Non−separated points in the dual spaces of semi−simple Lie groups

Let G be a connected semi−simple Lie group with finite center, $\mathfrak{g}$ its Lie algebra, $\mathfrak{g}_c = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, $\mathcal{G}$ the universal enveloping algebra of $\mathfrak{g}_c$ and K a maximal compact subgroup of G.

Let π be a continuous representation of G on a Banach space $\mathcal{H}$. For $\delta \in \hat{K}$ let $\mathcal{H}_\delta$ denote the δ−isotypic K−submodule of $\mathcal{H}$; $\mathcal{H}_\delta$ is the range of the continuous projection

$$P_\delta^\pi = \dim(\delta) \int_K \overline{\xi_\delta(k)} \pi(k) \, \mathrm{d}k$$

where ξ_δ is the character of δ and $\mathrm{d}k$ is the normalized Haar measure on K. We call π **admissible** if the subspace $\mathcal{H}_\delta$ is finitedimensional for every $\delta \in \hat{K}$. In this case set

$$(\pi : \delta) = \frac{\dim \mathcal{H}_\delta}{\dim \delta} \qquad \text{and} \qquad \mathcal{H}_K = \sum_{\delta \in \hat{K}} \dot{+} \mathcal{H}_\delta$$

(algebraic direct sum). Then $\mathcal{H}_K$ is a $\mathcal{G}$−module anbd we denote the corresponding representation of $\mathcal{G}$ again by π. For any finite set $S \subseteq \hat{K}$ we set

$$\mathcal{H}_S = \sum_{\delta \in S} \dot{+} \mathcal{H}_\delta.$$

Let π and π' be admissible representations of G on $\mathcal{H}$ and $\mathcal{H}'$, respectively. They are called **infinitesimally equivalent** if the corresponding $\mathcal{G}$−modules $\mathcal{H}_K$ and $\mathcal{H}'_K$ are isomorphic. If π and π' are unitary and irreducible, this is equivalent to the unitary equivalence.

Let $\mathfrak{g} = \mathfrak{k} \dot{+} \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ coirresponding to K. Choose a maximal Abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$ and set $M = Z_K(\mathfrak{a})$ and $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$ (Z denotes centralizers). Let $\mathfrak{d}$ be a Cartan subalgebra of $\mathfrak{m}$. Set

$$R = R(\mathfrak{g}_c, \mathfrak{h}_c) \subseteq \mathfrak{h}_0^*, \qquad \mathfrak{h} = \mathfrak{d} \dot{+} \mathfrak{a}, \qquad \mathfrak{h}_0 = i\mathfrak{d} \dot{+} \mathfrak{a},$$

$$R_0 = R(\mathfrak{m}_c, \mathfrak{d}_c) \subseteq \mathfrak{d}^*, \qquad \Sigma = R(\mathfrak{g}, \mathfrak{a}) \subseteq \mathfrak{a}^*.$$

Let W be the Weyl group of R acting on $\mathfrak{h}_c$ and $\mathfrak{h}_c^*$. Choose compatible orders on $\mathfrak{h}_0^*$, $i\mathfrak{d}^*$ and $\mathfrak{a}^*$ and let R^+, R_0^+ and Σ^+ denote the corresponding positive roots. Set

$$\mathfrak{n} = \sum_{\alpha \in \Sigma^+} \dot{+} \mathfrak{g}^\alpha, \qquad N = \exp \mathfrak{n}, \qquad A = \exp \mathfrak{a};$$

$$m_\alpha = \dim \mathfrak{g}^\alpha, \quad \alpha \in \Sigma;$$

$$\rho_c = \frac{1}{2} \sum_{\alpha \in R^+} \alpha, \qquad \rho_0 = \frac{1}{2} \sum_{\alpha \in R_0^+} \alpha, \qquad \rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha.$$

