Perturbation and Location of the Singular Values of Symmetrically Scaled Matrices

Vjeran Hari

Department of Mathematics, University of Zagreb, P.O. Box 335, 10002 Zagreb, Croatia.

Abstract. In this report two new results on the relative perturbations of the singular values of square matrices are presented. The first one considers the scaled diagonally dominant matrices of the form $G = D^*BD$, where D is diagonal and nonsingular. A simple and sharp estimate uses the norm of B in the assumption and in the bound. The second result deals with a general square matrix and uses in the assumption and in the bound the scaled polar factor of the matrix.

In addition a new location result for the singular values of a general matrix is presented. The intervals containing the singular values are defined using the absolute values of the diagonal elements of the triangular matrix R which is obtained by the QR factorization with column pivoting of the original matrix. In the bounds for the intervals, the norms of some principal sub-matrices of DRD are used, where D is such that $|\operatorname{diag}(DRD)| = I$.

Keywords: singular values, relative perturbations, symmetric scaling, location of singular values
PACS: 02.60.Dc

INTRODUCTION

We consider the perturbation and location problem of the singular values of *symmetrically scaled matrices* (SSM). If D_L and D_R are nonsingular diagonal matrices, D_LBD_R is referred to as the *scaled* matrix B. In the case of one-sided scaling, we have either $D_L = I$ or $D_R = I$ and the nontrivial scaling matrix is usually chosen to make the norms of the columns or of the rows of D_LBD_R one. Here we study the two-sided scaling under the constraint $D_L^* = D_R = D$. We refer to $G = D^*BD$ as the *symmetrically scaled* B. The matrix D is usually chosen to make the absolute value of the diagonal elements of G one. We shall address two problems for the singular values of such a G: the relative perturbation problem and the location problem.

From [1] and from the overview papers [2, 3], one finds out that for the relative perturbations of the singular values, the known results use either the one-sided scaling or the general two-sided scaling $G = D_1BD_2$, combined with the assumptions on (all minors of) B which guarantee the existence of a rank-revealing decomposition of G. These results define some classes of *well behaved* matrices for the singular value computation. The other classes of well behaved matrices include the matrices which satisfy some analytic conditions or some sparsity and sign pattern, then some rationally structured matrices and some finite element matrices.

In this paper we present two new relative perturbation results. The first one refers to scaled diagonally dominant (s.d.d.) SSM (see [4]). This result enlarges the set of well-behaved matrices to those square s.d.d. matrices G which have the form $G = D^*BD$ with D diagonal and B of small condition. The second result is more general, but the bound and the assumption use symmetrically scaled polar factor of G.

The second problem refers to finding the intervals which contain the singular values of a given matrix. To this end we use the simplest rank revealing QR factorization: the QR with column pivoting (QRP). Let G be a given rectangular matrix and let $G = QRP^T$ be the QRP of G. The task is to find the lower and upper bounds for the quotients $|r_{ii}|/\sigma_i$ in terms of the symmetrically scaled factor R.

Although, the results presented here have their own significance, typical applications lie in the accurate computation of the singular values of square matrices. The both problems are naturally linked with the accuracy problem of the Kogbetliantz diagonalization method for triangular matrices.

Throughout the paper, the following notation is used. For any square matrix X, $\operatorname{diag}(X)$ stands for the diagonal part of X, and $\Omega(X) = X - \operatorname{diag}(X)$ for the off-diagonal part of X. By $\|X\|$ and $\|X\|_F$ the spectral and the Frobenius (Euclidean) matrix norm of X is denoted, respectively. If not specified otherwise, in this paper, the default choice of norm is the spectral norm. The Euclidean vector norm is also denoted by $\|\cdot\|$. For a Hermitian matrix H, $\lambda_i(H)$ denotes the ith largest eigenvalue of H. The largest and the smallest eigenvalues of H are also denoted by $\lambda_{\max}(H)$ and $\lambda_{\min}(H)$. For a general matrix G, $\sigma_i(G)$ denotes the ith largest singular value of G. The largest and the smallest singular values of G are also denoted by $\sigma_{\max}(G)$ and $\sigma_{\min}(G)$. The absolute value of $X = (x_{ij})$ is the matrix $|X| = (|x_{ij}|)$.

