On the Convergence of Parallelized Eberlein Methods

D. Pupovci
Department of Mathematics, University of Prishtina,
M. Tita bb., 38000 Prishtina, Yugoslavia
and
Vjeran Hari*
Department of Mathematics, University of Zagreb,
Bijenička cesta 30, 10000 Zagreb, Croatia

Abstract. The well-known methods of Eberlein for diagonalizing general complex, real and symmetric complex matrices are modified to cope with parallel computing. The convergence is proved for all pivot strategies which are weakly equivalent to the modulus strategy. The obtained convergence results are quite similar to the known ones for the sequential methods under the serial strategies.

Subject Classifications: AMS (MOS): 65F15; CR: 5.14

Key Words and Phrases: Jacobi-type method, norm-reduction, modulus strategy

0 Introduction

The current arising interest in Jacobi-type methods is due to their adaptability for parallel processing. In this paper we consider the global conver-

*This paper was written in 1990 while this author was visiting Department of Mathematics, University of Kansas, Lawrence, Kansas.
gence of some norm-reducing Jacobi-type methods under a class of pivot strategies which includes the most important sequential (i.e. the serial) and parallel strategies.

The first efficient norm-reducing Jacobi-type methods have been proposed by Eberlein [1], [2] and Eberlein and Boothroyd [4]. Later Veselić [14] and Hari [5] have considered the global convergence of one of these methods. The first attempt to adapt a norm-reducing method for parallel computers is due to Sameh [12]. In their recent papers Luk and Park [10] and Hari [8] have considered convergence of Jacobi-type methods under the most common parallel strategies. Here we collect the ideas from all these papers to parallelize the methods of Eberlein and to prove appropriate convergence results which are not worse than the best known ones for the serial methods.

The paper is organized as follows. In Section 1 we provide a resume of the so called generalized cyclic Jacobi-type methods together with a general convergence result from [8]. In Section 2 we discuss the global convergence of the sequential methods of Eberlein, whereas in Section 3 we define the generalized cyclic (which include the parallel) versions of these methods. In Section 4 we prove convergence results of the new methods under a class of pivot strategies which are weakly-equivalent to the modulus one.

The results presented in Section 2 and especially in Section 3 and Section 4 are parts of an M.S.thesis [11].

The authors are indebted to the anonymous referee for helpful comments and suggestions.

1 Sequential and Parallel Jacobi-type Methods

Sequential Jacobi-type methods for solving eigenproblems of general matrices are iterative processes of the form

\[ A^{(k+1)} = T_k^{-1} A^{(k)} T_k, \quad k \geq 1 \]  

where \( A^{(1)} = A \) is the initial matrix of order \( n \), and \( T_k, k \geq 1 \) are nonsingular elementary plane matrices. Each \( T_k \) is defined by a pair of indices \( (p_k, q_k) \)
called pivot pair and by a $2 \times 2$ matrix $\hat{T}_k$,

$$\hat{T} = \begin{bmatrix} t_{p,k}^{(k)} & t_{p,q}^{(k)} \\ t_{q,p}^{(k)} & t_{q,k}^{(k)} \end{bmatrix}, k \geq 1$$
called pivot submatrix of $T_k$ or the $(p_k, q_k)$-restriction of $T_k$. Here $T_k = (t_{p,q}^{(k)})$ and except for its $(p_k, q_k)$-restriction, $T_k$ is equal to the identity matrix $I_n$. Each pivot submatrix $\hat{T}_k$ is determined by the algorithm of the method. If $\hat{T}_k$ depends only on $\hat{A}^{(k)}$, i.e. on the elements $a_{p_kp_k}^{(k)}, a_{p_kq_k}^{(k)}, a_{q_kp_k}^{(k)}$ and $a_{q_kq_k}^{(k)}$ for all $k$, the algorithm (or the method itself) is called simple. Typical examples of simple methods are (pure) Jacobi methods which diagonalize $A^{(k)}$. Jacobi-type methods are well suited for parallel processing. Here it is besides by its algorithm each Jacobi-type method is defined by a pivot strategy. The most common are the serial strategies, i.e. the row- and the column-cyclic ones. They belong to a larger class of cyclic strategies. Each cyclic strategy is defined by an ordering of the set $P_n = \{(i, j); 1 \leq i < j \leq n\}$. If $O = (i_1, j_1), \ldots, (i_N, j_N)$, where $N = n(n-1)/2$, is an ordering of $P_n$ then it defines a cyclic strategy namely the one where the pivot pair runs through the sequence $O$ in the cyclic way. For example, the row-cyclic strategy is defined by the row-cyclic ordering $O_R = (1, 2), (1, 3), \ldots, (1, n), (2, 3), \ldots, (2, n), \ldots, (n-1, n)$.

Two pairs $(i, j)$ and $(p, q)$ from $P_n$ are called disjoint or commuting provided that $i \neq p, i \neq q$ and $j \neq p, j \neq q$. In the set $O(P_n)$ of all orderings of $P_n$ are defined the three equivalence relations: equivalence, shift-equivalence and weak-equivalence (cf. [13]). Two orderings from $O(P_n)$ are equivalent if one can be derived from the other by a sequence of transpositions of adjacent disjoint pairs. If $O$ is as above and $O' = (i_{\sigma+1}, j_{\sigma+1}), \ldots, (i_N, j_N), (i_1, j_1), \ldots, (i_\sigma, j_\sigma)$ for $0 \leq \sigma < N$, then $O$ and $O'$ are shift-equivalent. The third equivalence relation is the join of the previous ones. Thus, $O$ and $\hat{O}$ are weakly-equivalent if there is a sequence $O_1 = O, O_2, \ldots, O_m = \hat{O}$ of orderings from $O(P_n)$, such that each $O_i, 1 \leq i < m$ is equivalent or shift-equivalent to $O_{i+1}$. Two cyclic strategies are equivalent (shift- or weak-equivalent) if their defining orderings from $O(P_n)$ are such.
essential that under certain pivot strategies several consecutive plane transformations can be performed concurrently provided that the corresponding pivot pairs are commuting. For simple Jacobi-type methods such are certain cyclic strategies (see [10],[12],[13]). In general, however, we can use the concept of generalized cyclic (GC) strategies which includes both the cyclic strategies and the parallel strategies (the latter ones enable optimal concurrency). Each GC strategy is defined by a partition of $P_n$ into subsets of commuting pairs (rotation sets) and by a numbering of this subset. A typical example of a parallel strategy is the modulus one (see [7],[10]) which is defined by the sequence $M_1, \ldots, M_n$ of subset, where

$$M_\tau = \{(i, j) \in P_n; i + j \equiv \tau + 1 (mod n)\}, 1 \leq \tau \leq n.$$  

Note that each $M_\tau$ contains around $\lceil n/2 \rceil$ pairs where generally $\lceil a \rceil$ denotes the largest integer $\leq a$.

In the sequel we assume that $R_1, \ldots, R_w$ is the defining sequence of a given GC strategy. For $\tau \geq 1$ let

$$Piv(\tau) = \begin{cases} R_w & \text{if } \tau \text{ is a multiple of } w \\ R_{\tau \mod w} & \text{otherwise} \end{cases}$$

where $\tau \mod w = \tau - \lceil \tau/w \rceil w$. A GC Jacobi-type method is defined as follows (cf. [8]).

