SUMMARY

We determine, for an elliptic curve $E/\mathbb{Q}$ and for all prime numbers p, all the possible torsion groups $E(\mathbb{Q}_{\infty,p})_{\text{tors}}$, where $\mathbb{Q}_{\infty,p}$ is the $\mathbb{Z}_p$-extension of $\mathbb{Q}$.

For a prime number p, denote by $\mathbb{Q}_{\infty,p}$ the unique $\mathbb{Z}_p$-extension of $\mathbb{Q}$ and for a positive integer n, denote by $\mathbb{Q}_{n,p}$ the n^{th} layer of $\mathbb{Q}_{\infty,p}$, i.e. the unique subfield of $\mathbb{Q}_{\infty,p}$ such that $\operatorname{Gal}(\mathbb{Q}_{n,p}/\mathbb{Q}) \simeq \mathbb{Z}/p^n\mathbb{Z}$.

Let, as always, $\mu_n = \{\omega \in \mathbb{C} \,:\, \omega^n = 1\}$ be the set of all n^{th} roots of unity. We also define

$$\mu_{p^\infty} = \bigcup_{k\in\mathbb{N}} \mu_{p^k}.$$

Note that $\mathbb{Q}(\mu_{p^k}) = \mathbb{Q}(\zeta_{p^k})$, where ζ_n is n^{th} primitive root of unity.

Recall that the $\mathbb{Z}_p$-extension of $\mathbb{Q}$ is the unique Galois extension $\mathbb{Q}_{\infty,p}$ of $\mathbb{Q}$ such that

$$\operatorname{Gal}(\mathbb{Q}_{\infty,p}/\mathbb{Q}) \simeq \mathbb{Z}_p,$$

where $\mathbb{Z}_p$ is the additive group of the p-adic integers and is constructed as follows:

Let

$$G = \operatorname{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) = \varprojlim_n \operatorname{Gal}\left(\mathbb{Q}(\mu_{p^{n+1}})/\mathbb{Q}\right) \xrightarrow{\sim} \varprojlim_n (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times = \mathbb{Z}_p^\times.$$

Here we know that $G = \Delta \times \Gamma$, where $\Gamma \simeq \mathbb{Z}_p$ and $\Delta \simeq \mathbb{Z}/(p-1)\mathbb{Z}$ for $p \geq 3$ and $\Delta \simeq \mathbb{Z}/2\mathbb{Z}$ (generated by complex conjugation) for $p = 2$, so we define

$$\mathbb{Q}_{\infty,p} := \mathbb{Q}(\mu_{p^\infty})^\Delta.$$

We also see that every layer is uniquely determined by (for $p \geq 3$)

$$\mathbb{Q}_{n,p} = \mathbb{Q}(\mu_{p^{n+1}}) \cap \mathbb{Q}_{\infty,p},$$

so for $p \geq 3$ it is the unique subfield of $\mathbb{Q}(\mu_{p^{n+1}})$ of degree p^n over $\mathbb{Q}$. More details and proofs of these facts about $\mathbb{Z}_p$-extensions and Iwasawa theory can be found in [56, Chapter 13].

Iwasawa theory for elliptic curves (see [19]) studies elliptic curves in $\mathbb{Z}_p$-extensions, in particular the growth of the rank and n-Selmer groups in the layers of the $\mathbb{Z}_p$-extensions.

In this paper we completely solve the problem of determining how the torsion of an elliptic curve defined over $\mathbb{Q}$ grows in the $\mathbb{Z}_p$-extensions of $\mathbb{Q}$. These results, interesting in their own right, might also find applications in other problems in Iwasawa theory for elliptic curves and in general. For example, to show that elliptic curves over $\mathbb{Q}_{\infty,p}$ are modular for all p, Thorne [55] needed to show that $E(\mathbb{Q}_{\infty,p})_{\text{tors}} = E(\mathbb{Q})_{\text{tors}}$ for two particular elliptic curves. In this work we did that thing in general case.

Our results are the following:

Let $E/\mathbb{Q}$ be an elliptic curve. Let $p \geq 5$ be a prime number. Then

$$E(\mathbb{Q}_{\infty,p})_{\text{tors}} = E(\mathbb{Q})_{\text{tors}}.$$

Group $E(\mathbb{Q}_{\infty,2})_{\text{tors}}$ is isomorphic to exactly one of the following groups:

$$\begin{aligned} \mathbb{Z}/N\mathbb{Z}, &\qquad 1 \leq N \leq 10, \text{ or } N = 12, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z}, &\qquad 1 \leq N \leq 4, \end{aligned}$$

and for each group G from the list above there exists an $E/\mathbb{Q}$ such that $E(\mathbb{Q}_{\infty,2})_{\text{tors}} \simeq G$.

