ON UNITAL $C(X)$-ALGEBRAS AND $C(X)$-VALUED CONDITIONAL EXPECTATIONS OF FINITE INDEX

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ABSTRACT. Let X be a compact Hausdorff space and let A be a unital $C(X)$-algebra, where $C(X)$ is embedded as a unital C^*-subalgebra of the centre of A. We consider the problem of characterizing the existence of a conditional expectation $E : A \to C(X)$ of finite index in terms of the associated C^*-bundle of A over X. More precisely, we show that if A admits a $C(X)$-valued conditional expectation of finite index, then A is necessarily a continuous $C(X)$-algebra, and there exists a positive integer N such that every fibre A_x of A is finite-dimensional, with $\dim A_x \leq N$. We also give some sufficient conditions on A that ensure the existence of a $C(X)$-valued conditional expectation of finite index.

1. INTRODUCTION

Let $B \subseteq A$ be two unital C^*-algebras with the same unit element. A *conditional expectation* (abbreviated by C.E.) from A to B is a completely positive contraction $E : A \to B$ such that $E(b) = b$ for all $b \in B$, and which is B-bilinear, i.e.

$$E(b_1 a b_2) = b_1 E(a) b_2$$

for all $a \in A$ and $b_1, b_2 \in B$. By a result of Y. Tomiyama (see [27, Theorem 1] or [4, Theorem II.6.10.2]), a map $E : A \to B$ is a C.E. if and only if E is a projection of norm one.

If $E(a^*a) = 0$ ($a \in A$) implies $a = 0$, E is said to be *faithful*. Every faithful conditional expectation $E : A \to B$ introduces a pre-Hilbert B-module structure on A, whose inner product is defined by

$$\langle a_1, a_2 \rangle_E := E(a_1^* a_2) \qquad (a_1, a_2 \in A). \tag{1.1}$$

The notion of finite index was introduced by V. F. R. Jones [18] in order to classify the subfactors of a type II_1 factor. Soon afterwards H. Kosaki [21] extended the Jones index theory to arbitrary factors. In order to generalize the results of [18, 21], M. Pimsner and S. Popa introduced in [24, 25] a definition for conditional expectations of finite index in the context of W^*-algebras: There must exist a constant $K \geq 1$ such that the map $K \cdot E - \mathrm{id}_A$ is positive on A. Then, following the idea of M. Baillet, Y. Denizeau and J.-F. Havet (see [3]), the index of E can be defined in the following way: Since the map $K \cdot E - \mathrm{id}_A$ is positive, E defines

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a (complete) Hilbert B-module structure on A, with respect to the inner product (1.1). If $\{x_i\}$ is a quasi-orthonormal basis in A, the *index* of E is the sum $\sum_{i=1}^{\infty} x_i^* x_i$, with respect to the ultraweak topology.

Y. Watatani also considered conditional expectations of (algebraically) finite index, when the original C^*-algebra A is a finitely generated Hilbert C^*-module over B (see [29]).

The results of M. Baillet, Y. Denizeau and J.-F. Havet in [3] also indicated that there might occur some difficulties in order to extend the notion of "finite index" for conditional expectations of C^*-algebras with arbitrary centres. However, this problem was solved by M. Frank and E. Kirchberg in [11]. The main result of their paper is [11, Theorem 1]:

Theorem 1.1 (M. Frank and E. Kirchberg)**.** *For a C.E. $E : A \to B$, where $B \subseteq A$ are unital C^*-algebras with the same unit element, the following conditions are equivalent:*

(i) *There exists a constant $K \geq 1$ such that the map $K \cdot E - \mathrm{id}_A$ is positive.*
(ii) *There exists a constant $L \geq 1$ such that the map $L \cdot E - \mathrm{id}_A$ is completely positive.*
(iii) *A becomes a (complete) Hilbert B-module when equipped with the inner product (1.1).*

Moreover, if

$$K(E) := \inf\{K \geq 1 \ : \ K \cdot E - \mathrm{id}_A \text{ is positive}\},$$
$$L(E) := \inf\{L \geq 1 \ : \ L \cdot E - \mathrm{id}_A \text{ is completely positive}\},$$

with $K(E) = \infty$ or $L(E) = \infty$ if no such number K or L exists, then

$$K(E) \leq L(E) \leq \lfloor K(E) \rfloor K(E),$$

where $\lfloor \cdot \rfloor$ denotes the integer part of a real number.

The importance of this result is that it gives the right general definition for conditional expectations on C^*-algebras to be of finite index:

Definition 1.2. If $B \subseteq A$ are two unital C^*-algebras with the same unit element, then a C.E. $E : A \to B$ is said to be of *finite index* (abbreviated C.E.F.I.) if E satisfies one of the equivalent conditions of Theorem 1.1.

In this case the index value of E can be calculated in the enveloping von Neumann algebra A^{**} (see [11, Definition 3.1]).

For a unital inclusion $A \subseteq B$ of unital C^*-algebras we introduce the following constant

$$K(A, B) := \inf\{K(E) \ : \ E : A \to B \text{ is C.E.F.I.}\},$$

with $K(A, B) = \infty$, if no such C.E.F.I. exists. This constant will play an important role in this paper.