At time $\tau = 1, 2, \cdots$ the nonsingular plane matrices $T_{pq}^{[\tau]}$, $(p, q) \in Piv(\tau)$ are determined from the elements of $A^{[\tau]}$ and the similarity transformation

$$A^{[\tau+1]} = T^{[\tau]-1} A^{[\tau]} T^{[\tau]}.$$  

is accomplished, where $A^{[1]} = A$ and

$$T^{[\tau]} = \prod_{(p,q) \in Piv(\tau)} T_{pq}^{[\tau]}.$$  

Since the pairs from $Piv(\tau)$, $\tau \geq 1$ are commuting the matrices $T_{pq}^{[\tau]}$, $(p, q) \in Piv(\tau)$ commute, hence $T^{[\tau]}$ is well defined by the relation (1.3). We call $Piv(\tau)$ pivot set at time $\tau$. Algorithm of a GC Jacobi-type method is just the algorithm for computing the matrices $T_{pq}^{[\tau]}$.

If $w = N$, then each $R_r$, $1 \leq r \leq w$, contains exactly one pair from $P_n$, hence we have a cyclic that is a sequential strategy. If each $R_i$ contains
around \([n/2]\) pairs we shall rather speak of a parallel than of a GC strategy (or Jacobi-type process). The transformations (1.2) shall be referred to as a (parallel or single) step of a GC process.

If each rotation set \(\mathcal{R}_r\) is somehow ordered, then \(\mathcal{R}_1, \ldots, \mathcal{R}_w\) represents an ordering of \(\mathcal{P}_n\). Thus, to each GC strategy one can associate more orderings. On the set of GC strategies one can define the three equivalence relations as above. Two GC strategies are equivalent (weakly-equivalent) if there exists a pair of associated orderings from \(O(\mathcal{P}_n)\) which are equivalent (weakly-equivalent). strategies can be found in [8]. In particular, It has been proved in [8] that the row cyclic strategy and the modulus(GC) strategy are weakly equivalent. In [10] it has been shown that certain orderings associated with today’s most common parallel strategies are weakly equivalent to an ordering associated with the modulus strategy. This fact shows the importance of the class of GC strategies which are weakly equivalent to the modulus one.

In this paper we consider convergence of some norm-reducing GC Jacobi-type methods. The convergence shall be measured by

\[
\| A \|_{	ext{off}} = \| A - \text{diag}(a_{11}, \ldots, a_{nn}) \| = \left[ \sum_{i,j=1}^{n} |a_{ij}|^2 \right]^{1/2}.
\]

Here \(\| \cdot \|\) denotes the Euclidean (Schur, Frobenius) matrix norm. The measure \(\| \cdot \|_{\text{off}}\) is called the off-norm or departure from the diagonal form. Later on we shall need

**Theorem 1.1** Let \(A\) be a matrix of order \(n\). Let \(A^{[1]} = A, A^{[2]}, \ldots\) be defined by the rule

\[
A^{[\tau+1]} = U^{[\tau]} A^{[\tau]} U^{[\tau]} + E^{[\tau]}, \quad \tau \geq 1,
\]

with

\[
U^{[\tau]} = \prod_{(p,q) \in \text{Piv}(\tau)} U^{[\tau]}_{pq}, \quad \tau \geq 1,
\]

and let the generalized cyclic strategy be weakly equivalent to the modulus one.

Let the conditions (i)—(v) hold.

(i) The sequence \((A^{[\tau]}, \tau \geq 1)\) is bounded;
\( \lim_{\tau \to \infty} \| E^{[r]} \|_{off} = 0; \)

(iii) \( U^{[r]} = (u^{[r]}_{rs}), \tau \geq 1 \) are unitary;

(iv) \( \lim_{\tau \to \infty} \min_{1 \leq r \leq n} \left| u^{[r]}_{rr} \right| > 0; \)

(v) \( \lim_{\tau \to \infty} \max_{(p,q) \in \Pi(v(\tau))} \left\{ |\tilde{a}^{[r]}_{pq}|^2 + |\tilde{a}^{[r]}_{qp}|^2 \right\} = 0 \)

where \( \tilde{A}^{[r]} = U^{[r]} A^{[r]} U^{[r]} = (\tilde{a}^{[r]}_{rs}), \tau \geq 1 \). Then

\[ \lim_{\tau \to \infty} \| A^{[r]} \|_{off} = 0. \]

**Proof.** The proof follows directly from Corollary 2.8 of [8].

Theorem 1.1. can be used for both the cyclic (when \( w=N \)) and parallel (when \( w \approx n \)) Jacobi-type methods. The first (second) case shall be treated in Section 2 (Section 3).

## 2 The Methods of Eberlein

Here we shortly discuss the convergence properties of the three well-known sequential methods of Eberlein. These are Jacobi-type methods for diagonalizing general complex, symmetric complex and general real matrices (see [1],[2],[3],[4]). First we consider a modification of the method for general complex matrices.

Let \( A \) be a complex matrix of order \( n \). The method generates the sequence \( A^{(1)} = A, A^{(2)}, \ldots \) by the rule (1.1), where \( T_k = R_k S_k \) and \( R_k, S_k \) are nonsingular plane matrices defined by their pivot submatrices

\[
\hat{R}_k = \begin{bmatrix}
\cos x_k & -e^{i\alpha_k} \sin x_k \\
e^{i\alpha_k} \sin x_k & \cos x_k
\end{bmatrix}, \quad \hat{S}_k = \begin{bmatrix}
\cosh y_k & -ie^{i\beta_k} \sinh y_k \\
-ie^{-i\beta_k} \sinh y_k & \cosh y_k
\end{bmatrix},
\]

respectively. Since \( R_k \) is unitary, we can rewrite (1.1) in the form

\[ A^{(k+1)} = S_k^{-1} \tilde{A}^{(k)} S_k, \quad \tilde{A}^{(k)} = R_k^{*} A^{(k)} R_k, \quad k \geq 1, \quad (2.1) \]
The transformation formulas are as follows (see [1]):

\[ B^{(k)} = \frac{1}{2}[A^{(k)} + A^{(k)*}] = (b_{rs}^{(k)}) \]
\[ C(\tilde{A}^{(k)}) = \tilde{A}^{(k)} - \tilde{A}^{(k)*} = (\tilde{c}_{rs}^{(k)}) \]

and set
\[ A^{(k)} = (a_{rs}^{(k)}), \quad \tilde{A}^{(k)} = (\tilde{a}_{rs}^{(k)}). \]

Next we consider the transformations involved at step \( k \). For simplicity, we denote the pivot indices \( p_k, q_k \) by \( p, q \). The parameters \( x_k \) and \( \alpha_k \) are chosen to annihilate the pivot element \( b_{pq}^{(k)} \) of the Hermitian matrix \( B^{(k)} \). The standard Jacobi formulas yield (see [15], [9])

\[ \alpha_k = \arg(b_{pq}^{(k)}), \quad \tan 2x_k = \frac{2 |b_{pq}^{(k)}|}{\tilde{a}_{pq}^{(k)} - b_{pq}^{(k)}}, \quad x_k \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]. \quad (2.2) \]