Let $H : G \to \mathfrak{a}$, $\nu : G \to N$, $\kappa : G \to K$ be the analytic maps corresponding to the Iwasawa decomposition $G = ANK$:

$$x = \exp(H(x))\nu(x)\kappa(x), \qquad x \in G.$$

Let $\sigma \in \hat{M}$ be realized on a finitedimensional Hilbert space V^σ and set

$$\mathcal{H}^\sigma = \{f \in L_2(K, V^\sigma);\ f(mk) = \sigma(m)f(k),\ m \in M,\ k \in K\}$$

(a closed subspace of the Hilbert space $L_2(K, V^\sigma)$). For any $\lambda \in \mathfrak{a}_c^*$ define the so called **elementary representation** $\pi^{\sigma,\lambda}$ of G on $\mathcal{H}^\sigma$ by

$$[\pi^{\sigma,\lambda}(x)f](k) = \mathrm{e}^{(\lambda+\rho)(H(kx))}f(\kappa(kx)), \qquad k \in K, \quad x \in G, \quad f \in \mathcal{H}^\sigma.$$

Then $\pi^{\sigma,\lambda}$ is an admissible representation of G. The restriction $\pi^{\sigma,\lambda}|K$ is independent of λ and it is the representation of K induced by σ. Furthermore, by Frobenius reciprocity

$$(\pi^{\sigma,\lambda} : \delta) = (\delta : \sigma), \qquad \delta \in \hat{K}, \quad \sigma \in \hat{M}, \quad \lambda \in \mathfrak{a}_c^*.$$

Let $\mathcal{Z}$ be the center of $\mathcal{G}$ and let $\nu \mapsto \chi_\nu$ be the mapping from $\mathfrak{h}_c^*$ to $Hom(\mathcal{Z}, \mathbb{C})$ which induces the Harish−Chandra's bijection from the set $\mathfrak{h}_c^*/W$ of $W-$orbits in $\mathfrak{h}_c^*$ onto $Hom(\mathcal{Z}, \mathbb{C})$. The representation $\pi^{\sigma,\lambda}$ admits infinitesimal character and it is equal to $\chi_{\nu(\sigma,\lambda)}$, where

$$\nu(\sigma, \lambda)|\mathfrak{d}_c = \Lambda_\sigma + \rho_0, \qquad \nu(\sigma, \lambda)|\mathfrak{a}_c = \lambda;$$

Λ_σ is the highest weight (with respect to R_0^+) of the representation of $\mathfrak{m}_c$ derived from σ.

We shall need the following result by W. Casselman:

Theorem 1. *Every irreducible admissible representation π of G admits an infinitesimal embedding into some $\pi^{\sigma,\lambda}$ (i.e. π is infinitesimally equivalent to a subrepresentation of some $\pi^{\sigma,\lambda}$).*

As far as I know the proof of this theorem has not been published yet, but my colleague D. Miličić has written down the complete proof, and it turns out to be considerably simpler then the proof of Harish−Chandra's subqotient theorem. It is based on the following:

1. (Osborne's theorem) $\mathcal{H}_K$ is finitely generated as $U(\mathfrak{n}_c)-$module, where $U(\mathfrak{n}_c)$ is the universal enveloping algebra of $\mathfrak{n}_c$.

2. The study of asymptotic behaviour of $K-$finite matrix coefficients of π shows that the subspace
$$\mathcal{H}_K(\mathfrak{n}) = span\,\{\pi(X)v;\ v \in \mathcal{H}_K,\ X \in \mathfrak{n}\}$$
is different from $\mathcal{H}_K$. Let V be the quotient space $\mathcal{H}_K/\mathcal{H}_K(\mathfrak{n})$. By 1. V if finite−dimensional and it is naturally a $MAN-$module on which N acts trivially.