Conditional expectations of finite index arise naturally when considering certain continuous group actions (of certain groups) on (locally) compact Hausdorff spaces. In these situations properties of group actions can be investigated using the structure of associated Hilbert C^*-modules (see e.g. [13, 14, 28]). Other interesting relations with the group theory can be found in [17]. In [19] Kajiwara et al. established an equivalence between Jones index theory for certain Hilbert C^*-bimodules, which can naturally arise from conditional expectations of finite index, and conjugation theory of certain tensor C^*-categories. Their results are linked to the quantum field theory. In [12] Frank and Larson generalized the theory of frames for (separable) Hilbert spaces to (finitely and countably generated) Hilbert C^*-modules over unital C^*-algebras. Quasi-bases for conditional expectations of (algebraically) finite index appear as special cases of their approach.

More recently, A. Pavlov and E. Troitsky considered in [22] the problem of existence of a C.E.F.I. $E : C(Y) \to C(X)$ for a unital inclusion $\varphi : C(X) \hookrightarrow C(Y)$ of unital commutative C^*-algebras. The main result of their paper is [22, Theorem 1.1], which shows that such a C.E.F.I. exists if and only if the transpose map $\varphi_* : Y \to X$ is a *branched covering*. This means that φ_* is an open map with uniformly bounded number of pre-images (i.e. $\sup_{x\in X} |\varphi_*^{-1}(x)| < \infty$). This result motivated A. Pavlov and E. Troitsky to define the *noncommutative branched coverings*, as unital inclusion $B \subseteq A$ of unital C^*-algebras such that there exists a C.E.F.I. from A to B (see [22, Definition 1.2]).

Using the above inclusion $\varphi : C(X) \hookrightarrow C(Y)$ we may consider $C(Y)$ as a $C(X)$-algebra. Then the map φ_* is open if and only if $C(Y)$ is a continuous $C(X)$-algebra, and φ_* has uniformly bounded number of pre-images if and only if $C(Y)$ is subhomogeneous $C(X)$-algebra. This means that there exists a positive integer N such that every fibre $C(Y)_x$ of $C(Y)$ is finite-dimensional with $\dim C(Y)_x \leq N$ (see Section 2). Therefore, we can restate [22, Theorem 1.1] in terms of $C(X)$-algebras as follows:

Theorem 1.3 (A. Pavlov and E. Troitsky)**.** *Let A be a unital commutative $C(X)$-algebra, where $C(X)$ is embedded as a unital C^*-subalgebra of A. Then A admits a $C(X)$-valued C.E.F.I. if and only if A is a continuous subhomogeneous $C(X)$-algebra.*

The purpose of the present paper is to consider a possible extension of Theorem 1.3 to the case when A is an arbitrary (not necessarily commutative) unital $C(X)$-algebra. The necessary condition for the existence of a $C(X)$-valued C.E.F.I. appears to be identical to the one of Theorem 1.3:

Theorem 1.4. *Let A be a unital $C(X)$-algebra, where $C(X)$ is embedded as a unital C^*-subalgebra of the centre of A. If A admits a $C(X)$-valued C.E.F.I., then A is a continuous subhomogeneous $C(X)$-algebra. Moreover, in this case the following inequality holds:*

$$K(A, C(X)) \geq r(A),$$

where $r(A)$ is the rank of A, i.e.

$$r(A) = \max \left\{ \sum_{[\pi_x] \in \widehat{A_x}} \dim \pi_x \ : \ x \in X \right\}.$$

We shall prove Theorem 1.4 in Section 3. At the moment we do not know if the converse of Theorem 1.4 also holds. However, if all the fibres of a continuous unital $C(X)$-algebra A are $*$-isomorphic to the same finite-dimensional C^*-algebra (i.e. A is a homogeneous $C(X)$-algebra), then there exists a unique C.E. $E : A \to C(X)$ such that the map $r(A) \cdot E - \mathrm{id}_A$ is positive (Proposition 3.4). In particular, we have the equality $K(A, C(X)) = r(A)$ in this case. Also, a direct consequence of this fact is that any unital $C(X)$-algebra A which can be embedded as a $C(X)$-subalgebra of some continuous homogeneous unital $C(X)$-algebra also admits a

$C(X)$-valued C.E.F.I.. However, this embedding condition is not necessary for the existence of such C.E.F.I.. Indeed, there exists a continuous unital $C(X)$-algebra A over a second-countable compact Hausdorff space X with fibres $M_2(\mathbb{C})$ or $\mathbb{C}$ which admits a $C(X)$-valued C.E.F.I., but which cannot be embedded as a $C(X)$-subalgebra into any continuous homogeneous unital $C(X)$-algebra (Example 3.6). At the end of this paper we also show that any continuous unital $C(X)$-algebra A of rank 2 admits a C.E. $E : A \to C(X)$ such that the map $2 \cdot E - \mathrm{id}_A$ is positive (Proposition 3.7). In particular, the equality $K(A, C(X)) = r(A)$ also holds in this class of $C(X)$-algebras.