The parameters \( y_k \) and \( z_k \) are chosen to reduce the Euclidean norm of \( A^{(k)} \), i.e. to increase \( \Delta_k \), where

\[ \Delta_k = \|A^{(k)}\|^2 - \|A^{(k+1)}\|^2, \quad k \geq 1. \]

The transformation formulas are as follows (see [1]):

\[
\begin{align*}
    a_{rs}^{(k+1)} &= a_{rs}^{(k)} \quad \text{if } (r, s) \text{ and } (p, q) \text{ are disjoint,} \\
    a_{pr}^{(k+1)} &= \tilde{a}_{rs}^{(k)} = \cosh y_k + i e^{i\beta_k} a_{pq}^{(k)} \sinh y_k \\
    a_{rp}^{(k+1)} &= \tilde{a}_{rp}^{(k)} = \cosh y_k + i e^{-i\beta_k} \tilde{a}_{rq}^{(k)} \sinh y_k \\
    a_{qr}^{(k+1)} &= \tilde{a}_{qr}^{(k)} = \cosh y_k - i e^{-i\beta_k} a_{rp}^{(k)} \sinh y_k \\
    a_{rq}^{(k+1)} &= \tilde{a}_{rq}^{(k)} = \cosh y_k - i e^{i\beta_k} \tilde{a}_{qr}^{(k)} \sinh y_k \\
    a_{pp}^{(k+1)} &= \frac{1}{2} (\tilde{a}_{pp}^{(k)} + \tilde{a}_{qq}^{(k)} + \tilde{D}_{pq}^{(k)} \cosh 2y_k + i \tilde{c}_{pq}^{(k)} \sinh 2y_k), \\
    a_{qq}^{(k+1)} &= \frac{1}{2} (\tilde{a}_{pp}^{(k)} + \tilde{a}_{qq}^{(k)} - \tilde{D}_{pq}^{(k)} \cosh 2y_k - i \tilde{c}_{pq}^{(k)} \sinh 2y_k), \\
    a_{pq}^{(k+1)} &= \frac{1}{2} e^{i\beta} (\tilde{a}_{pq}^{(k)} - \tilde{D}_{pq}^{(k)} \sinh 2y_k + \tilde{c}_{pq}^{(k)} \cosh 2y_k), \\
    a_{qp}^{(k+1)} &= \frac{1}{2} e^{-i\beta} (-\tilde{a}_{pq}^{(k)} - \tilde{D}_{pq}^{(k)} \sinh 2y_k + \tilde{c}_{pq}^{(k)} \cosh 2y_k),
\end{align*}
\]
where \( r \notin \{p, q\} \) and
\[
\tilde{\xi}^{(k)}_{pq} = (\tilde{a}^{(k)}_{pq} + \tilde{a}^{(k)}_{qp}) \cos \beta_k - i(\tilde{a}^{(k)}_{pq} - \tilde{a}^{(k)}_{qp}) \sin \beta_k
\]
\[
\tilde{\eta}^{(k)}_{pq} = (\tilde{a}^{(k)}_{pq} - \tilde{a}^{(k)}_{qp}) \cos \beta_k - i(\tilde{a}^{(k)}_{pq} + \tilde{a}^{(k)}_{qp}) \cos \beta_k
\]
\[
\tilde{D}^{(k)}_{pq} = \tilde{a}^{(k)}_{pp} - \tilde{a}^{(k)}_{qq}.
\]

In [1] Eberlein proved that
\[
\Delta_k = \tilde{G}^{(k)}_{pq} (1 - \cosh 2y_k) - \tilde{H}^{(k)}_{pq} \sinh 2y_k
\]
\[
+ \frac{1}{2} \left( |\tilde{D}^{(k)}_{pq}|^2 + |\tilde{\xi}^{(k)}_{pq}|^2 \right) (1 - \cosh 4y_k) + \text{Im}(\tilde{\xi}^{(k)}_{pq} \tilde{D}^{(k)*}_{pq}) \sinh 4y_k,
\]
where
\[
\tilde{G}^{(k)}_{pq} = \sum_{r \neq p,q} (|\tilde{a}^{(k)}_{pr}|^2 + |\tilde{a}^{(k)}_{rp}|^2 + |\tilde{a}^{(k)}_{qr}|^2 + |\tilde{a}^{(k)}_{rq}|^2),
\]
\[
\tilde{H}^{(k)}_{pq} = -\text{Re}(\tilde{K}^{(k)}_{pq}) \sin \beta_k + \text{Im}(\tilde{K}^{(k)}_{pq}) \cos \beta_k,
\]
\[
\tilde{K}^{(k)}_{pq} = 2 \sum_{r \neq p,q} (\tilde{a}^{(k)}_{pr} \tilde{a}^{(k)*}_{qr} - \tilde{a}^{(k)*}_{rp} \tilde{a}^{(k)}_{rq}).
\]

Note that generally, \( z^* \) denotes the complex conjugate of \( z \). Also, \( \text{Re}(y) \) and \( \text{Im}(z) \) denote the real and the imaginary parts of \( z \).

In [1] Eberlein has proved that the choice
\[
\frac{\text{tanh} y_k}{2} = \frac{1}{2} \left\{ \frac{2 \text{Im}(\tilde{\xi}^{(k)}_{pq} \tilde{D}^{(k)*}_{pq}) - \tilde{H}^{(k)}_{pq}}{G^{(k)}_{pq} + 2(|\tilde{\xi}^{(k)}_{pq}|^2 + |\tilde{D}^{(k)}_{pq}|^2)} \right\} - \frac{\text{Re}(\tilde{\xi}^{(k)}_{pq})}{\text{Im}(\tilde{\xi}^{(k)}_{pq})} \right\}
\]
implies
\[
\Delta_k = \|A^{(k)}\|^2 - \|A^{(k+1)}\|^2 = \|\tilde{A}^{(k)}\|^2 - \|A^{(k+1)}\|^2
\]
\[
\geq \frac{1}{3} \frac{|\tilde{c}^{(k)}_{pq}|^2}{\|A^{(k)}\|^2}, \quad k \geq 1.
\]

Since for \( k \geq 1, \|A^{(k)}\| \geq \|A^{(k+1)}\| \geq 0 \), the sequence \( \{\|A^{(k)}\|, k \geq 1\} \) is convergent. It implies (we return to notation \( p_k, q_k \)) that
\[
\tilde{c}^{(k)}_{p_kq_k} \to 0 \quad \text{as} \quad k \to \infty.
\]

The relation (2.4) plays an essential part in the proof of the following theorem.
Theorem 2.1 Let \(A\) be a matrix of order \(n\) and \(A^{(1)} = A, A^{(2)}, \ldots\) be a sequence of matrices defined by the relations (2.1)–(2.3). If the pivot strategy is cyclic and weakly-equivalent to the row-cyclic one then the assertions I to III hold.