3. (Casselman's reciprocity theorem) Let $\sigma \in \hat{M}$ and $\lambda \in \mathfrak{a}_c^*$ and let V^σ be endowed with $MAN-$module structure by
$$man \mapsto \mathrm{e}^{(\lambda+\rho)(\log a)}\sigma(m), \qquad m \in M, \quad a \in A, \quad n \in N.$$
For $T \in Hom_{\mathcal{G}}(\mathcal{H}_K, \mathcal{H}_K^{\sigma,\lambda})$ let $\tilde{T} : V \to V^\sigma$ be defined by
$$\tilde{T}(v + \mathcal{H}_K(\mathfrak{n})) = (Tv)(e), \qquad v \in \mathcal{H}_K.$$
Then $T \mapsto \tilde{T}$ is an isomorphism from $Hom_{\mathcal{G}}(\mathcal{H}_K, \mathcal{H}_K^{\sigma,\lambda})$ onto $Hom_{MAN}(V, V^\sigma)$.

4. By 2. we can find σ and λ such that $Hom_{MAN}(V, V^\sigma) \neq \{0\}$. By 3. it follows that $Hom_{\mathcal{G}}(\mathcal{H}_K, \mathcal{H}_K^{\sigma,\lambda}) \neq \{0\}$. π being irreducible, there exists an infinitesimal embeding of π into $\pi^{\sigma,\lambda}$.

Let $\mathcal{D}(G)$ be the space of compactly supported $C^\infty-$functions on G and let $\mathcal{D}'(G)$ denote the space of distributions on G endowed with the weak topology. Let π be an irreducible admissible representation of G on a Banach space $\mathcal{H}$. By theorem 1. up to infinitesimal equivalence we can suppose that $\mathcal{H}$ is a Hilbert space. By Harish−Chandra's results for any $f \in \mathcal{D}(G)$ $\pi(f)$ is a trace−class operator and if we set

$$\Theta_\pi(f) = \mathrm{Tr}\, \pi(f), \qquad f \in \mathcal{D}(G),$$

then $\Theta_\pi \in \mathcal{D}'(G)$. Furthermore, if π and π' are infinitesimally equivalent, then $\Theta_\pi = \Theta_{\pi'}$, and if $\pi_1, \pi_2, \ldots, \pi_n$ are pairwise infinitesimally inequivalent, then $\Theta_{\pi_1}, \Theta_{\pi_2}, \ldots, \Theta_{\pi_n}$ are linearly independent.

For an admissible representation π on a Banach space $\mathcal{H}$ and for $\delta \in \hat{K}$ let Φ_δ^π be the corresponding trace spherical function

$$\Phi_\delta^\pi(x) = \mathrm{Tr}\, \left(P_\delta^\pi \pi(x) P_\delta^\pi\right), \qquad x \in G.$$

For $f \in \mathcal{D}(G)$ set

$$\Phi_\delta^\pi(f) = \mathrm{Tr}\, \left(P_\delta^\pi \pi(f) P_\delta^\pi\right) = \int_G f(x)\Phi_\delta^\pi(x)\mathrm{d}x.$$

Of course, $\mathrm{d}x$ denotes a fixed Haar measure on G.

If π is of finite length, Θ_π is also well−defined and it equals the sum of characters of the successive subquotients in any Jordan−Hölder series of π. Obvously we have

$$\Theta_\pi(f) = \sum_{\delta \in \hat{K}} \Phi_\delta^\pi(f), \qquad f \in \mathcal{D}(G).$$

We shall need the following result from PhD thesis of D. Miličić:

Theorem 2. *(i) Let $\xi : \hat{G} \to \mathcal{D}'(G)$ be the canonical injection, i.e. $\xi(\pi) = \Theta_\pi$. The closure $\overline{\xi(\hat{G})}$ of $\xi(\hat{G})$ in $\mathcal{D}'(G)$ consists of distributions of the form*

$$\sum_{\omega \in \Omega} m(\omega)\Theta_\omega, \qquad m(\omega) \in \mathbb{N}, \quad \Omega \subseteq \hat{G} \text{ finite}.$$

(ii) If $\Theta = \sum_{\omega\in\Omega} m(\omega)\Theta_\omega$ is in the closure $\overline{\xi(\hat{G})}$ of $\xi(\hat{G})$ set $T(\Theta) = \Omega$. For any subset S of $\hat{G}$ its closure in $\hat{G}$ is

$$\overline{S} = \bigcup \left\{ T(\Theta);\ \Theta \in \overline{\xi(S)} \right\}.$$