2. Notation and preliminaries

Throughout this paper A will be a C^*-algebra. We denote by A_{sa} and A_+ the self-adjoint and the positive parts of A. The centre of A is denoted by $Z(A)$. By $\hat{A}$ and $\mathrm{Prim}(A)$ we respectively denote the *spectrum* of A (i.e. the set of all classes of irreducible representations of A) and the *primitive spectrum* of A (i.e. the set of all primitive ideals of A), equipped with the Jacobson topology. By a *dimension* of $[\pi] \in \hat{A}$, which is denoted by $\dim \pi$, we mean the dimension of the underlying Hilbert space of some representative of $[\pi]$.

Let X be a compact Hausdorff space. For each point $x \in X$ let

$$C_x(X) := \{f \in C(X) \ : \ f(x) = 0\}$$

be the corresponding maximal ideal of $C(X)$.

Definition 2.1. A $C(X)$-*algebra* is a C^*-algebra A endowed with a unital $*$-homomorphism ψ_A from $C(X)$ to the centre of the multiplier algebra of A.

Remark 2.2. Given $f \in C(X)$ and $a \in A$, we write fa for the product $\psi_A(f) \cdot a$ if no confusion is possible.

There is a natural connection between $C(X)$-algebras and upper semicontinuous C^*-bundles over X. We first give a formal definition of such bundles:

Definition 2.3. Following [30] by an *upper semicontinuous C^*-bundle* we mean a triple $\mathfrak{A} = (p, \mathcal{A}, X)$ where $\mathcal{A}$ is a topological space with a continuous open surjection $p : \mathcal{A} \to X$, together with operations and norms making each *fibre* $\mathcal{A}_x := p^{-1}(x)$ into a C^*-algebra, such that the following conditions are satisfied:

- (A1) The maps $\mathbb{C} \times \mathcal{A} \to \mathcal{A}$, $\mathcal{A} \times_X \mathcal{A} \to \mathcal{A}$, $\mathcal{A} \times_X \mathcal{A} \to \mathcal{A}$ and $\mathcal{A} \to \mathcal{A}$ given in each fibre by scalar multiplication, addition, multiplication and involution, respectively, are continuous ($\mathcal{A} \times_X \mathcal{A}$ denotes the Whitney sum over X).
- (A2) The map $\mathcal{A} \to \mathbb{R}$, defined by norm on each fibre, is upper semicontinuous.
- (A3) If $x \in X$ and if (a_α) is a net in $\mathcal{A}$ such that $\|a_\alpha\| \to 0$ and $p(a_\alpha) \to x$ in X, then $a_\alpha \to 0_x$ in $\mathcal{A}$ (0_x denotes the zero-element of $\mathcal{A}_x$).

If "upper semicontinuous" in (A2) is replaced by "continuous", then we say that $\mathfrak{A}$ is a *continuous* C^*-bundle.

By a *section* of an upper semicontinuous C^*-bundle $\mathfrak{A}$ we mean a map $s : X \to \mathcal{A}$ such that $p(s(x)) = x$ for all $x \in X$. We denote by $\Gamma(\mathfrak{A})$ the set of all continuous sections of $\mathfrak{A}$. Then $\Gamma(\mathfrak{A})$ becomes a $C(X)$-algebra with respect to the natural pointwise operations and sup-norm.

On the other hand, given a $C(X)$-algebra A, one can always associate an upper semicontinuous C^*-bundle $\mathfrak{A}$ over X such that $A \cong \Gamma(\mathfrak{A})$, as follows. Set $J_x := C_x(X) \cdot A$ and note that J_x is a closed two-sided ideal in A (by Cohen factorization theorem [7], [6, Theorem A.6.2])). The quotient $A_x := A/J_x$ is called the *fibre* at the point x, and we denote by a_x the image in A_x of an element $a \in A$. Let

$$\mathcal{A} := \bigsqcup_{x\in X} A_x,$$

and let $p : \mathcal{A} \to X$ be the canonical associated projection. For $a \in A$ we define the map $\hat{a} : X \to \mathcal{A}$ by $\hat{a}(x) := a_x$, and let $\Omega := \{\hat{a} \ : \ a \in A\}$. Since for each $a \in A$ we have

$$\|a_x\| = \inf\{\| [1 - f + f(x)] \cdot a\| \ : \ f \in C(X)\},$$

the norm function $x \mapsto \|a_x\|$ is upper semicontinuous on X. Hence, by Fell's theorem [30, Theorem C.25] there exists a unique topology on $\mathcal{A}$ for which $\mathfrak{A} := (p, \mathcal{A}, X)$ becomes an upper semicontinuous C^*-bundle such that $\Omega \subseteq \Gamma(\mathfrak{A})$. Moreover, by Lee's theorem [30, Theorem C.26], $\Omega = \Gamma(\mathfrak{A})$, and the *generalized Gelfand transform* $\mathcal{G} : a \in A \mapsto \hat{a} \in \Gamma(\mathfrak{A})$, is an isomorphism of $C(X)$-algebras, from A onto $\Gamma(\mathfrak{A})$.