I \( C(A^{(k)}) \rightarrow 0 \) as \(k \rightarrow \infty\).

II \( B^{(k)} \rightarrow \text{diag}(\text{Re} \lambda_1, \ldots, \text{Re} \lambda_n) \) as \(k \rightarrow \infty\), where \(\text{Re} \lambda_1, \ldots, \text{Re} \lambda_n\), is an ordering of the real parts of the eigenvalues of \(A\).

III \( a_{rs}^{(k)} \rightarrow 0\) and \(a_{rs}^{(k)} \rightarrow 0\) as \(k \rightarrow \infty\), for every pair \((r, s)\) satisfying \(\text{Re} \lambda_r \neq \text{Re} \lambda_s\).

Proof. The complete proof can be found in [11]. It is very similar to the proof of Theorem 2.1 of [5]. However, it can be also deduced from Theorem 4.3 in this paper, by assuming \(w = N\), i.e. a cyclic strategy weakly equivalent to the modulus (cyclic) strategy.

The choice \(\alpha_k = \beta_k = 0\) (\(\alpha_k = \pi, \beta_k = \pi/2\)) yields the known method of Eberlein for complex symmetric (general real) matrices. Hence, Theorem 1.2 holds for these methods as well.

To ensure that \(A^{(k)}\) always tends to a diagonal matrix, Eberlein [2] has proposed a more sophisticated choice for \(\alpha_k\) and \(x_k\). For the choice in question, however, the above considerations can ensure (without further investigation) only the convergence to normality, i.e., the assertion I.

3 The Parallelized Algorithm

In order to parallelize the methods of Eberlein we must modify the algorithm to cope with parallel processing. In addition we modify it in such a way that the convergence proof becomes feasible. Again we concentrate to the case of a general complex matrix.

The matrix \(T^{[\tau]}\) of the relation (1.3), is sought in the form

\[
T^{[\tau]} = R^{[\tau]}M^{[\tau]}S^{[\tau]}, \quad \tau \geq 1,
\]

(3.1)

where \(R^{[\tau]}, M^{[\tau]}\) and \(S^{[\tau]}\) are defined as follows:
(a) \[ R^{[\tau]} = \prod_{(p,q) \in Piv(\tau)} R^{[\tau]}_{pq}, \]

where each \( R^{[\tau]}_{pq} \) is a unitary plane matrix defined by its \((p,q)\)-restriction

\[
\hat{R}^{[\tau]}_{pq} = \begin{bmatrix}
\cos x^{[\tau]}_{pq} & -e^{i\alpha^{[\tau]}_{pq}} \sin x^{[\tau]}_{pq} \\
e^{-i\alpha^{[\tau]}_{pq}} \sin x^{[\tau]}_{pq} & \cos x^{[\tau]}_{pq}
\end{bmatrix}.
\] (3.2)

(b) \[ M^{[\tau]} = \prod_{(p,q) \in Piv(\tau)} M^{[\tau]}_{pq}, \]

where each \( M^{[\tau]}_{pq} \) is a unitary diagonal plane matrix defined by

\[
\hat{M}^{[\tau]}_{pq} = \begin{bmatrix}
\sigma^{[\tau]}_{pq} \\
1
\end{bmatrix}.
\]

(c) \[ S^{[\tau]} = \prod_{(p,q) \in Piv(\tau)} S^{[\tau]}_{pq}, \]

where each \( S^{[\tau]}_{pq} \) is nonsingular plane matrix defined by

\[
\hat{S}^{[\tau]}_{pq} = \begin{bmatrix}
\cosh y^{[\tau]} & -ie^{i\beta^{[\tau]}} \sinh y^{[\tau]} \\
ie^{-i\beta^{[\tau]}} \sinh y^{[\tau]} & \cosh y^{[\tau]}
\end{bmatrix}.
\]

Note that \( \hat{S}^{[\tau]}_{pq} \) does not depend on \((p,q) \in Piv(\tau)\). The idea for this choice is due to Sameh [12]. This choice does not guarantee the maximal norm-reduction at each parallel step, but it facilitates the convergence considerations.

Using the factorization (3.1), we can rewrite the relation (1.2) in the form

\[
A^{[\tau+1]} = S^{[\tau]} A^{[\tau]} S^{[\tau]}, \\
\hat{A}^{[\tau]} = M^{[\tau]} A^{[\tau]} M^{[\tau]}, \\
A^{[\tau]}' = R^{[\tau]} A^{[\tau]} R^{[\tau]}, \
\tau \geq 1.
\]

To define the algorithm we fix \( \tau \) and omit the superscript \( \tau \).

The parameters \( \alpha_{pq} \) and \( x_{pq} \) are chosen to annihilate the pivot elements \( b_{pq} \) of \( B^{[\tau]} = (A^{[\tau]} + A^{[\tau]}')/2 = (b_{rs}) \). Thus

\[
\alpha_{pq} = \arg(b_{pq}), \quad \tan 2x_{pq} = \frac{2 |b_{pq}|}{b_{pp} - b_{qq}}, \quad x \in \left[ -\frac{\pi}{4}, \frac{\pi}{4} \right] \quad (3.3)
\]
holds for \((p, q) \in \text{Piv}(\tau)\).

Let \(C(A^{[r]}) = (c'_{pq})\), \(C(\tilde{A}^{[r]}) = (\tilde{c}_{rs})\) where generally \(C(X) = XX^* - X^*X\) is the commutator of \(X\).

The elements \(\sigma_{pq}\) of the matrices \(\hat{M}^{[r]}\) are chosen to meet the requirement

\[
Re(\tilde{c}_{pq}) \geq 0, \quad Im(\tilde{c}_{pq}) \geq 0, \quad (p, q) \in \text{Piv}(\tau).
\]

Since \(C(\tilde{A}^{[\tau]}) = M^{[\tau]} C(A^{[\tau]}) M^{[\tau]}\) we obtain

\[
\sigma_{pq} = \begin{cases} 
- i \text{sign}(Im(c'_{pq})) & \text{if } Re(c'_{pq}) Im(c'_{pq}) < 0 \\
\text{sign}(Re(c'_{pq}) + Im(c'_{pq})) & \text{otherwise}
\end{cases}
\]

for \((p, q) \in \text{Piv}(\tau)\). Here

\[
\text{sign}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\
-1 & \text{otherwise}
\end{cases}
\]

Let us consider the transformation \(\tilde{A}^{[\tau]} \to S^{[\tau]-1} \tilde{A}^{[\tau]} S^{[\tau]} = A^{[\tau + 1]}\). To this end let \(\mathcal{L}_{\tau}\) be a set of indices involved at time \(\tau\), i.e.,

\[
\mathcal{L}_{\tau} = \{i; (i, j) \in \text{Piv}(\tau) \text{ or } (j, i) \in \text{Piv}(\tau) \text{ for some } j\}.
\]

