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ON JACOBI-TYPE CONGRUENCE TRANSFORMATIONS FOR A POSITIVE DEFINITE MATRIX

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ABSTRACT:

Let B be a positive definite matrix of order two. We consider the general form of a nonsingular matrix F such that F^*BF is diagonal. The obtained result is a generalization of the known one for the case of real matrices.

O TRANSFORMACIJAMA KONGRUENCIJE JACOBIJEVOG TIPA ZA POZITIVNO DEFINITNU MATRICU. Promatra se opći oblik nesingularne matrice F koja kroz transformaciju F^*BF dijagonalizira pozitivno definitnu matricu B reda dva. Dobiveni rezultat je generalizacija postojećeg rezultata za slučaj realnih matrica.

1. INTRODUCTION

Let B denote a positive definite complex matrix of order two and let $\mathcal{C}(B)$ be a class of nonsingular matrices F such that F^*BF is diagonal. Here F^* denotes the conjugate transpose of F . Obviously. $\mathcal{C}(B)$ is not empty. It contains a unitary matrix since B can be diagonalized via a unitary similarity transformation. Let

$$B^{-1} = GG^*$$

be the Cholesky factorization of B^{-1}. Then $G \in \mathcal{C}(B)$ hence $\mathcal{C}(B)$ contains also a triagular matrix. Since $F \in \mathcal{C}(B)$ implies $FD \in \mathcal{C}(B)$ for every nonsingular diagonal D the set $\mathcal{C}(B)$ is infinite.

In this paper we investigate the general form of an element from $\mathcal{C}(B)$. The result, which is a part of the author's Ph.D. thesis [3], is presented below. This result is used in [3] for constructing the quadratically convergent cyclic Jacobi algorithms for the positive definite generalized eigenproblem

$$Ax = \lambda Mx \ , \ x \neq 0 \ .$$

Here A and M are n by n hermitian matrices and M is positive definite (see [3], sec.2.2. and sec.3.6.). Recently, we have used this result for proving the global convergence of certain cyclic Jacobi methods for the same eigenproblem (see [2]).

2. THE FORM OF $F \in \mathcal{C}(B)$

The following theorem is a generalization of a result of Gose [1] to complex matrices.

<u>THEOREM.</u> Let

$$(1) \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

be an arbitrary hermitian positive definite matrix of order two. If F is a nonsingular matrix satisfying

$$(2) \qquad F^*BF = \begin{bmatrix} b'_{11} & \\ & b'_{22} \end{bmatrix}$$

then

$$(3) \quad F = \frac{1}{\cos\gamma} \begin{bmatrix} \frac{1}{\sqrt{b_{11}}} & \\ & \frac{1}{\sqrt{b_{22}}} \end{bmatrix} \begin{bmatrix} \cos\varphi & e^{i\alpha}\sin\varphi \\ -e^{-i\beta}\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} e^{i\sigma_1}\sqrt{b'_{11}} & \\ & e^{i\sigma_2}\sqrt{b'_{22}} \end{bmatrix} .$$

where $\sigma_1, \sigma_2, \alpha, \beta$. are real, $\varphi, \phi \in [0, \frac{\pi}{2})$ and

(4) $$\sin\gamma = \frac{|b_{12}|}{\sqrt{b_{11} \cdot b_{22}}}, \quad \gamma \in [0, \tfrac{\pi}{2}) .$$

In addition the equality

(5) $$|\cos\varphi \cos\phi + e^{i(\alpha-\beta)} \sin\varphi \sin\phi| = \cos\gamma$$

holds.

Proof. Let (cf [1])

(6) $$D = \frac{1}{\cos\gamma} \begin{bmatrix} e^{i\sigma_1}\sqrt{b'_{11}} & \\ & e^{i\sigma_2}\sqrt{b'_{22}} \end{bmatrix}$$

where γ is defined by (4) and σ_1, σ_2 are chosen so that the diagonal elements of FD^{-1} are real and nonnegative. For the nonsingular matrix FD^{-1} there is a unitary matrix U and an upper triangular matrix T such that

(7) $$FD^{-1} = TU .$$

We can assume

(8) $$T = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}, \quad x > 0, \ z > 0$$

an

(9) $$\begin{bmatrix} \cos\phi & e^{i\alpha_1}\sin\phi \\ -e^{-i\alpha_1}\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} e^{i\beta_1} & \\ & e^{i\beta_2} \end{bmatrix}, \quad \phi \in [0, \tfrac{\pi}{2}] .$$