Definition 2.4. Let A be a $C(X)$-algebra. If all the norm functions $x \mapsto \|a_x\|$ ($a \in A$) are continuous on X, we say that A is a *continuous $C(X)$-algebra.*

Note that the $C(X)$-algebra A is continuous if and only if $\mathfrak{A}$ is continuous as a C^*-bundle.

The C^*-algebra A is said to be

- (n-)homogeneous ($n \in \mathbb{N}$), if $\dim \pi = n$ for all $[\pi] \in \widehat{A}$,
- (n-)subhomogeneous ($n \in \mathbb{N}$), if $\sup_{[\pi]\in\widehat{A}} \dim \pi = n$.

We shall now define the similar notions for $C(X)$-algebras. To do this, first recall that if D is a finite-dimensional C^*-algebra, then there is a finite number of central pairwise orthogonal projections $p_1, \dots, p_m \in Z(D)$ with $\sum_{i=1}^m p_i = 1_D$, such that

$$D = p_1 D \oplus \cdots \oplus p_m D, \tag{2.1}$$

and each $p_i D$ is $*$-isomorphic to the matrix algebra $M_{n_i}(\mathbb{C})$ (see e.g. [26, Theorem I.11.9]). We define the *rank* of D as

$$r(D) := \sum_{i=1}^m n_i = \sum_{[\pi]\in\hat{D}} \dim \pi.$$

Definition 2.5. Let A be a $C(X)$-algebra. We say that A is

- *homogeneous* if all the fibres of A are $*$-isomorphic to the same finite-dimensional C^*-algebra.
- *subhomogeneous* if there exists a positive integer N such that every fibre A_x of A is finite-dimensional with $\dim A_x \le N$.

Remark 2.6. Let A be a $C(X)$-algebra.

(i) A is subhomogeneous if and only if

$$r(A) := \sup\{r(A_x) \ : x \in X\} < \infty.$$

As in the finite-dimensional case, we call the number $r(A)$ the *rank* of A.

(ii) If A is continuous and homogeneous, then by [10, Lemma 3.1] the associated C^*-bundle $\mathfrak{A}$ is locally trivial.

3. Results

Remark 3.1. If A is a unital $C(X)$-algebra, we always assume in this section that the map $\psi_A : C(X) \to Z(A)$ is injective, so that we can identify $C(X)$ with the unital C^*-subalgebra $\psi_A(C(X))$ of $Z(A)$.

In order to prove Theorem 1.4 we shall need the following two auxiliary results.

Lemma 3.2. *Let D be a unital C^*-algebra. Then $K(D,\mathbb{C}) := K(D,\mathbb{C}1_D) < \infty$ if and only if D is finite-dimensional. In this case we have:*

(i) *The constant $K(\omega)$ is finite for every faithful state ω on D, which we identify with the corresponding faithful C.E.*

$$d \mapsto \omega(d) \cdot 1_D \in \mathbb{C} \cdot 1_D \quad (d \in D).$$

(ii) $K(D,\mathbb{C}) = r(D)$. *Moreover, there exists a unique state τ on D such that*

$$r(D) \cdot \tau(d) 1_D \geq d \quad \text{for all } d \in D_+. \tag{3.1}$$

Proof. The equivalence $K(D,\mathbb{C}) < \infty \Leftrightarrow \dim D < \infty$ follows from [16, Lemma 4.5]. Hence, suppose that D is finite-dimensional and let ω be a faithful state on D. The proof will now proceed in two steps.

Step 1. Assume that D is simple, i.e. $D = M_n(\mathbb{C})$ for some n. If $\mathrm{tr}(\cdot)$ is the standard trace of $M_n(\mathbb{C})$, then there exists a strictly positive matrix $a \in M_n(\mathbb{C})$ with $\mathrm{tr}(a) = 1$ such that

$$\omega(d) = \mathrm{tr}(ad) \quad (d \in M_n(\mathbb{C})).$$

Let $a = u^* \cdot \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \cdot u$ be a diagonalisation of a, where $u \in M_n(\mathbb{C})$ is a unitary and $\lambda_1, \ldots, \lambda_n > 0$ are the eigenvalues of a. Then for all $d \in M_n(\mathbb{C})$ one has

$$\omega(u^* du) = \mathrm{tr}(au^* du) = \mathrm{tr}(uau^* d) = \mathrm{tr}(\mathrm{diag}(\lambda_1, \ldots, \lambda_n) d). \tag{3.2}$$

The constant $K(\omega)$ is by definition the smallest number $K \geq 1$ satisfying

$$K \cdot \omega(d) 1_D \geq d \quad \text{for all } d \in D_+. \tag{3.3}$$

Thus, (3.2) and (3.3) for rank 1 projections in D imply that

$$K(\omega) = \max\{\lambda_i^{-1} \ : \ 1 \leq i \leq n\}.$$

As $1 = \omega(1) = \sum_{i=1}^n \lambda_i$, one has $K(\omega) \geq n$ for any faithful state ω on D. Also, $K(\omega) = n$ if and only if $\omega = \tau := \frac{1}{n}\mathrm{tr}(\cdot)$. In particular, if $D = M_n(\mathbb{C})$, we have $K(D,\mathbb{C}) = r(D) = n$, and τ is the unique state on D satisfying (3.1).