Obviously, \(\mathcal{L}_{\tau} \subseteq \{1, 2, \ldots, n\}\) and the inclusion can be proper. To express the elements of \(A^{[\tau + 1]}\) in terms of the elements of \(\tilde{A}^{[\tau]}\) we note that some elements of \(\tilde{A}^{[\tau]}\) are transformed twice, some once, and some may remain unchanged. Let \((p, q), (r, s) \in \text{Piv}(\tau)\) and \(j, k \notin \mathcal{L}_{\tau}\). Then a straightforward calculation yields

\[
\begin{align*}
a^{[\tau + 1]}_{jk} &= \tilde{a}_{jk} \quad (= \tilde{a}^{[\tau]}_{jk}) \\
a^{[\tau + 1]}_{pk} &= \tilde{a}_{pk} \cosh y + i e^{i\beta} \tilde{a}_{qk} \sinh y \\
a^{[\tau + 1]}_{kp} &= \tilde{a}_{kp} \cosh y + i e^{-i\beta} \tilde{a}_{kq} \sinh y \\
a^{[\tau + 1]}_{qk} &= \tilde{a}_{qk} \cosh y - i e^{-i\beta} \tilde{a}_{pk} \sinh y \\
a^{[\tau + 1]}_{kq} &= \tilde{a}_{kq} \cosh y - i e^{i\beta} \tilde{a}_{kp} \sinh y \\
a^{[\tau + 1]}_{pr} &= \frac{1}{2}(\tilde{a}_{pr} + \tilde{a}_{qs} + \tilde{D}_{pq,rs} \cosh 2y + i \tilde{\xi}_{pq,rs} \sinh 2y)
\end{align*}
\]
\[ a_{qs}^{[\tau+1]} = \frac{1}{2}(\tilde{a}_{pr} + \tilde{a}_{qs} - \tilde{D}_{pq,rs} \cosh 2y - i\tilde{\xi}_{pq,rs} \sinh 2y) \]
\[ a_{ps}^{[\tau+1]} = \frac{1}{2} e^{i\beta}(\tilde{\eta}_{pq,rs} - i\tilde{D}_{pq,rs} \sinh 2y + \tilde{\xi}_{pq,rs} \cosh 2y) \]
\[ a_{qr}^{[\tau+1]} = \frac{1}{2} e^{-i\beta}(-\tilde{\eta}_{pq,rs} - i\tilde{D}_{pq,rs} \sinh 2y + \tilde{\xi}_{pq,rs} \cosh 2y) \]

where

\[ \tilde{\xi}_{pq,rs} = (\tilde{a}_{ps} + \tilde{a}_{qr}) \cos \beta - i(\tilde{a}_{ps} - \tilde{a}_{qr}) \sin \beta \]
\[ \tilde{\eta}_{pq,rs} = (\tilde{a}_{ps} - \tilde{a}_{qr}) \cos \beta - i(\tilde{a}_{ps} + \tilde{a}_{qr}) \sin \beta \]
\[ \tilde{D}_{pq,rs} = \tilde{a}_{pr} - \tilde{a}_{qs}. \]

The formulas for \( a_{rp}^{[\tau+1]}, a_{sq}^{[\tau+1]}, \ldots \) can be obtained from the given ones by interchanging (replacing) the indices \( p, q \) by \( r, s \), respectively. Note that the case \( i = p \) and \( j = q \) is contained in the relations (3.6) and (3.7).

A straightforward calculation (the details can be found in [11]) yields

\[ \Delta = \Delta^{[\tau]} = \| A^{[\tau]} \|^2 - \| A^{[\tau+1]} \|^2 \]
\[ = \tilde{G}(1 - \cosh 2y) - \tilde{H} \sinh 2y \]
\[ + \frac{1}{2} \tilde{\rho}_1 (1 - \cosh 4y) + \tilde{\rho}_2 \sinh 4y, \quad (3.8) \]

where

\[ \tilde{G} = \sum_{(p,q) \notin \mathcal{L}_r} (|\tilde{a}_{pk}|^2 + |\tilde{a}_{qk}|^2 + |\tilde{a}_{kp}|^2 + |\tilde{a}_{kq}|^2) \]
\[ \tilde{K} = 2 \sum_{(p,q) \notin \mathcal{L}_r} (\tilde{a}_{pk} \tilde{a}_{qk}^* - \tilde{a}_{kp}^* \tilde{a}_{qk}) \]
\[ \tilde{H} = -Re(\tilde{K}) \sin \beta + Im(\tilde{K}) \cos \beta \]
\[ \tilde{\rho}_1 = \sum_{(p,q) \in \mathcal{P}_{iv}(\tau)} (|\tilde{D}_{pq,rs}|^2 + |\tilde{\xi}_{pq,rs}|^2) \]
\[ \tilde{\rho}_2 = \sum_{(p,q) \in \mathcal{P}_{iv}(\tau)} Im(\tilde{\xi}_{pq,rs} \tilde{D}_{pq,rs}^*) \]

In the relation (3.9) and later on \( \sum_{(p,q)}, \sum_{(r,s)}, \sum_{k \notin \mathcal{L}_r} \) denote \( \sum_{(p,q) \in \mathcal{P}_{iv}(\tau)}, \sum_{(r,s) \in \mathcal{P}_{iv}(\tau)}, \sum_{k=1}^n \sum_{k \notin \mathcal{L}_r} \), respectively.
The relations (3.8) and (3.9) show that $\Delta$ depends only on $y$ and $\beta$. Our aim is to define $y$ and $\beta$ so that $\Delta$ could be bounded from below by a suitable nonnegative quantity. The choice is

$$\tanh y = \frac{\bar{\rho}_2 - \bar{H}/2}{G + 2\bar{\rho}_1}, \quad (3.10)$$

$$\tanh \beta = -\frac{\text{Re}(\bar{c}_\tau)}{\text{Im}(\bar{c}_\tau)}, \quad \sin \beta \geq 0 \quad (3.11)$$

where

$$\bar{c}_\tau = \sum_{(p,q) \in \text{Piv}(\tau)} \bar{c}_{pq}. \quad (3.12)$$

In particular, if $\text{Im}(\bar{c}_\tau) = 0$ we assume (even when $\bar{c}_\tau = 0$), $\beta = \frac{\pi}{2}$. The relations (3.11), (3.4) and (4.2) imply

$$\bar{\rho}_2 - \bar{H}/2 = |\bar{c}_\tau|, \quad (3.13)$$

yielding the simpler form for $\tanh y$.

The relations (3.1)—(3.3), (3.5), (3.10)—(3.12) together with the definitions (3.7) and (3.9) define our norm-reducing algorithm. This algorithm generalizes the algorithm of Section 2 since for the case $w = N$ the two algorithms are almost identical.

**Remark 3.1** A similar algorithm can be obtained by assuming $\hat{M}^{[\tau]}_{pq} = \text{diag}(1, \sigma^{[\tau]}_{pq})$. Our choice for $\alpha_{pq}$ and $x_{pq}$ is the same as that for the sequential method of Section 2. A more sophisticated choice, analogous to that of Eberlein [2] could have been assumed, but then our convergence result would be weaker. A similar argument could be stated for the choice of matrices $\hat{S}^{[\tau]}_{pq}$ (cf. [12]).

