E_{ij}\circ E_{jk})=\phi(E_{ij})\circ\phi(E_{jk})\\ &=\frac{1}{2}g(i,j)g(j,k)E_{ik},\end{aligned}$$

which implies $g(i,k) = g(i,j)g(j,k)$. On the other hand, if $i = k$, then

$$\frac{1}{2}(E_{ii} + E_{jj}) \overset{\text{Lemma 3.3,(3.6)}}{=} \phi\left(\frac{1}{2}(E_{ii} + E_{jj})\right) = \phi(E_{ij} \circ E_{ji}) = \phi(E_{ij}) \circ \phi(E_{ji})$$
$$= \frac{1}{2} g(i,j)g(j,i)(E_{ii} + E_{jj})$$

which implies $g(i,i) = 1 = g(i,j)g(j,i)$. Following [3], denote by

$$g^* : M_n \to M_n, \qquad g^*(E_{ij}) := g(i,j)E_{ij}$$

the induced (algebra) automorphism of M_n. Since every automorphism of M_n is inner (see e.g. [2, Theorem 1.30]), by passing to the map $X \mapsto (g^*)^{-1}(\phi(X))$, without loss of generality we can assume that

$$\phi(E_{ij}) = E_{ij}, \quad \text{for all } (i,j) \in [n]^2.$$

In view of Lemma 3.3, denote by $\omega : \mathbb{F} \to \mathbb{F}$ the induced ring monomorphism that satisfies (3.1). We claim that

$$\phi(X) = \omega(X), \quad \text{for all } X \in M_n.$$

Fix $(i,j) \in [n]^2$. If $i = j$, we have

$$\omega(X_{ii})E_{ii} = \phi(X_{ii}E_{ii}) = \phi(E_{ii}XE_{ii}) \overset{\text{Lemma 3.4}}{=} E_{ii}\phi(X)E_{ii} = \phi(X)_{ii}E_{ii},$$

so $\phi(X)_{ii} = \omega(X_{ii})$. Now assume $i \neq j$. Since ω is multiplicative and acts as the identity on the prime subfield $\mathbb{K} \subseteq \mathbb{F}$, we obtain

$$\frac{1}{2}\omega(X_{ij})E_{ji} = \phi\left(\frac{1}{2}X_{ij}E_{ji}\right) = \phi\left(\frac{1}{2}E_{ji}XE_{ji}\right) = \phi((E_{ji} \circ X) \circ E_{ji})$$
$$= (\phi(E_{ji}) \circ \phi(X)) \circ \phi(E_{ji}) = (E_{ji} \circ \phi(X)) \circ E_{ji}$$
$$= \frac{1}{2}\phi(X)_{ij}E_{ji}.$$

This implies $\phi(X)_{ij} = \omega(X_{ij})$, which completes the proof of the theorem. □

It is also worth noting that the first part of the proof of Theorem 1.1 immediately yields the following corollary:

Corollary 3.5. *Let $m < n$. The map $\phi : M_n \to M_m$ is Jordan multiplicative if and only if it is constant and equal to a fixed idempotent.*

In contrast to the matrix algebra M_n, the next two examples illustrate that the non-constant Jordan multiplicative maps defined on general central SMAs (i.e. those with a trivial centre), or on the C^*-algebra $\mathrm{B}(H)$ of bounded linear operators on an infinite-dimensional Hilbert space H, are no longer automatically additive.

Example 3.6. Let $\mathcal{T}_n \subseteq M_n$ be the upper-triangular subalgebra of M_n. Choose an arbitrary non-additive multiplicative map $\omega : \mathbb{F} \to \mathbb{F}$ (e.g. $\omega(x) := x^2$) and define a map

$$\phi : \mathcal{T}_n \to \mathcal{T}_n, \qquad \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ 0 & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{nn} \end{bmatrix} \mapsto \begin{bmatrix} \omega(x_{11}) & 0 & \cdots & 0 \\ 0 & \omega(x_{22}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega(x_{nn}) \end{bmatrix}.$$

Then ϕ is clearly $\circ$-multiplicative, but is neither constant nor additive.

Example 3.7. Let H be an infinite-dimensional Hilbert space. In view of the identification $H \cong H \oplus H$, for any fixed nonzero idempotent $P \in \mathrm{B}(H)$, the map

$$\phi : \mathrm{B}(H) \to \mathrm{B}(H \oplus H), \qquad X \mapsto \begin{bmatrix} X & 0 \\ 0 & P \end{bmatrix}$$

is $\circ$-multiplicative, but is neither constant nor additive.

Finally, the next simple example demonstrates that Theorem 1.1 does not extend to $\diamond$-multiplicative maps over fields $\mathbb{F}$ of characteristic two.

Example 3.8. Assume that $\mathrm{char}(\mathbb{F}) = 2$ and $n \geq 2$. Fix an arbitrary trace-one matrix $A \in M_n(\mathbb{F})$ and define a map $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ that sends A to a fixed nonzero matrix and all other matrices to zero. As the trace of any matrix in $M_n(\mathbb{F})$ of the form $X \diamond Y = XY + YX$ is zero, it follows that ϕ is $\diamond$-multiplicative, but is neither constant nor additive.

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