(iii) If $(\pi_n)_{n\in\mathbb{N}}$ is a sequence in $\hat{G}$ and if $\Omega \neq \emptyset$ denotes the set of all its limits, there exists a subsequence $(\pi_{n_k})_{k\in\mathbb{N}}$ such that the sequence $\left(\Theta_{\pi_{n_k}}\right)_{k\in\mathbb{N}}$ converges in $\mathcal{D}'(G)$. For any such subsequence

$$\Omega = T\left(\lim_{k\to\infty} \Theta_{\pi_{n_k}}\right).$$

(iv) *Let* $(\pi_n)_{n\in\mathbb{N}}$ *be a sequence in* $\hat{G}$, *let* Ω *be a finite subset of* $\hat{G}$ *and* $m(\omega)\in\mathbb{N}$ *for any* $\omega\in\Omega$. *Suppose that*

$$\lim_{n\to\infty}\Theta_{pi_n}=\sum_{o\in\Omega}m(\omega)\Theta_\omega.$$

Then

$$\lim_{n\to\infty}\Phi_\delta^{\pi_n}(f)=\sum_{o\in\Omega}m(\omega)\Phi_\delta^\omega(f),\qquad \forall\delta\in\hat{K},\quad\forall f\in\mathcal{D}(G).$$

This theorem is deduced from Theorem 7. in [1] by showing that $C^*(G)$ is a C^*-algebra with bounded trace and that $\mathcal{D}(G)$ is an admissible subalgebra of $C^*(G)$. The proof of these facts is an adaptation of Harish$-$Chandra's proof that Θ_π are distributions.

Let α denote the representation $k\mapsto(Ad\,k)|\mathfrak{p}_c$ of K on $\mathfrak{p}_c$. For $\delta\in\hat{K}$ set

$$S(\delta)=\{\gamma\in\hat{K};\ (\alpha\otimes\delta:\gamma)>0\}.$$

Lemma 1. *Let* π *be an admissible representation of* G *on* $\mathcal{H}$, $\delta\in\hat{K}$ *and* $X\in\mathfrak{g}_c$. *Then* $\pi(X)\mathcal{H}_\delta\subseteq\mathcal{H}_{S(\delta)}$.

Proof: This follows from $\mathfrak{g}_c=\mathfrak{k}_c\dot{+}\mathfrak{p}_c$ and from the fact that

$$\sum_i X_i\otimes v_i\mapsto\sum_i\pi(X_i)v_i,\qquad X_i\in\mathfrak{p}_c,\quad v_i\in\mathcal{H}_\delta,$$

is a $K-$morphism from $\mathfrak{p}_c\otimes\mathcal{H}_\delta$ into $\mathcal{H}_K$.

Lemma 2. *Let* $\sigma\in\hat{M}$, $\delta\in\hat{K}$ *and* $X\in\mathfrak{g}_c$. *Then* $(\lambda,f)\mapsto\pi^{\sigma,\lambda}(X)f$ *is a continuous mapping from* $\mathfrak{a}_c^*\times\mathcal{H}_\delta^\sigma$ *into* $\mathcal{H}_{S(\delta)}^\sigma$.

Proof: For $k\in K$, $f\in\mathcal{H}_K^\sigma$ and $X\in\mathfrak{g}_c$ we have

$$\left[\pi^{\sigma,\lambda}(X)f\right](k)=\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{(\lambda+\rho)(H(k\exp tX))}f(\mathfrak{k}(k\exp tX))\Big|_{t=0}=$$

$$=(\lambda+\rho)(H(k,X))f(k)+(X(f\circ\mathfrak{k}))(k),$$

where

$$H(k,X)=\frac{\mathrm{d}}{\mathrm{d}t}H(k\exp tX))\Big|_{t=0},\qquad k\in K,\quad X\in\mathfrak{g}_c.$$

Now, $\mathcal{H}_K^\sigma$ consists of C^∞ (even analytic) functions $K\to V^\sigma$. Furthermore, the space $\mathcal{H}_S^\sigma$ is finite$-$dimensional for any finite $S\subseteq\hat{K}$, hence the usual topology on this finite$-$dimensional space is the one induced from $C^\infty(K,V^\sigma)$. The assertion follows from this and from Lemma 1.