It is easily seen that the choice $\alpha_{pq} = \beta_{pq} = 0$ ($\alpha_{pq} = \pi, \beta_{pq} = \pi/2$) for all $(p,q) \in \text{Piv}(\tau)$ makes the above algorithm suitable for complex symmetric (general real) matrices.
4 Convergence

Here we prove convergence properties of a GC Jacobi-type norm-reducing method defined by the algorithm of Section 3 and by a pivot strategy which is weakly equivalent to the modulus one. In the sequel $A$ is a general matrix of order $n$ and $A^{[1]} = A, A^{[2]}, \ldots$ matrices defined by the recursion (1.2). First we prove some auxiliary results.

Lemma 4.1 Let $y, \beta, \tilde{c}_r$ and $\tilde{G}, \tilde{\rho}_1$ be defined by the relations (3.10), (3.11), (3.12) and (3.9), respectively. Then $\Delta$ in (3.8) satisfies

$$\Delta \geq \frac{4}{3} \frac{|\tilde{c}_r|^2}{\tilde{G} + 2\tilde{\rho}_1}. \quad (4.1)$$

Proof. First we prove the relation

$$\tilde{\rho}_2 - \tilde{H}/2 = \sin \beta \text{ Re}(\tilde{c}_r) - \cos \beta \text{ Im}(\tilde{c}_r). \quad (4.2)$$

Let

$$\tilde{c}_r = \frac{1}{2} \tilde{K} + \tilde{\chi} \quad (4.3)$$

where

$$\tilde{\chi} = \sum_{(p,q)} \sum_{k \in L_r} (a_{pk} \tilde{a}_{qk}^* - \tilde{a}_{kp}^* a_{qk}). \quad (4.4)$$

From the definition of $L_r$ it follows that

$$\tilde{\chi} = \sum_{(p,q)} \sum_{(r,s)} \tilde{\chi}_{pq,rs}, \quad (4.5)$$

where

$$\tilde{\chi}_{pq,rs} = a_{pr} \tilde{a}_{qrs}^* - \tilde{a}_{qsr}^* a_{prs} + \tilde{a}_{qps}^* a_{rsq} - \tilde{a}_{qsp} a_{rsq}. \quad (4.6)$$

Let

$$\tilde{\omega}_{pq,rs} = a_{qrs}^* \tilde{D}_{pq,rs} - a_{qsp} \tilde{D}_{pq,rs}^*, \quad (4.7)$$

where $\tilde{D}_{pq,rs}^*$ is defined by the relation (3.7). By a simple calculation we obtain

$$\tilde{\chi}_{pq,rs} + \tilde{\chi}_{rs,pq} = \tilde{\omega}_{pq,rs} + \tilde{\omega}_{rs,pq}, \quad (p,q), (r,s) \in \text{Piv}(\tau). \quad (4.8)$$
The relations (4.4)-(4.7) imply
\[
\tilde{\chi} = \frac{1}{2} \sum_{(p,q)} \sum_{(r,s)} \left( \tilde{\chi}_{pq,rs} + \tilde{\chi}_{rs,pq} \right)
\]
\[
= \frac{1}{2} \sum_{(p,q)} \sum_{(r,s)} \left( \tilde{\omega}_{pq,rs} + \tilde{\omega}_{rs,pq} \right)
\]
\[
= \frac{1}{2} \sum_{(p,q)} \sum_{(r,s)} \tilde{\omega}_{pq,rs}. \tag{4.8}
\]

From the relations (4.6) and (3.7) we obtain
\[
\tilde{\omega}_{pq,rs} = -Re\left( [\tilde{a}_{ps} - \tilde{a}_{qr}] \tilde{D}_{pq,rs}^* \right) - i \text{Im} \left( [\tilde{a}_{ps} + \tilde{a}_{qr}] \tilde{D}_{pq,rs}^* \right). \tag{4.9}
\]

Using the relations (4.9) and (3.7) we obtain
\[
\sin \beta \ Re\left( \tilde{\omega}_{pq,rs} \right) - \cos \beta \ Im\left( \tilde{\omega}_{pq,rs} \right) = \text{Im} \left( \tilde{\xi}_{pq,rs} \tilde{D}_{pq,rs}^* \right). \tag{4.10}
\]

The relations (3.9), (4.10) and (4.8) yield
\[
\tilde{\rho}_2 = \sin \beta \ Re\left( \tilde{\chi} \right) - \cos \beta \ Im\left( \tilde{\chi} \right).
\]

The obtained equality, together with the relations (4.3) and (3.9) imply the relation (4.2).

The rest of the proof is quite similar to the proof of Lemma 1 of [3], so we discuss it very briefly. Setting
\[
\Theta = \tanh y, \quad ch = \cosh y, \quad sh = \sinh y, \tag{4.11}
\]
one obtains
\[
\Delta = 2 \left( 2 \tilde{\rho}_2 - \tilde{H} \right) sh \ ch - 2 \tilde{G} \ sh^2 - 4 \tilde{\rho}_1 \ (sh^4 + sh^2) + 8 \tilde{\rho}_2 \ sh^3 \ ch. \tag{4.12}
\]

Since \( sh \ ch = \Theta \ ch^2, sh^2 = \Theta^2 \ ch^2 \) the relations (3.10) and (4.12) imply
\[
\Delta = 2 \ ch^2 \ \frac{|\tilde{c}_r|^2}{\tilde{G} + 2\tilde{\rho}_1} \rho \tag{4.13}
\]
where
\[
\rho = 1 - \frac{2 \tilde{\rho}_1 \ sh^2 - 4 \tilde{\rho}_2 \ sh \ ch}{\tilde{G} + 2\tilde{\rho}_1}.
\]
Using the corresponding part of the proof of [3], one obtains
\[ \rho \geq \frac{2}{3}. \]  \hfill (4.14)

This completes the proof of Lemma 4.1, since the inequality (4.1) follows from the relations (4.13), (4.14) and the inequality \( ch^2 \geq 1 \). \hfill \blacksquare

**Lemma 4.2.** If \( y, \beta \) and \( \tilde{c}_r \) are defined by the relations (3.10), (3.11) and (3.12), respectively, then
\[ |a_{j}^{\tau+1} - \tilde{a}_{j}^{\tau}| \leq |\tilde{c}_{r}|^{1/2}, 1 \leq j, k \leq m. \]  \hfill (4.15)

**Proof.** The proof is similar to that of Veselić [14]. For the sake of simplicity we omit the superscript \( \tau \) and write \( a''_{st} \) for \( a_{\tau st}^{\tau+1} \).