Note that every unitary matrix can be written in the form (9). Here α_1, ϕ are chosen to triangularize FD^{-1} and β_1, β_2 are chosen to make x, y positive. Set (cf.[1])

(10) $$V = T^*BT, \quad \tilde{V} = U^*VU .$$

The relations (10). (7) and (2) imply

$$\tilde{V} = (TU)^* B(TU) = (FD^{-1})^* B(FD^{-1})$$

$$= D^{-*} (F^*BF) D^{-1} = \cos^2\gamma I_2 .$$

hence using again (10) we obtain

$$V = U \tilde{V} U^* = \cos^2\gamma \; UU^* = \tilde{V} . \tag{11}$$

From the relations (10), (11), (8) and (1) we obtain

$$\begin{bmatrix} \cos^2\gamma & \\ & \cos^2\gamma \end{bmatrix} = \begin{bmatrix} x & 0 \\ \bar{y} & z \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \tag{12}$$

where $\bar{y}$ denotes the complex conjugate of y .

Since $b_{21} = \bar{b}_{12}$ we have

$$\begin{aligned} \cos^2\gamma &= x^2 \, b_{11} \\ 0 &= x\,(y\, b_{11} + z\, b_{12}) \\ \cos^2\gamma &= (y\, b_{11} + z\, \bar{b}_{12})\, y + (\bar{y}\, b_{12} + z\, b_{22})\, z \\ &= |y|^2 \, b_{11} + z^2 \, b_{22} + 2z \,\mathrm{Re}\, (\bar{y}\, b_{12}) . \end{aligned} \tag{13}$$

The first two equalities in (13) and $x > 0$ imply

$$x = \frac{1}{\sqrt{b_{11}}} \cos\gamma \quad . \quad y = -z \frac{b_{12}}{b_{11}} \quad . \tag{14}$$

Since $z > 0$ we obtain from (13).(14) and (4)

$$\cos^2\gamma = |y|^2 \, b_{11} + z^2 \, b_{22} - 2 z^2 \frac{|b_{12}|^2}{b_{11}}$$

$$= z^2 \, b_{22} \left(1 - \frac{|b_{12}|^2}{b_{11} b_{22}}\right) = z^2 \, b_{22} \cos^2\gamma ,$$

hence

$$z = \frac{1}{\sqrt{b_{22}}} . \tag{15}$$

The equalities (14), (15) and (4) imply

$$(16) \qquad y = -\frac{1}{\sqrt{b_{22}}}\ \frac{b_{12}}{b_{11}} = -\frac{e^{i\alpha_2}}{\sqrt{b_{11}}}\ \sin\gamma$$

where

$$\alpha_2 = \arg(b_{12}) .$$

Next we calculate the elements of TU. From the equalities (14) , (16), (8) and (9) we have

$$(TU)_{11} = e^{i\beta_1}(x\cos\phi - y\,e^{-i\alpha_1}\sin\phi)$$
$$= \frac{e^{i\beta_1}}{\sqrt{b_{11}}}(\cos\phi\cos\gamma + e^{i(\alpha_2-\alpha_1)}\sin\phi\ \sin\gamma).$$

The choice of σ_1 implies

$$(TU)_{11} = (FD^{-1})_{11} \geq 0$$

hence

$$(TU)_{11} = \frac{1}{\sqrt{b_{11}}}\ |\cos\phi\cos\gamma + e^{i(\alpha_2-\alpha_1)}\sin\phi\sin\gamma| .$$

Using the equalities (14) and (16) we obtain

$$(TU)_{12} = e^{i\beta_2}(x\,e^{i\alpha_1}\sin\phi + y\cos\phi)$$
$$= \frac{e^{i(\alpha_1+\beta_2)}}{\sqrt{b_{11}}}\ (\sin\phi\ \cos\gamma - e^{i(\alpha_2-\alpha_1)}\cos\phi\ \sin\gamma) .$$