Step 2. Suppose that D is an arbitrary finite-dimensional C^*-algebra. We decompose D as in (2.1). For each $1 \leq i \leq m$

$$\omega_i(p_i d) := \frac{1}{\omega(p_i)} \cdot \omega(p_i d) \quad (d \in D)$$

defines a faithful state on $p_i D$. By Step 1 we have $n_i \leq K(\omega_i) < \infty$ for all $1 \leq i \leq m$. Put

$$K_\omega := \max\left\{ \frac{K(\omega_i)}{\omega(p_i)} \ : \ 1 \leq i \leq m \right\}.$$

We claim that $K(\omega) = K_\omega$. Indeed, for all $d \in D_+$ we have

$$\begin{aligned} K_\omega \cdot \omega(d)1_D &= \sum_{i=1}^m K_\omega \cdot \omega(p_i)\omega_i(p_i d)1_D \geq \sum_{i=1}^m K(\omega_i) \cdot \omega_i(p_i d)p_i \\ &\geq \sum_{i=1}^m p_i d = d, \end{aligned}$$

which shows $K(\omega) \leq K_\omega$. On the other hand, for each $d \in D_+$ we have

$$[\omega(p_i)K(\omega)] \cdot \omega_i(p_i d)p_i \geq p_i d\,,$$

so that

$$\omega(p_i)K(\omega) \geq K(\omega_i) \quad (1 \leq i \leq m). \tag{3.4}$$

This shows $K(\omega) = K_\omega$, as wanted. Also,

$$K(\omega) = \sum_{i=1}^m \omega(p_i)K(\omega) \geq \sum_{i=1}^m K(\omega_i) \geq \sum_{i=1}^m n_i = r(D)\,,$$

so that $K(D, \mathbb{C}) \geq r(D)$.

It remains to show that there exists a unique state τ on D satisfying (3.1). To prove the existence, suppose that $r(D) = n$, and for each $1 \leq i \leq m$ let τ_i be the only faithful tracial state on $p_i D \cong M_{n_i}(\mathbb{C})$. Define the state τ on D by

$$\tau(d) := \frac{1}{n}\sum_{i=1}^m n_i \cdot \tau_i(p_i d) \quad (d \in D). \tag{3.5}$$

As $\tau(p_i) = \frac{n_i}{n}$ and $K(\tau_i) = n_i$ for all $1 \leq i \leq m$, we have $K(\tau) = K_\tau = n$. In particular, $K(D, \mathbb{C}) = n = r(D)$.

To show the uniqueness of this state τ, suppose that ω is another state on D with $K(\omega) = n$. Then using (3.4) we have

$$\sum_{i=1}^m K(\omega_i) \leq \sum_{i=1}^m \omega(p_i)K(\omega) = K(\omega) = n.$$

But since $K(\omega_i) \geq n_i$ and $\sum_{i=1}^m n_i = n$, we must have $K(\omega_i) = n_i$ for all $1 \leq i \leq m$. By the uniqueness part of Step 1 we conclude that

$$\omega_i = \tau_i \quad \text{for all } \ 1 \leq i \leq m. \tag{3.6}$$

Also, $K_\omega = K(\omega) = n$ and $K(\omega_i) = n_i$ imply $\omega(p_i) \geq \frac{n_i}{n}$ for all $1 \leq i \leq m$. Since ω is a state on D and $\sum_{i=1}^m p_i = 1_D$, we must have

$$\omega(p_i) = \frac{n_i}{n} \quad \text{for all } \ 1 \leq i \leq m. \tag{3.7}$$

Finally, (3.6) and (3.7) imply that

$$\omega(d) = \sum_{i=1}^m \omega(p_i)\omega_i(p_i d) = \frac{1}{n}\sum_{i=1}^m n_i \cdot \tau_i(p_i d) = \tau(d),$$

for all $d \in D$, which finishes the proof. □

Proposition 3.3. *Let A be a unital $C(X)$-algebra. If A admits a faithful $C(X)$-valued C.E., then A is a continuous $C(X)$-algebra.*

Proof. This can be deduced from [5, Section 2]. For completeness, we include a short proof of this fact. It suffices to show that all norm functions $x \mapsto \|a_x\|$ $(a \in A)$ are lower semicontinuous on X. To prove this, let $E : A \to C(X)$ be a faithful C.E. and let $L^2(A, E)$ be the completion of the pre-Hilbert $C(X)$-module A, with respect to the inner product (1.1). For $a \in A$ let $\Phi(a) : L^2(A, E) \to L^2(A, E)$ denote the continuous extension of the left multiplication map $a_1 \mapsto aa_1$ $(a \in A)$. Since E is faithful and since

$$\begin{aligned}\langle \Phi(a)(a_1), a_2\rangle_E &= \langle aa_1, a_2\rangle_E = E(a_1^* a^* a_2) = \langle a_1, a^* a_2\rangle_E \\ &= \langle a_1, \Phi(a^*)(a_2)\rangle_E,\end{aligned}$$