Using the Cauchy-Schwarz inequality and the definitions from the relation (3.9), we have
\[ |\tilde{H}| \leq |\tilde{K}| \leq \sum_{(p,q)} \sum_{k \notin L_{\tau}} (2 |\tilde{a}_{pqk}| |\tilde{a}_{qpk}| + 2 |\tilde{a}_{kp}| |\tilde{a}_{kq}|) \leq \tilde{G} \]
and
\[ 2 |\tilde{\rho}_{2}| \leq \tilde{\rho}_{1}. \]

Thus, from the definition (3.10) of \( y \) we conclude that \( |\tanh y| \leq 1/2 \). The relation (4.11) and the estimates from [14] imply
\[ \Theta^2 \leq \Theta/2, \quad ch^2 \leq 4/3, \]
hence
\[ \begin{align*}
(ch - 1)^2 + sh^2 &\leq \Theta^2 \leq \frac{10}{9} \left( 1 + \frac{1}{4} \right) |\Theta| = \frac{10}{9} |\Theta|. \\
sh^2(sh^2 + ch^2) &\leq \frac{10}{9} \left( 1 + \frac{1}{4} \right) \left( 1 + \frac{1}{4} \right) |\Theta| \leq \frac{10}{9} |\Theta|.
\end{align*} \]  \hfill (4.16)

For \( (p,q), (r,s) \in \text{Piv}(\tau) \) and \( k \notin L_{\tau} \) we have
\[ \begin{align*}
|\tilde{a}_{pqk}|^2 + |\tilde{a}_{qpk}|^2 + |\tilde{a}_{kp}|^2 + |\tilde{a}_{kq}|^2 &\leq \tilde{G} \\
|\tilde{D}_{pq,rs}|^2 + |\tilde{\zeta}_{pq,rs}|^2 &\leq \tilde{\rho}_{1}.
\end{align*} \]  \hfill (4.17)
Using the relations (3.6), (3.9), (4.16), (4.17) and the Cauchy-Schwarz inequality we obtain

\[
|a_{pk}'' - \tilde{a}_{pk}| \leq |\tilde{a}_{pk}| |ch - 1| + |\tilde{a}_{qk}| |sh| \\
\leq [(ch - 1)^2 + sh^2]^{1/2} [||\tilde{a}_{pk}|^2 + |\tilde{a}_{qk}|^2]^{1/2} \\
\leq [||\Theta||^2]^{1/2} \leq |\tilde{c}_r|^{1/2}.
\]

The last inequality follows directly from the relations (3.10) and (3.13). A similar proof holds for the estimates involving \(\tilde{a}_{qk}, \tilde{a}_{kp}\) and \(\tilde{a}_{kq}\), where \((p, q) \in Piv(\tau), k \notin L_\tau\).

Using the relations (3.6), (3.7), (4.16), (4.17), (3.10) and (3.13) we obtain

\[
|a_{pr}'' - \tilde{a}_{pr}| \leq |\tilde{D}_{pq,rs}| sh^2 + |\tilde{\xi}_{pq,rs}|^2 |ch sh| \\
\leq [sh^2(sh^2 + ch^2)]^{1/2} [||\tilde{D}_{pq,rs}|^2 + |\tilde{\xi}_{pq}|^2]^{1/2} \\
\leq \left(\frac{10}{9}||}\Theta||\tilde{b}_1\right)^{1/2} \leq \left(\frac{10}{18}||}\tilde{c}_r||\right)^{1/2} \leq |\tilde{c}_r|^{1/2},
\]

where \((p, q), (r, s) \in Piv(\tau)\). In a similar way we prove the estimates involving \(\tilde{a}_{ps}, \tilde{a}_{qr}\) and \(\tilde{a}_{qs}\).

The following result is a proper generalization of Theorem 1.2.

**Theorem 4.3** Let \(A\) be a matrix of order \(n\) and let \(A^{[1]} = A, A^{[2]}, \ldots\) be the sequence of matrices defined by the relations (1.2), (3.1), (3.3), (3.5), (3.10) and (3.11). If the GC strategy is weakly equivalent to the modulus one, then as \(\tau \to \infty\).

I \(C(A^{[\tau]}) \to 0\),

II \(A^{[\tau]} + A^{[\tau]^*} \to 2 \text{ diag}(Re\lambda_1, \ldots, Re\lambda_n)\), where \(Re\lambda_1, \ldots, Re\lambda_n\) is an ordering of the real parts of the eigenvalues of \(A\),

III \(a^{[\tau]}_{rs} \to 0\) and \(a^{[\tau]}_{sr} \to 0\), provided that \(Re\lambda_r \neq Re\lambda_s\).

**Proof.** The relation (1.2) can be written as

\[
A^{[\tau+1]} = U^{[\tau]-1} A^{[\tau]} U^{[\tau]} + E^{[\tau]}, \quad \tau \geq 1
\]  
(4.18)

with

\[
U^{[\tau]} = R^{[\tau]} M^{[\tau]}, \quad E^{[\tau]} = A^{[\tau+1]} - A^{[\tau]}, \quad \tau \geq 1.
\]
By Lemma 4.2 we have
\[ \| E^{[\tau]} \|^2 = \sum_{r,s=1}^n | a^{[\tau+1]}_{rs} - \tilde{a}^{[\tau]}_{rs} |^2 \leq n^2 | \tilde{c}_\tau |, \quad \tau \geq 1. \tag{4.19} \]

From Lemma 4.1 we conclude that \((\| A^{[\tau]} \|, \tau \geq 1)\) is a nonincreasing sequence. Since it is bounded below by 0 it is convergent. The relation (4.1) implies \(| \tilde{c}_\tau | \to 0\) as \(\tau \to \infty\). Hence, from the relation (4.19) we obtain
\[ \| E^{[\tau]} \| \to 0 \quad \text{as} \quad \tau \to \infty. \tag{4.20} \]

Since \(U^{[\tau]}\) are unitary matrices, the relations (4.18) and (4.20) imply that for \(\tau \geq 1\)
\[ B^{[\tau+1]} = U^{[\tau]} B^{[\tau]} U^{[\tau]} + F^{[\tau]} \]
\[ C(A^{[\tau+1]}) = U^{[\tau]} C(A^{[\tau]}) U^{[\tau]} + W^{[\tau]} \tag{4.21} \]
holds with
\[ \| F^{[\tau]} \| \leq \| E^{[\tau]} \| \to 0 \quad \text{as} \quad \tau \to \infty \]
\[ \| W^{[\tau]} \| \leq 4 \| A^{[\tau]} \| \| E^{[\tau]} \| + 2 \| E^{[\tau]} \|^2 \leq 4 \| A^{[\tau]} \| \| E^{[\tau]} \| + 2 \| E^{[\tau]} \|^2 \to 0 \quad \text{as} \quad \tau \to \infty. \tag{4.22} \]

Next we apply Theorem 1.1 to the sequences \((B^{[\tau]}, \tau \leq 1)\) and \((C(A^{[\tau]}), \tau \geq 1)\). In order to obtain
\[ \| B^{[\tau]} \|_{off} \to 0 \quad \text{as} \quad \tau \to \infty \tag{4.23} \]
and
\[ \| C(A^{[\tau]}) \|_{off} \to 0 \quad \text{as} \quad \tau \to \infty, \tag{4.24} \]
we must check the conditions (i) to (v) for each sequence. The conditions (i) and (iii) are trivial. The condition (ii) is implied by the relation (4.22) whereas the condition (iv) holds since
\[ u_{kk}^{[\tau]} = 1 \quad \text{if} \quad k \not\in \mathcal{L}_\tau \]
\[ | u_{pp}^{[\tau]} | = | u_{qq}^{[\tau]} | = | \cos x_{pq}^{[\tau]} | \geq \frac{\sqrt{2}}{2} \quad \text{if} \quad (p,q) \in \text{Piv}(\tau) \]
holds for $U^r = (u^r_{rs})$. Finally, the condition (v) is implied by Lemma 4.2 and the choice of angles which implies

$$(U^{r-1}) B^r U^r)_{pq} = 0 \quad \text{and} \quad |(U^{r-1}) C(A^r) U^r)_{pq}| \leq |\tilde{c}_\tau| \quad \text{for all } (p, q) \in Piv(\tau).$$

In conclusion, the relations (4.23) and (4.24) hold.