Lemma 3. *For* $\varphi\in\mathcal{D}(G)$, $\sigma\in\hat{M}$ *and* $\delta\in\hat{K}$ *the mapping* $(\lambda,f,g)\mapsto\left(\pi^{\sigma,\lambda}(\varphi)f|g\right)$ *from* $\mathfrak{a}_c^*\times\mathcal{H}_\delta^\sigma\times\mathcal{H}_\delta^\sigma$ *into* $\mathbb{C}$ *is continuous.*

Proof: We have

$$\left(\pi^{\sigma,\lambda}(\varphi)f|g\right)=\int_G\int_K\varphi(x)\mathrm{e}^{(\lambda+\rho)(H(kx))}(f(\mathfrak{k}(kx))|g(k))_{V^\sigma}\mathrm{d}k\,\mathrm{d}x.$$

The assertion follows by easy estimation.

We are now able to prove a result on non$-$separated points in $\hat{G}$.

Theorem 3. *Let Ω be a finite subset of $\hat{G}$ such that there exists a sequence $(\pi_n)_{n\in\mathbb{N}}$ in $\hat{G}$ having Ω as the set of all limits. Then there exist $\sigma \in \hat{M}$ and $\lambda_0 \in \mathfrak{a}_c^*$ and a subrepresentation π_0 of $\pi^{\sigma,\lambda}$ such that Ω is the set of all infinitesimal equivalence classes of irreducible subquotients of π_0.*

Proof: Let $(\pi_n)_{n\in\mathbb{N}}$ be a sequence in $\hat{G}$ such that Ω is the set of all its limits. By (iii) of Theorem 2. we can suppose that

$$\lim_{n\to\infty} \Theta_{\pi_n} = \sum_{o\in\Omega} m(\omega)\Theta_\omega, \qquad m(\omega) \in \mathbb{N},$$

and by (iv) of the same theorem we have then

$$\lim_{n\to\infty} \Phi_\delta^{\pi_n}(\varphi) = \sum_{o\in\Omega} m(\omega)\Phi_\delta^\omega(\varphi), \qquad \forall\delta \in \hat{K}, \quad \forall\varphi \in \mathcal{D}(G). \tag{1}$$

By Theorem 1. for every $n \in \mathbb{N}$ there exist $\sigma_n \in \hat{M}$ and $\lambda_n \in \mathfrak{a}_c^*$ such that π_n is infinitesimally equivalent to a subrepresentation of π^{σ_n,λ_n}. Hence, we can and shall identify $\mathcal{H}_K(\pi_n)$ with a $\mathcal{G}-$submodule (with respect to $\pi^{\sigma_n,\lambda:n}$) of $\mathcal{H}_K^{\sigma_n}$. Thus, $\chi_{\nu(\sigma_n,\lambda_n)}$ is the uinfinitesimal character of the representation π_n. Let χ_0 be the infinitesimal character of the members of Ω. Due to a theorem by P. Bernat and J. Dixmier

$$\chi_0(z) = \lim_{n\to\infty} \chi_{\nu(\sigma_n,\lambda_n)}(z) \qquad \forall z \in \mathcal{Z}.$$