The paper is organized as follows. In the next section two new results are presented, which shed light to the solution of the perturbation problem. After that we consider the location problem.

RELATIVE PERTURBATIONS FOR SSM

We first recall the notion of *scaled diagonally dominant* matrices from [5].

Scaled Diagonally Dominant Matrices

Let $\mathbf{C}^{n\times n}$ denote the set of complex matrices of order n and let $G \in \mathbf{C}^{n\times n}$. If $G = \operatorname{diag}(G) + N$, N has zero diagonal, then $G = (g_{ij})$ is *α-diagonally dominant* with respect to a norm $\|\cdot\|$ if $\|N\| \leq \alpha \min_{1\leq i\leq n} |g_{ii}|$, with $0 \leq \alpha < 1$. Now, let $B = \operatorname{diag}(B) + N$ with $|b_{ii}| = 1$, $1 \leq i \leq n$ and let D_L, D_R be arbitrary nonsingular diagonal matrices. Then $G = D_L B D_R$ is *α–scaled diagonally dominant* (α–s.d.d.) with respect to a given norm, if B is α–diagonally dominant with respect to that norm. If G is Hermitian, it is presumed that $D_L = D_R^*$. Note that an α–s.d.d matrix has nonzero diagonal elements.

Let $H \in \mathbf{C}^{n\times n}$ be a Hermitian matrix with non-zero diagonal elements. Then the eigenvalues and the diagonal elements of H are real and

$$H_S = |\operatorname{diag}(H)|^{-1/2} H |\operatorname{diag}(H)|^{-1/2} \tag{1}$$

is the scaled matrix H. The diagonal elements of H_S are ones or minus ones. If $\|\Omega(H_S)\| \leq \alpha < 1$, then H is α-s.d.d. Since $\alpha < 1$, H_S and consequently H must be nonsingular.

If one assumes that H in (1) is any square matrix with non-zero diagonal elements, then the absolute values of the diagonal elements of H_S have to be one. If $\|\Omega(H_S)\| \leq \alpha < 1$, then the classical perturbation result for the singular values implies that H_S and consequently H must be non-singular.

Each α-s.d.d. matrix has a special structure (see [6],[7],[8]). This structure has impact on the rate of convergence of the appropriate diagonalization methods (see [9],[10],[11],[12]). Here we consider only the perturbations properties of the singular values of such matrices.

A Relative Perturbation Result for the Symmetrically Scaled s.d.d. Matrices

Here we present the new relative perturbation result for the singular values of an α-s.d.d. square matrix G from [4].

Theorem 1 *Let G and δG be square matrices of order n and let $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$ and $\sigma_1' \geq \sigma_2' \geq \ldots \geq \sigma_n'$ be the singular values of G and $G' = G + \delta G$, respectively. Let $D = |\operatorname{diag}(G)|^{1/2}$ be nonsingular and let $B = D^{-1}GD^{-1}$, $\delta B = D^{-1}\delta G D^{-1}$. Let α and η be real numbers such that $\|\Omega(B)\| \leq \alpha$ and $\|\delta B\| \leq \eta$.*

If $\eta + 2\alpha < 1$, then G and G' are nonsingular and

$$\left|\frac{\sigma_i'}{\sigma_i} - 1\right| \leq \frac{\eta}{1 - \left(1 + \frac{\eta}{1-2\alpha}\right)\alpha} \leq \frac{\eta}{1-2\alpha}, \quad 1 \leq i \leq n. \tag{2}$$

If $\delta G \neq 0$, then the first inequality is strict provided that $\|\delta B\| < \eta$ or $\alpha > 0$. If $\eta > 0$, the second inequality is strict if and only if $\alpha > 0$.