Let us now prove the assertion I. It remains to prove that $\text{diag}(c^1_{11}, \ldots, c^n_{nn}) \to O$ as $\tau \to \infty$, where $C(A^r) = (c^r_{rs})$. Setting $A^r = B^r + G^r$, $\tau \geq 1$, we have $B^r = B^r$, $G^r = -G^r$ and consequently

$$C(A^r) = 2(G^r B^r - B^r G^r), \quad \tau \geq 1. \quad (4.25)$$

Hence

$$\max_{1 \leq r \leq n} |c^r_{rr}| \leq 2\|G^r\| \|B^r\|_{off} \leq 2\|A\| \|B^r\|_{off}, \quad \tau \geq 1.$$

The relation (4.23) shows that $c^r_{rr} \to 0$ as $\tau \to \infty$ for all $1 \leq r \leq n$. To prove the assertion II it remains to show that $\text{diag}(b^1_{11}, \ldots, b^n_{nn}) \to \text{diag}(\text{Re}\lambda_1, \ldots, \text{Re}\lambda_n)$ as $\tau \to \infty$. The proof is almost identical to the corresponding part of the proof of Theorem 1.1 of [5]. In [5] it is shown that

$$\sum_{r=1}^n |b^r_{rr} - \text{Re}\lambda_r|^2 \leq 2\|B^r\|^2_{off} + \sqrt{\frac{n^3 - n}{3}} \|C(A^r)\|,$$  \quad (4.26)

holds for an ordering of $\lambda_1, \ldots, \lambda_n$. By the assertion I and by the relation (4.23) the right-hand side of the inequality (4.26) tends to zero. Hence the sequence of vectors $((b^1_{11}, \ldots, b^n_{nn})^T, \tau \geq 1)$ has only finitely many accumulation points and remains to prove that for large $\tau$ the changes in each $b^r_{rr}$ are arbitrarily small.

Let $(p, q) \in Piv(\tau), \tau \geq 1$. The choice of angles $x^r_{pq}, \alpha^r_{pq}$ and the first equation in the relation (4.21) give

$$\begin{align*}
{b^r_{pp}}^{r+1} &= b^r_{pp} - \tan x^r_{pq} |b^r_{pq}| + f^r_{pp} \\
{b^r_{qq}}^{r+1} &= b^r_{qq} + \tan x^r_{pq} |b^r_{pq}| + f^r_{qq}
\end{align*}
\quad (4.27)$$
where \( F^{[\tau]} = (f_{rs}^{[\tau]}). \) Since \(| \tan x_{pq}^{[\tau]} | \leq 1\), the relations (4.22), (4.23) and (4.24) show that
\[
| b_{rr}^{[\tau+1]} - b_{rr}^{[\tau]} | \to 0 \quad \text{as} \quad \tau \to \infty
\]
holds for \( r \in \mathcal{L}_\tau. \) For \( r \notin \mathcal{L}_\tau \) we have \( b_{rr}^{[\tau+1]} = b_{rr}^{[\tau]} \), hence \((b_{11}^{[\tau]}, \ldots, b_{nn}^{[\tau]})^T\) has only one accumulation point. This completes the proof of II.

To prove the assertion III we recall the relations (4.25) and (4.23). They imply
\[
0 = \lim_{\tau \to \infty} c_{rs}^{[\tau]} = 2 \lim_{\tau \to \infty} (g_{rs}^{[\tau]} b_{ss}^{[\tau]} - b_{rr}^{[\tau]} g_{rs}^{[\tau]}) = 2 (\text{Re}\lambda_s - \text{Re}\lambda_r) \lim_{\tau \to \infty} g_{rs}^{[\tau]}, \quad 1 \leq r, s \leq n.
\]
If \( \text{Re}\lambda_s \neq \text{Re}\lambda_r \), then \( g_{rs}^{[\tau]} \to 0 \) as \( \tau \to \infty. \) Since \( b_{rs}^{[\tau]} \to 0 \) as \( \tau \to \infty \) because of the assertion II, we have
\[
a_{rs}^{[\tau]} = b_{rs}^{[\tau]} + g_{rs}^{[\tau]} \to 0,
\]
\[
a_{sr}^{[\tau]} = b_{rs}^{[\tau]} - g_{rs}^{[\tau]} \to 0
\]
as \( \tau \to \infty. \) This proves assertion III and Theorem 4.3.

The convergence results stated by Theorem 4.3 concern the diagonalization method for general complex matrices. This result can be used in connection with the GC Jacobi-type methods for complex-symmetric and general real matrices.

In the first case the appropriate choice is \( \alpha_{pq}^{[\tau]} = 0 \) for \((p, q) \in \text{Piv}(\tau)\) and \( \beta^{[\tau]} = 0, \tau \geq 1. \) Here the assertion I of Theorem 4.3 holds for all \( x_{pq}^{[\tau]}, (p, q) \in \text{Piv}(\tau) \) provided that \( x_{pq}^{[\tau]} \) is in a closed interval interior to \((-\frac{\pi}{2}, \frac{\pi}{2})\) or \((\frac{\pi}{2}, \frac{3}{2}\pi)). \) If however, \( x_{pq}^{[\tau]} \) is chosen as indicated by the relation (3.3) then the assertions II and III of Theorem 4.3 also hold.

In the second case the appropriate choice is \( \alpha_{pq}^{[\tau]} = \pi \) for \((p, q) \in \text{Piv}(\tau)\) and \( \beta^{[\tau]} = \frac{\pi}{2}, \tau \geq 1. \) In this case Theorem 4.3 can be extended by the following assertion

(iv) If \( \text{Re}\lambda_r = \text{Re}\lambda_s \) for a fixed pair \( r \neq s \) and \( \text{Re}\lambda_t \neq \text{Re}\lambda_s \) for all \( t \neq r, s, \) then \( a_{rs}^{[\tau]} \to \text{Im}\lambda_r, a_{sr}^{[\tau]} \to -\text{Im}\lambda_s \) as \( \tau \to \infty. \) The proof is identical to that in [5].

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References


O KONVERGENCIJI PARALELIZIRANIH METODA OD P. EBERLEIN
(Dukagjin Pupovci, Priština, Vjeran Hari, Zagreb)

SADRŽAJ

Dobro poznate metode od Eberlein za diagonalizaciju opće kompleksne, opće realne i kompleksne simetrične matrice su modificirane za rad na paralelnom računalu. Konvergencija je dokazana za sve pivotne strategije slabo ekvivalentne s modularnom strategijom. Rezultati su prilično slični poznatim rezultatima za sekvencijalne metode pod serijskom strategijom.