for all $a_1, a_2 \in A$, the map Φ defines an injective $C(X)$-linear morphism from A to the $C(X)$-algebra $\mathbb{B}_{C(X)}(L^2(A, E))$ of bounded adjointable $C(X)$-linear operators on $L^2(A, E)$. Therefore, for $a \in A$ and $x \in X$ we have

$$\begin{aligned}\|a_x\| &= \|\Phi(a)_x\| \\ &= \sup\{|\langle \Phi(a)(a_1), a_2\rangle_E(x)| \ : \ a_1, a_2 \in A, \ \|a_1\|_E = \|a_2\|_E = 1\} \\ &= \sup\{|E(a_1^* a^* a_2)(x)| \ : \ a_1, a_2 \in A, \ \|a_1\|_E = \|a_2\|_E = 1 \ \}.\end{aligned}$$

In particular, the function $x \mapsto \|a_x\|$ is a supremum of continuous functions $x \mapsto |E(a_1^* a^* a_2)(x)|$ $(\|a_1\|_E = \|a_2\|_E = 1)$, so it must be lower semicontinuous on X. □

Proof of Theorem 1.4. Let $E : A \to C(X)$ be a C.E.F.I.. As the conditional expectation E is faithful, Proposition 3.3 implies that the $C(X)$-algebra A is continuous (note that in this case $(A, \langle \cdot, \cdot\rangle_E)$ is already a complete Hilbert $C(X)$-module by Theorem 1.1). It remains to show that each fibre A_x $(x \in X)$ is finite-dimensional and satisfies $r(A_x) \le K(E)$. Indeed, for a fixed point $x \in X$ and $\varepsilon > 0$,

$$\omega_x : a_x \mapsto E(a)(x)$$

defines a state on a fibre A_x satisfying

$$(K(E) + \varepsilon) \cdot \omega_x(a_x) 1_x \ge a_x$$

for all $a_x \in (A_x)_+$. Lemma 3.2 now yields $r(A_x) \le K(E)$, as wanted. □

We shall now give some sufficient conditions on a continuous unital subhomogeneous $C(X)$-algebra A to ensure the existence of a $C(X)$-valued C.E.F.I..

Proposition 3.4. *Every continuous homogeneous unital $C(X)$-algebra A admits a unique C.E. $E : A \to C(X)$ such that the map $r(A) \cdot E - \mathrm{id}_A$ is positive. In particular, $K(A, C(X)) = r(A)$ in this case.*

Proof. The construction of such a C.E. $E : A \to C(X)$ can be deduced from the proof of [16, Lemma 4.6]. But we include here the main steps of the proof for completeness. By assumption all fibres of A are $*$-isomorphic to a fixed finite-dimensional C^*-algebra D. Suppose that $r(D) = n$, and let τ be a state on D defined by (3.5). It is easy check that τ is invariant under the group $\mathrm{Aut}(D)$ of $*$-automorphisms of D. Since the $C(X)$-algebra A is continuous and homogeneous, its associated bundle $\mathfrak{A}$ is locally trivial by Remark 2.6. Hence, there exists an open covering $\{U_\alpha\}$ of X such that $\Phi_\alpha : \mathfrak{A}|_{U_\alpha} \cong U_\alpha \times D$, where

- Φ_α is an isomorphism of C^*-bundles, and
- $\mathfrak{A}|_U$ is the restriction bundle over a subset $U \subseteq X$.

Fix an element $a \in A$. For $x \in X$ choose an index α such that $x \in U_\alpha$, and define

$$E(a)(x) := \tau(\Phi_\alpha(a_x)).$$

Since τ is invariant under the group $\mathrm{Aut}(D)$, the value $E(a)(x)$ is well defined, and the local triviality of $\mathfrak{A}$ implies that the function $E(a) : x \mapsto E(a)(x)$ is continuous on X. It is now easy to see that the map $E : a \to E(a)$ defines a $C(X)$-valued C.E.F.I. on A. Moreover, by (3.1) we have

$$n \cdot E(a)(x)1_x \geq a_x, \quad \text{for all } a \in A_+ \text{ and } x \in X.$$

Thus, the map $n \cdot E - \mathrm{id}_A$ is positive and E is the only C.E. with this property (Lemma 3.2). In particular, $K(A, C(X)) \leq r(A)$, so Theorem 1.4 yields that $K(A, C(X)) = n$. □

Corollary 3.5. *If the unital $C(X)$-algebra A admits a $C(X)$-linear embedding into some continuous homogeneous unital $C(X)$-algebra A', then A admits a $C(X)$-valued C.E.F.I..*

Proof. By Proposition 3.4 there exists a C.E. $E' : A' \to C(X)$ of finite index. Then the restriction $E'|_A : A \to C(X)$ defines a convenient C.E.F.I.. □

Note that the embedding condition of Corollary 3.5 is not necessary for the existence of a $C(X)$-valued C.E.F.I.. Indeed, in Example 3.6 we show that there exists a continuous unital $C(X)$-algebra A of rank 2 which does not admit a $C(X)$-linear embedding into any continuous homogeneous unital $C(X)$-algebra. On the other hand, a direct consequence of Proposition 3.7 is that A admits a $C(X)$-valued C.E.F.I..