Example 2 *Let us consider the following matrices G and δG:*

$$G = \begin{bmatrix} 1e{-}12 & -7e\text{-}13 & -9e{-}11 & 5e{-}10 & -3e{-}06 & 8e{-}03 \\ 7e{-}13 & 1e{-}08 & -8e{-}09 & -4e{-}06 & -8e{-}04 & 2e{-}01 \\ 9e{-}11 & 8e{-}09 & 1e{-}04 & -9e{-}04 & -2e{-}01 & 2e{+}02 \\ 5e{-}10 & 4e{-}06 & 9e{-}04 & 2e{+}01 & -2e{+}02 & -1e{+}03 \\ 3e{-}06 & 8e{-}04 & 2e{-}01 & 2e{+}02 & 5e{+}06 & 1e{+}07 \\ -8e{-}03 & -2e{-}01 & -2e{+}02 & 3e{+}03 & -1e{+}07 & 1e{+}12 \end{bmatrix}, \quad \delta G = \begin{bmatrix} 1e{-}15 & 1e{-}13 & 1e{-}11 & 4e{-}09 & 2e{-}06 & 1e{-}03 \\ 1e{-}13 & 1e{-}11 & 1e{-}09 & 4e{-}07 & 2e{-}04 & 1e{-}01 \\ 1e{-}11 & 1e{-}09 & 1e{-}07 & 4e{-}05 & 2e{-}02 & 1e{+}01 \\ 4e{-}09 & 4e{-}07 & 4e{-}05 & 2e{-}02 & 1e{+}01 & 4e{+}03 \\ 2e{-}06 & 2e{-}04 & 2e{-}02 & 1e{+}01 & 5e{+}03 & 2e{+}06 \\ 1e{-}03 & 1e{-}01 & 1e{+}01 & 4e{+}03 & 2e{+}06 & 1e{+}09 \end{bmatrix}$$

One can check that $\alpha = 3.6411e-02$ *and* $\eta = 5.7203e-03$*, so the conditions of the theorem are satisfied and the first (smaller) bound for* $|\sigma_i'/\sigma_i - 1|$ *in (2) is* $5.9378e-03$*. Using multiple precision arithmetic with 80 decimal digits, one can verify that* $\max_{1\le i\le 6}|\sigma_i'/\sigma_i - 1| = 9.9916e-04$.

For the spectral conditions holds: $\mathrm{cond}(B) = \mathrm{cond}(D^{-1}GD^{-1}) = 1.0007$ *while* $\mathrm{cond}(GD_R) = 2.6940e+10$, $\mathrm{cond}(D_LG) = 2.7638e+10$*, where* D_R *(*D_L*) is a diagonal matrix which makes the norms of the columns (rows) equal to one.*

Hence, for this example, the existing relative perturbations results based on $\mathrm{cond}(GD_R)$ *and* $\mathrm{cond}(D_LG)$ *(or on* $\mathrm{cond}(G^{-1}\delta G)$ *and* $\mathrm{cond}(\delta G G^{-1})$*) cannot be used.*

A typical application of this result lies in proving sharp accuracy estimates for the Kogbetliantz method when applied to the current s.d.d. iterate generated by the same method.

A GENERAL PERTURBATION RESULT FOR SSM

Here we present a new relative perturbation result for the singular values of a square matrix which uses diagonal scaling of its polar factors. The lengthy proof extends the scope of this report.

Theorem 3 *Let* $G, \delta G \in \mathbf{C}^{n\times n}$ *and let* $D \in \mathbf{C}^{n\times n}$ *be real, diagonal and nonsingular. Let* $\sigma_1 \ge \cdots \ge \sigma_n$ *and* $\sigma_i' \ge \cdots \ge \sigma_n'$ *be the singular values of* G *and* $G' = G + \delta G$*, respectively. If*

$$\|D^{-1}\delta G D^{-1}\| \le \eta \min\{\lambda_{\min}(D^{-1}\sqrt{GG^*}D^{-1}), \lambda_{\min}(D^{-1}\sqrt{G^*G}D^{-1})\},$$

then for each $1 \le i \le n$*, either* $\sigma_i = 0$ *and* $\sigma_i' = 0$ *or*

$$1-\eta \le \frac{\sigma_i'}{\sigma_i} \le 1+\eta\,.$$

Theorem obviously holds with

$$\eta = \frac{\|D^{-1}\delta G D^{-1}\|}{\min\{\lambda_{\min}(D^{-1}\sqrt{GG^*}D^{-1}), \lambda_{\min}(D^{-1}\sqrt{G^*G}D^{-1})\}}\,.$$

