A NOTE ON THE ROOT SUBSPACES OF REAL SEMISIMPLE LIE ALGEBRAS

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Abstract. In this note we prove that for any two restricted roots α, β of a real semisimple Lie algebra $\mathfrak{g}$, such that $\alpha + \beta \neq 0$, the corresponding root subspaces satisfy $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

Let $\mathfrak{g}$ be a real semisimple Lie algebra, $\mathfrak{a}$ a Cartan subspace of $\mathfrak{g}$ and R the (restricted) root system of the pair $(\mathfrak{g}, \mathfrak{a})$ in the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$. For $\alpha \in R$ denote by $\mathfrak{g}_\alpha$ the corresponding root subspace of $\mathfrak{g}$:

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g};\ [h, x] = \alpha(h)x \ \ \forall h \in \mathfrak{a}\}.$$

The aim of this note is to prove the following theorem:

THEOREM. *Let* $\alpha, \beta \in R$ *be such that* $\alpha + \beta \neq 0$. *Then either* $[x, \mathfrak{g}_\alpha] = \mathfrak{g}_{\alpha+\beta} \ \ \forall x \in \mathfrak{g}_\beta \setminus \{0\}$ *or* $[x, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta} \ \ \forall x \in \mathfrak{g}_\alpha \setminus \{0\}$.

Although the proof is very simple and elementary, the assertion does not seem to appear anywhere in the literature. The argument for the proof is from [2], where it is used to prove $[\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] = \mathfrak{g}_{2\alpha}$ (a fact which is also proved in [3], 8.10.12), and also to prove that the nilpotent constituent in an Iwasawa decomposition is generated by the root subspaces corresponding to the simple roots.

Let B be the Killing form of $\mathfrak{g}$:

$$B(x, y) = \mathrm{tr}\,(\mathrm{ad}\, x\, \mathrm{ad}\, y), \qquad x, y \in \mathfrak{g}.$$

Choose a Cartan involution ϑ of $\mathfrak{g}$ in accordance with $\mathfrak{a}$, i.e. such that $\vartheta(h) = -h \ \forall h \in \mathfrak{a}$. Denote by $(\cdot|\cdot)$ the inner product on $\mathfrak{g}$ defined by

$$(x|y) = -B(x, \vartheta(y)), \qquad x, y \in \mathfrak{g}.$$

We shall use the same notation $(\cdot|\cdot)$ for the induced inner product on the dual space $\mathfrak{a}^*$ od $\mathfrak{a}$. Let $\|\cdot\|$ denote the corresponding norms on $\mathfrak{g}$ and on $\mathfrak{a}^*$. For $\alpha \in R$ let h_α be the unique element of $\mathfrak{a}$ such that

$$B(h, h_\alpha) = \alpha(h) \qquad \forall h \in \mathfrak{a}.$$

LEMMA. *Let $\alpha, \beta \in R$ be such that $(\alpha|\alpha + \beta) > 0$. Then*

$$[x, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta} \qquad \forall x \in \mathfrak{g}_\alpha \setminus \{0\}.$$

Proof. Take $x \in \mathfrak{g}_\alpha$, $x \neq 0$. We can suppose that $\|x\|^2\|\alpha\|^2 = 2$. Put

$$h = \frac{2}{\|\alpha\|^2} h_\alpha \qquad \text{and} \qquad y = -\vartheta(x).$$

Then

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h$$

([3], 8.10.12). Therefore, the subspace $\mathfrak{s}$ of $\mathfrak{g}$ spanned by $\{x, y, h\}$ is a simple Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. From the representation theory of $\mathfrak{sl}(2, \mathbb{R})$ ([1],1.8) we know that if π is any representation of $\mathfrak{s}$ on a real finite dimensional vector space V, then $\pi(h)$ is diagonalizable, all eigenvalues of the operator $\pi(h)$ are integers, and if for $n \in \mathbb{Z}$ V_n denotes the $n-$eigenspace of $\pi(h)$, then

$$n \geq -1 \qquad \Longrightarrow \qquad \pi(x)V_n = V_{n+2}.$$

Put

$$V = \sum_{j \in \mathbb{Z}} \mathfrak{g}_{\beta + j\alpha}.$$

Then V is an $\mathfrak{s}-$module for the adjoint action and

$$\mathfrak{g}_{\beta + j\alpha} = V_{n+2j} \qquad \text{where} \qquad n = 2\frac{(\beta|\alpha)}{\|\alpha|^2} \in \mathbb{Z}.$$

Especially,

$$V_n = \mathfrak{g}_\beta, \qquad V_{n+2} = \mathfrak{g}_{\alpha+\beta}.$$

Now

$$n+2=2\frac{(\alpha|\alpha+\beta)}{\|\alpha\|^2}>0 \quad\Longrightarrow\quad n\geq -1 \quad\Longrightarrow\quad (\mathrm{ad}\,x)V_n=V_{n+2}.$$

Proof of the Theorem. It is enough to notice that if $\alpha+\beta\neq 0$ then

$$0<(\alpha+\beta|\alpha+\beta)=(\alpha|\alpha+\beta)+(\beta|\alpha+\beta),$$

hence, either $(\alpha|\alpha+\beta)>0$ or $(\beta|\alpha+\beta)>0$.

Let m_α denote the multiplicity of $\alpha\in R$ ($m_\alpha=\dim\,\mathfrak{g}_\alpha$). An immediate consequence of the Theorem is:

COROLLARY. *If* $\alpha,\beta\in R$, $\alpha+\beta\neq 0$, *then* $m_{\alpha+\beta}\leq\max\,(m_\alpha,m_\beta)$.

Literatura

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[3] N. Wallach, *Harmonic Analysis on Homogeneous Spaces*, Marcel Dekker, New York, 1973.