To do this, first recall that a C^*-algebra A is said to be *central* if it satisfies the following two conditions:

(i) A is *quasi-central* (i.e. no primitive ideal of A contains $Z(A)$);
(ii) If $P, Q \in \mathrm{Prim}(A)$ and $P \cap Z(A) = Q \cap Z(A)$, then $P = Q$

(see [1, 8, 15, 20]). By [8, Proposition 3] a quasi-central C^*-algebra A is central if and only if $\mathrm{Prim}(A)$ is Hausdorff.

Example 3.6. By [23, Example 3.5] there exists a continuous $M_2(\mathbb{C})$-bundle $\mathfrak{A}_0$ over the second countable locally compact space $X_0 := \bigsqcup_{n=1}^{\infty} \mathbb{C}P^n$, where $\mathbb{C}P^n$ is the complex projective space of dimension n, which is not of finite type (that is, X_0 does not admit a finite open cover $\{U_i\}$ such that each restriction bundle $\mathfrak{A}_0|_{U_i}$ is trivial, as a C^*-bundle). Let A_0 be the C^*-algebra $\Gamma_0(\mathfrak{A}_0)$ consisting of all continuous sections of $\mathfrak{A}_0$ which vanish at infinity. Then A_0 is a 2-homogeneous C^*-algebra with $\mathrm{Prim}(A_0) = X_0$. In particular A_0 is a central C^*-algebra with centre $C_0(X_0)$. Let $X := X_0 \sqcup \{\infty\}$ be the one-point compactification of X_0, and let A be the minimal unitisation of A_0. By [8, Proposition 3] (or [15, Proposition 3.12]) A is also a central C^*-algebra with $\mathrm{Prim}(A) = X$ and centre $C(X)$. In particular, by [4, II.6.5.8] all norm functions $x \mapsto \|a_x\|$ $(a \in A)$ are continuous on X, so that A is a continuous unital $C(X)$-algebra with fibres $A_x = M_2(\mathbb{C})$ $(x \in X_0)$ and $A_\infty = \mathbb{C}$. Suppose that A is $C(X)$-subalgebra of some continuous homogeneous $C(X)$-algebra A'. Then the associated C^*-bundle $\mathfrak{A}$ of A over X is a C^*-subbundle of the associated C^*-bundle $\mathfrak{A}'$ of A' over X. Since A' is continuous and homogeneous, $\mathfrak{A}'$ is locally trivial by Remark 2.6. Hence, since X is compact, $\mathfrak{A}'$ is of finite type. Using [23, Lemma 2.6] we conclude that $\mathfrak{A}$ is of finite type as a vector bundle. In particular, $\mathfrak{A}_0$ is of finite type as a vector bundle, since

$\mathfrak{A}_0 = \mathfrak{A}|_{X_0}$. As $\mathfrak{A}_0$ is a $M_2(\mathbb{C})$-bundle, this implies by [23, Proposition 2.9] that $\mathfrak{A}_0$ is also of finite type as a C^*-bundle; a contradiction.

On the other hand, the $C(X)$-algebra A of Example 3.6 also admits a $C(X)$-valued C.E.F.I.. This follows from the following more general fact:

Proposition 3.7. *Let A be a continuous unital $C(X)$-algebra. If $r(A) = 2$, then there exists a conditional expectation $E : A \to C(X)$ such that the map $2 \cdot E - \mathrm{id}_A$ is positive. In particular, $K(A, C(X)) = r(A)$ in this case.*

In order to prove Proposition 3.7, let us first make the following observation:

Lemma 3.8. *Let A be a unital $C(X)$-algebra and let $a \in A_{sa}$. For each point $x \in X$ let $\lambda_{\max}(a_x)$ and $\lambda_{\min}(a_x)$ respectively denote the largest and the smallest numbers in the spectrum of a_x. Then the functions $x \mapsto \lambda_{\max}(a_x)$ and $x \mapsto \lambda_{\min}(a_x)$ are upper semicontinuous on X. Furthermore, these functions are continuous on X, whenever A is a continuous $C(X)$-algebra.*

Proof. This follows directly from the equations

$$\lambda_{\max}(a_x) = \| \|a\| 1_x + a_x \| - \|a\| \quad \text{and} \quad \lambda_{\min}(a_x) = \|a\| - \| \|a\| 1_x - a_x \|.$$