A LOCATION RESULT FOR THE SINGULAR VALUES

Suppose $R \in \mathbf{C}^{n\times n}$ is obtained by the QR factorization with column pivoting. Then for the elements of R

$$|\, r_{ii}\,|^2 \ge \sum_{k=i}^{j} |\, r_{kj}\,|^2, \quad 1 \le i \le j \le n, \tag{3}$$

holds. The following new results are closely connected to those from [13, Theorem 2.1]. The difference lies in the scaling.

Lemma 4 *Let* $R \in \mathbf{C}^{n\times n}$ *be a nonsingular upper-triangular matrix satisfying the condition (3). Let* $R = DR_SD$*, where* $D = |\,\mathrm{diag}(R)|^{\frac{1}{2}}$*. Then*

$$1 \le \frac{|r_{nn}|}{\sigma_n} \le \|R_S^{-1}\|\,.$$

This lemma is just a special case of the following general result

Theorem 5 *Let* $R \in \mathbf{C}^{n\times n}$ *be a nonsingular upper-triangular matrix satisfying the condition (3). Let* $R = DR_SD$*, where* $D = |\,\mathrm{diag}(R)|^{\frac{1}{2}}$*. Let* $R^{(i)} = {E^{(i)}}^T R E^{(i)}$*,* $T^{(i)} = E_i^T R E_i$*, where* $E^{(i)} = [e_1,\ldots,e_i]$*,* $E_i = [e_i,\ldots,e_n]$*. Here* $I_n = [e_1,\ldots,e_n]$*. If* $R_S^{(i)}$ *and* $T_S^{(i)}$ *are the appropriate sub-matrices of* R_S*, then*

$$\max\left\{\frac{\sigma_n}{\sigma_i}, \frac{1}{\|T_S^{(i)}\|}\right\} \le \frac{|r_{ii}|}{\sigma_i} \le \|{R_S^{(i)}}^{-1}\|\,. \tag{4}$$

REFERENCES

1. J. Demmel, M. Gu, S. Eisenstat, I. Slapničar, K. Veselić, and Z. Drmač, *Linear Algebra Appl.* **299**, 21–80 (1999).
2. I. Ipsen, "Relative Perturbation Results for Matrix Eigenvalues and Singular Values," in *Acta Numerica 1998*, Cambridge University Press, Cambridge, 1998, pp. 151–201.
3. I. Slapničar, *Math. Commun.* **1**, 153–167 (1996).
4. J. Matejaš, and V. Hari, Relative eigenvalue and singular value perturbations of scaled diagonally dominant matrices (2007), proposed for publications in SIAM J. Matrix Anal. Appl.
5. J. Barlow, and J. Demmel, *SIAM J. Numer. Anal.* **27**, 762–791 (1990).
6. V. Hari, and Z. Drmač, *SIAM J. Matrix Anal. Appl.* **18**, 1000–1012 (1997).
7. J. Matejaš, and V. Hari, *Z. Angew. Math. Mech.* **78**, 121–131 (1998).
8. V. Hari, *Math. Commun.* **4**, 135–158 (1999).
9. J. Matejaš, *Numer. Math.* **87**, 171–199 (2000).
10. J. Matejaš, *Linear Algebra Appl.* **349**, 17–53 (2002).
11. J. Matejaš, and V. Hari, "Scaled Iterates by Kogbetliantz Method," in *Proceedings of the Conference on Applied Mathematics and Computation, Dubrovnik, Croatia, September 13–18, 1999*, edited by M. Rogina, V. Hari, N. Limić, and Z. Tutek, Department of Mathematics, University of Zagreb, Zagreb, 2001, pp. 1–20, iSBN 953–6076–70–5.
12. V. Hari, and J. Matejaš, *Computing (Suppl.)* **16**, 83–105 (2003).
13. Z. Drmač, *IMA J. Numer. Anal.* **19**, 191–213 (1999).