Since

$$|\cos\phi\ \cos\gamma + e^{i(\alpha_2-\alpha_1)}\sin\phi\ \sin\gamma|^2 +$$
$$+ |\sin\phi\ \cos\gamma - e^{i(\alpha_2-\alpha_1)}\cos\phi\ \sin\gamma|^2 = 1$$

we can define $\varphi \in [0, \frac{\pi}{2})$ so that

$$(TU)_{11} = \frac{1}{\sqrt{b_{11}}}\ \cos\ \varphi, \quad (TU)_{12} = \frac{e^{i\alpha}}{\sqrt{b_{22}}}\ \sin\varphi$$

where

$$\alpha = \arg ((TU)_{12}) .$$

From the relations (8), (9) and (15) we obtain

$$(TU)_{21} = - \frac{e^{-i(\alpha_1 - \beta_1)}}{\sqrt{b_{22}}} \sin \phi ,$$

$$(TU)_{22} = \frac{e^{i\beta_2}}{\sqrt{b_{22}}} \cos \phi .$$

The choice of σ_2 implies $(TU)_{22} \geq 0$ and since $\cos \phi \geq 0$ we have $\beta_2 = 0$. Therefore we obtain

$$(17) \quad TU = \begin{bmatrix} \frac{1}{\sqrt{b_{11}}} & \\ & \frac{1}{\sqrt{b_{22}}} \end{bmatrix} \begin{bmatrix} \cos \varphi & e^{i\alpha} \sin \varphi \\ -e^{-i\beta} \sin \phi & \cos \phi \end{bmatrix}$$

where

$$\beta = \alpha_1 - \beta_1 .$$

Now the relation (3) follows from the relations (17), (7) and (6).

To prove the equality (5) we apply the determinant to the matrix equalities (2) and (3). We have

$$|\det F|^2 = \frac{b'_{11} \ b'_{22}}{\det B} = \frac{b'_{11} \ b'_{22}}{b_{11} \ b_{22}} \ \frac{1}{\cos^2 \gamma}$$

and

$$|\det F|^2 = \frac{b'_{11} \ b'_{22}}{b_{11} \ b_{22}} \ |\cos \varphi \cos \phi + e^{i(\alpha - \beta)} \sin \varphi \sin \phi|^2 \ \frac{1}{\cos^4 \gamma} \ )$$

hence (5) is otbained by equating the expressions on the right hand sides.

Q.E.D.

Set

$$\eta = \phi - \varphi$$

where φ and ϕ are such that the relations (3) and (5) hold. Then

$$\cos\gamma \leq \cos\varphi\cos\phi + \sin\varphi\sin\phi = \cos\eta$$

hence

(18) $$|\eta| \leq \gamma \ .$$

In the real case it holds $|\eta| = \gamma$. Namely, Gose [1] proved that in the real case

$$F = \frac{1}{\cos\tilde{\gamma}}\begin{bmatrix} \frac{1}{\sqrt{b_{11}}} & \\ & \frac{1}{\sqrt{b_{22}}} \end{bmatrix}\begin{bmatrix} \cos\tilde{\varphi} & \sin\tilde{\varphi} \\ -\sin(\tilde{\varphi}+\tilde{\gamma}) & \cos(\tilde{\varphi}+\tilde{\gamma}) \end{bmatrix}\begin{bmatrix} \sqrt{b_{11}} & \\ & \sqrt{b_{22}} \end{bmatrix}$$

where

$$\sin\tilde{\gamma} = \frac{b_{12}}{\sqrt{b_{11}\,b_{22}}} \ .$$

Note that in the real case the angles $\tilde{\varphi}$, $\tilde{\gamma}$ are not restricted to the interval $[0, \frac{\pi}{2}]$.

From the relations (3), (5) and (18) it is easy to prove that $\varphi - \phi \to 0$ and $\alpha - \beta \to 0$ as $b_{12} \to 0$. This means that the principal part of each matrix F , namely the matrix

$$\frac{1}{\cos\gamma}\begin{bmatrix} \cos\varphi & e^{i\alpha}\sin\varphi \\ -e^{-i\beta}\sin\phi & \cos\phi \end{bmatrix}$$

tends to a unitary matrix (a complex rotation) as b_{12} tends to zero. This fact is essentially used in the global convergence proof in [2].

<u>Remark</u>. If F satisfies the relation (2) and if Φ is unitary and diagonal then $F\Phi$ also satisfies (2). Therefore, σ_1 and σ_2 are free parametres of F . Using the theorem on the simultaneous diagonalization of two hermitian forms we conjecture that one element from the set $\{\varphi,\phi\}$ and one element from the set $\{\alpha,\beta\}$ are free parametres of F.

On the other hand, if F from the relation (3) is inserted into (2) then one obtains a set of necessary conditions for F to satisfy (2). It is an open question whether these necessary conditions can be simplified.

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