□

Proof of Proposition 3.7. As $r(A) = 2$, any fibre A_x is isomorphic to $\mathbb{C}$, $\mathbb{C} \oplus \mathbb{C}$ or $M_2(\mathbb{C})$. Therefore, for each point $x \in X$ we can choose a unital embedding $\varphi_x : A_x \hookrightarrow M_2(\mathbb{C})$. For $a \in A$ and $x \in X$ we define

$$E(a)(x) := \frac{1}{2}\mathrm{tr}(\varphi_x(a_x)).$$

Obviously $E(a)$ is a $C(X)$-linear map. If $a \in A_{sa}$, note that

$$E(a)(x) = \frac{1}{2}(\lambda_{\min}(a_x) + \lambda_{\max}(a_x)) \tag{3.8}$$

for all $x \in X$. By Lemma 3.8, $E(a)$ is a continuous function on X for all $a \in A_{sa}$. As A is the linear span of A_{sa}, we conclude that $E(a) \in C(X)$ for all $a \in A$. Therefore, E defines a C. E. from A onto $C(X)$. Further, by (3.8) for all $a \in A_+$ and $x \in X$ we have

$$2 \cdot E(a)(x) 1_x = (\lambda_{\min}(a_x) + \lambda_{\max}(a_x)) \cdot 1_x \geq a_x.$$

This shows that the map $2 \cdot E - \mathrm{id}_A$ is positive, so that $K(A, C(X)) = 2$ by Theorem 1.4. □

Let A be a unital C^*-algebra and let $\check{Z}$ be the maximal ideal space of $Z(A)$. We may consider A as a $C(\check{Z})$-algebra, with respect to the action

$$f \cdot a := \mathcal{G}^{-1}(f)a \quad (f \in C(X),\ a \in A),$$

where $\mathcal{G} : Z(A) \to C(\check{Z})$ is the Gelfand transform. We say that A is *quasi-standard* if A is a continuous $C(\check{Z})$-algebra and each (Glimm) ideal $J_x = C_x(\check{Z})A$ is primal (see [2]).

Corollary 3.9. *For a unital C^*-algebra A the following conditions are equivalent:*

(i) *There exist a C.E. $E : A \to Z(A)$ such that the map $2 \cdot E - \mathrm{id}_A$ is positive.*
(ii) *A is either commutative or quasi-standard and 2-subhomogeneous.*

Proof. (i) $\Rightarrow$ (ii). Suppose that there exists a C.E. $E : A \to Z(A)$ such that the map $2 \cdot E - \mathrm{id}_A$ is positive. Then by Theorem 1.4 A is a continuous $C(\check{Z})$-algebra and $r(A_x) \leq 2$ for all $x \in \check{Z}$. In particular, A as a C^*-algebra is n-subhomogeneous, where $n \in \{1, 2\}$. Hence, by [16, Proposition 4.1] every Glimm ideal of A is primal. Also, $n = 1$ if and only if A is commutative.

(ii) $\Rightarrow$ (i). If A is commutative we have nothing to prove, so suppose that A is quasi-standard and 2-subhomogeneous. Then by [9, Corollary 1, p. 388] for each point $x \in X$ we have

$$r(A_x) = \sum_{[\pi_x] \in \widehat{A_x}} \dim \pi_x \leq 2.$$

It remains to apply Proposition 3.7. □

Remark 3.10. At the end of this paper we note that every separable continuous unital $C(X)$-algebra A admits a faithful C.E. $E : A \to C(X)$ (see e.g. [5]). In particular, this result applies to continuous subhomogeneous unital $C(X)$-algebras, when X is second-countable. In this case for each point $x \in X$, the map $E_x : a_x \mapsto E(a)(x)$ defines a faithful state on A_x, so Lemma 3.2 implies $K(E_x) < \infty$. However, this does not imply that E is of finite index. That is, it may happen that $\sup_{x \in X} K(E_x) = \infty$. Consider for instance the following example:

- Let X be the closed compact subset $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ of $[0, 1]$.
- Let A be the continuous $C(X)$-subalgebra of $C(X) \oplus C(X)$ consisting of all pairs $(f, g) \in C(X) \oplus C(X)$ such that $f(0) = g(0)$.
- Let $E : A \to C(X)$ be a C.E. fixed by the relations

$$E(f \oplus g)\left(\frac{1}{n}\right) = \begin{cases} \frac{n}{n+1} f(\frac{1}{n}) + \frac{1}{n+1} g(\frac{1}{n}) & \text{if } n \text{ is odd} \\ \\ \frac{1}{n+1} f(\frac{1}{n}) + \frac{n}{n+1} g(\frac{1}{n}) & \text{otherwise} \end{cases}$$

where $(f, g) \in A$.

Then E is a faithful C.E. which is not of finite index. Indeed, one has

$$E(f \oplus 0)\left(\frac{1}{2n}\right) = \frac{1}{2n+1} f\left(\frac{1}{2n}\right)$$

for all $f \in C_0(X \setminus \{0\})$ and all integers $n \in \mathbb{N}$. Consequently, a convenient constant K would satisfy $K \geq 2n + 1$ for all $n \in \mathbb{N}$, which is impossible.

We end this paper with some unresolved problems:

Problem 3.11. Is the converse of Theorem 1.4 also true? Moreover, does every continuous subhomogeneous unital $C(X)$-algebra A admit a C.E. $E : A \to C(X)$ such that the map $r(A) \cdot E - \mathrm{id}_A$ is positive? In particular, do we always have $K(A, C(X)) = r(A)$?

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