Group $E(\mathbb{Q}_{\infty,3})_{\text{tors}}$ is isomorphic to exactly one of the following groups:

$$\begin{aligned} \mathbb{Z}/N\mathbb{Z}, &\qquad 1 \leq N \leq 10, \text{ or } N = 12, 21 \text{ or } 27, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z}, &\qquad 1 \leq N \leq 4. \end{aligned}$$

and for each group G from the list above there exists an $E/\mathbb{Q}$ such that $E(\mathbb{Q}_{\infty,3})_{\text{tors}} \simeq G$.

By Mazur's theorem we see that

$$\begin{aligned} &\{E(\mathbb{Q}_{\infty,2})_{\text{tors}} : E/\mathbb{Q} \text{ elliptic curve}\} = \{E(\mathbb{Q})_{\text{tors}} : E/\mathbb{Q} \text{ elliptic curve}\}, \\ &\{E(\mathbb{Q}_{\infty,3})_{\text{tors}} : E/\mathbb{Q} \text{ elliptic curve}\} = \{E(\mathbb{Q})_{\text{tors}} : E/\mathbb{Q} \text{ elliptic curve}\} \cup \{\mathbb{Z}/21\mathbb{Z}, \mathbb{Z}/27\mathbb{Z}\}. \end{aligned}$$

However, given a specific $E/\mathbb{Q}$ it is not necessarily the case that $E(\mathbb{Q}_{\infty,p})_{\text{tors}} = E(\mathbb{Q})_{\text{tors}}$. Indeed there are many elliptic curves for which torsion grows from $\mathbb{Q}$ to $\mathbb{Q}_{\infty,p}$, and we investigate this question further in Section 3.6. Specifically, for each prime p we find for which groups G there exists infinitely many j-invariants j such that there exists an elliptic curve $E/\mathbb{Q}$ with $j(E) = j$ and such that $E(\mathbb{Q})_{\text{tors}} \subsetneq E(\mathbb{Q}_{\infty,p})_{\text{tors}} \simeq G$.

Furthermore, after we understood the behaviour of the torsion of elliptic curve $E/\mathbb{Q}$ over the field $\mathbb{Q}_{\infty,p}$, we tried to find out what will happen if we look at the compositum of all of those fields. We answered that question completely too.

Let

$$\mathscr{K}_{\geq 5} = \prod_{p \geq 5 \text{ prime}} \mathbb{Q}_{\infty,p}$$

and let

$$\mathscr{K} = \prod_{p \text{ prime}} \mathbb{Q}_{\infty,p}.$$

We proved that for an elliptic curve $E/\mathbb{Q}$ it holds that

$$E(\mathscr{K}_{\geq 5})_{\text{tors}} = E(\mathbb{Q})_{\text{tors}}$$

and also that $E(\mathscr{K})_{\text{tors}}$ is isomorphic to one of the following groups

$$\begin{aligned} \mathbb{Z}/n\mathbb{Z}, &\quad 1 \leq n \leq 10 \text{ or } n \in \{12, 13, 21, 27\}, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2n\mathbb{Z}, &\quad 1 \leq n \leq 4. \end{aligned}$$

For each group G from the list above there exists an $E/\mathbb{Q}$ such that $E(\mathbb{Q}_{\infty,3})_{\text{tors}} \simeq G$.

At the end, in chapter 5 we state some results about the behaviour of the torsion of elliptic curve $E/\mathbb{Q}$ over the fields

$$\mathbb{Q}(\mu_{p^\infty}) = \bigcup_{k=1}^{\infty} \mathbb{Q}(\mu_{p^k}).$$

More precisely, we prove the following result

Let $E/\mathbb{Q}$ be an elliptic curve, then for a prime number $p \geq 5$ it holds that

$$E(\mathbb{Q}(\mu_{p^\infty}))_{\text{tors}} = E(\mathbb{Q}(\mu_p))_{\text{tors}}.$$

Furthermore

$$E(\mathbb{Q}(\mu_{3^\infty}))_{\text{tors}} = E(\mathbb{Q}(\mu_{3^3}))_{\text{tors}} \qquad \text{and} \qquad E(\mathbb{Q}(\mu_{2^\infty}))_{\text{tors}} = E(\mathbb{Q}(\mu_{2^4}))_{\text{tors}}.$$

In chapter 6 we exhibit all `magma` [2] codes that we used for computations.

Keywords: elliptic curve, Iwasawa theory, $\mathbb{Z}_p$-extension, torsion, torsion growth, cyclotomic extension