Now, $\nu \mapsto \chi_\nu$ induces a homeomorphism of $\mathfrak{h}_c^*/W$ (with the quotient topology) onto $Hom(\mathcal{Z},\mathbb{C})$ (with the pointwise convergence topology). Therefore, the sequence $(W\nu(\sigma_n,\lambda_n))_{n\in\mathbb{N}}$ in $\mathfrak{h}_c^*/W$ converges to the $W-$orbit in $\mathfrak{h}_c^*$ associated to χ_0. W being finite, we can suppose (by passing to a subsequence of (π_n) if necessary) that the sequence $(\nu(\sigma_n,\lambda_n))_{n\in\mathbb{N}}$ converges in $\mathfrak{h}_c^*$. Set

$$\nu_0 = \lim_{n\to\infty} \nu(\sigma_n,\lambda_n).$$

Then $\nu(\sigma_n,\lambda_n)|\mathfrak{d}_c = \Lambda_{\sigma_n} + \rho_0$ converges to $\nu_0|\mathfrak{d}_c$ in $\mathfrak{d}_c^*$. But the set $\{\Lambda_\sigma;\ \sigma \in \hat{M}\}$ is discrete in $\mathfrak{d}_c^*$, hence we can assume (by passing to a subsequence again) that $\Lambda_{\sigma_n} + \rho_0 = \nu_o|\mathfrak{d}_c$ for every $n \in \mathbb{N}$. The group M has finitely many connected components, hence for a given $\sigma \in \hat{M}$ the set $N(\sigma) = \{\tau \in \hat{M};\ \Lambda_\tau = \Lambda_\sigma\}$ is finite and

$$\sup\{\#N(\sigma);\ \sigma \in \hat{M}\} < +\infty.$$

Therefore, by taking a subsequence again we can suppose that $\sigma_n = \sigma_m$ for all $n,m \in \mathbb{N}$. Set $\sigma = \sigma_n$, $n \in N$. Furthermore, set $\pi^\lambda = \pi^{\sigma,\lambda}$, $\lambda \in \mathfrak{a}_c^*$, and $\mathcal{H} = \mathcal{H}^\sigma$. Now, $\mathcal{H}_K^{\pi_n} \subseteq \mathcal{H}_K$ for every $n \in \mathbb{N}$ and $\mathcal{H}_\delta^{\pi_n} = \mathcal{H}_K^{\pi_n} \cap \mathcal{H}_\delta$, $\delta \in \hat{K}$. Let

$$\lambda_0 = \lim_{n\to\infty} \lambda_n = \nu_0|\mathfrak{a}_c.$$

Fix $\delta \in \hat{K}$. The space $Grass(\mathcal{H}_\delta)$ of all subspaces of $\mathcal{H}_\delta$ with the usual topology is compact. Hence, a subsequence of $(\mathcal{H}_\delta^{\pi_n})_{n\in\mathbb{N}}$ convereges in $Grass(\mathcal{H}_\delta)$. $\hat{K}$ being countabel, using a diagonal procedure we can pass to a subsequence of $(\pi_n)_{n\in\mathbb{N}}$ (denoted again by $(\pi_n)_{n\in\mathbb{N}}$) such that the sequence $(\mathcal{H}_\delta^{\pi_n})_{n\in\mathbb{N}}$ converges in $Grass(\mathcal{H}_\delta)$. Denote by $\mathcal{H}_\delta^0$ its limit and set

$$\mathcal{V} = \sum_{\delta\in\hat{K}} \dot{+}\mathcal{H}_\delta^0.$$

Let $\delta \in \hat{K}$ and $f_0 \in \mathcal{H}_\delta^0$ be arbitrary. Choose $f_n \in \mathcal{H}_\delta^{\pi_n}$, $n \in \mathbb{N}$, so that $f_0 = \lim_{n\to\infty} f_n$. For any $X \in \mathfrak{g}_c$ we have by Lemma 1.

$$\pi^{\lambda_0}(X)f_0 = \lim_{n\to\infty} \pi^{\lambda_n}(X)f_n \in \lim_{n\to\infty} \mathcal{H}_{S(\delta)}^{\pi_n} = \sum_{\gamma\in S(\delta)} \dot{+}\mathcal{H}_\gamma^0 \subseteq \mathcal{V}.$$

It follows that $\mathcal{V}$ is a $\mathcal{G}-$submodule of $\mathcal{H}_K$ eith respect to π^{λ_0}, hence its closure $\mathcal{H}^0$ in $\mathcal{H}$ is a $G-$submodule for π^{λ_0}. Let π_0 be the corresponding subrepresentation of π^{λ_0}.

Let Ω_1 be the set of all infinitesimal equivalence classes of irreducible subquotients of π_0 and for $\omega \in \Omega_1$ let $n(\omega)$ denote the multiplicity of ω in a Jordan$-$Hölder series of π_0.

Fix $\delta \in \hat{K}$. Let $n_0 \in \mathbb{N}$ be such that

$$\dim \mathcal{H}_\delta^{\pi_n} = \dim \mathcal{H}_\delta^0 = m \qquad \forall n \geq n_0.$$

For any $n \geq n_0$ choose an orthonormal basis $(f_1^n, f_2^n, \ldots, f_m^n)$ of $\mathcal{H}_\delta^{\pi_n}$ in such a way that this sequence of orthornormal bases converges to an orthonormal basis $(f_1, f_2, \ldots, f_n)$ of $\mathcal{H}_\delta^0$:

$$f_j = \lim_{n\to\infty} f_j^n, \qquad j = 1, 2, \ldots, m.$$

By Lemma 3. for any $\varphi \in \mathcal{D}(G)$ we have

$$\lim_{n\to\infty} \Phi_\delta^{\pi_n}(\varphi) = \lim_{n\to\infty} \sum_{j=1}^m (\pi_n(\varphi)\varphi_j^n | f_j^n) = \lim_{n\to\infty} \sum_{j=1}^m (\pi^{\lambda_n}(\varphi) f_j^n | \varphi_j^n) =$$

$$= \sum_{j=1}^m (\pi^{\lambda_0}(\varphi) f_j | f_j) = \Phi_\delta^{\pi_0}(\varphi) = \sum_{\omega\in\Omega_1} n(\omega)\Phi_\delta^\omega(\varphi).$$

Therefore, for arbitrary $\delta \in \hat{K}$ and $\varphi \in \mathcal{D}(G)$ it follows from (1)

$$\sum_{\omega\in\Omega} m(\omega)\Phi_\delta^\omega(\varphi) = \sum_{\omega\in\Omega_1} n(\omega)\Phi_\delta^\omega(\varphi).$$

Summing over all $\delta \in \hat{K}$ we get

$$\sum_{\omega\in\Omega} m(\omega)\Theta_\omega = \sum_{o\in\Omega_1} n(\omega)\Theta_\omega.$$

Thus, $\Omega_1 = \Omega$.

REFERENCES:

[1] D. Miličić, *On C^*-algebras with bounded trace*, Glasnik Mat. 8(1973),7−21

Comment: In my first letter I have mentioned that $\pi, \pi' \in \hat{G}$ are non−separated if and only if they are infinitesimally equivalent to subquotients of the same elementary representation. The *only if* part follows from Theorem 3. But unfortunately the *if* part is not true. The simplest counter−example is found in the case $G = Sl(2, \mathbb{R})$. Namely, in this case dte discrete series representations occur in elementary representations in pairs which are separated in $\hat{G}$. Similar thing happens in the cases $G = SU(n, 1)$ and $G = SO(2n, 1)$, $n \geq 2$. But in all these cases *if and only if* holds true in $\hat{G}$ for representations with trivial infinitesimal character, even in $\hat{G} \setminus \hat{G}_d$, where $\hat{G}_d$ denotes the set of infinitesimal equivalence classes of discrete series representations.

If we consider the non−unitary dual space of G introduced by Fell instead of $\hat{G}$, the *if* part is easily proven, because for any $\sigma \in \hat{M}$ the set $\{\lambda \in \mathfrak{a}_c^*;\ \pi^{\sigma,\lambda}$ is irreducible$\}$ is dense in $\mathfrak{a}_c^*$. Probably the *only if* part holds true also in the non−unitary dual space, but to prove this it is necessary to generalize the Miličić's theorem to include the case of non−unitary representations.