Rank one reducibility for metapletic groups via theta correspondence

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Abstract

We calculate reducibility for the representations of metaplectic groups induced from cuspidal representations of maximal parabolic subgroups via theta correspondence, in terms of the analogous representations of the odd orthogonal groups. We also describe the lifts of all relevant subquotients.

1 Introduction

In this paper we study rank-one reducibility for the non–trivial double $\widetilde{Sp(n)}$ cover of symplectic group $Sp(n)$ over a non–Archimedean local field of characteristic different than two using theta correspondence. This paper combined with [8] is a fundamental step in a systematic study of smooth complex representations of metaplectic groups. We expect application in the theory of automorphic forms where metaplectic groups play a prominent role.

We recall that the group $\widetilde{Sp(n)}$ is not a linear algebraic group. Thus, it is not in the framework of the usual theory for p-adic groups. Nevertheless, some basic algebraic facts [8] are true here. More precisely, based on a fundamental work of Bernstein and Zelevinsky ([2], [3], [26]) we checked that the basic notions of the representation theory of p–adic groups hold for metaplectic groups (some of that is already well–known from the previous works of Kudla [10], [9] and the book [15]). As usual, a parabolic subgroup of $\widetilde{Sp(n)}$ is the preimage $\widetilde{P}$ of a parabolic subgroup P of $Sp(n)$. If we write $P = MN$ for a Levi decomposition, then the unipotent radical N lifts to $\widetilde{Sp(n)}$. So, we have a decomposition $\widetilde{P} = \widetilde{M}N$. In [8] (see (1.2)) we describe the parametrization of irreducible smooth complex representations of the

Levi factor $\widetilde{M}$ of a maximal proper parabolic subgroup $\widetilde{P}$ of $\widetilde{Sp(n)}$. Roughly speaking, to $\widetilde{M}$ is attached an integer j , $0 < j \leq n$, and there is an epimorphism $\widetilde{GL(j)} \times \widetilde{Sp(n-j)} \to \widetilde{M}$ such that irreducible representations of $\widetilde{M}$ can be seen as $\rho \otimes \sigma$, where ρ is an irreducible representation of $\widetilde{GL(j, F)}$ and σ an irreducible representation of $\widetilde{Sp(n-j)}$. The goal of the present paper is to understand the reducibility and composition series of $\mathrm{Ind}_{\widetilde{P}}^{\widetilde{Sp(n)}}(\rho \otimes \sigma)$ where ρ and σ are cuspidal representations. This is a hard problem for linear groups and it is not completely solved yet (the case of generic representaions is covered by Shahidi [21], [22], and some conjectural description is known for classical groups due to many people (see for example [14])). One possible approach is to develop the theory for metaplectic groups from scratch. The other one (i.e., the one adopted in this paper) is to use the theta correspondence for the dual pair $\widetilde{Sp(n-j)} \times O(2(r-j)+1)$, where $O(2(r-j)+1)$ is a F–split (full) odd–orthogonal group. The approach is based on refining and further developing methods of [16], [17]. To simplify the notation and precisely describe our results, we shift indexes. Let σ be an irreducible cuspidal representation of $\widetilde{Sp(n)}$. So, we fix a non–trivial additive character ψ of F and let $\omega_{n,r}$ be the Weil representations attached to the dual pair $\widetilde{Sp(n)} \times O(2r+1)$. We write $\Theta(\sigma, r)$ for the smooth isotypic component of σ in $\omega_{n,r}$. Since σ is cuspidal, for the smallest r such that $\Theta(\sigma, r) \neq 0$ we have that $\Theta(\sigma, r)$ is an irreducible cuspidal representation of $O(2r+1)$. We denote it by τ. Let ρ be a self–contragredient irreducible cuspidal representation of $GL(j, F)$. Finally, let $\chi_{V,\psi}$ be a character of $\widetilde{GL(1)}$ defined at the end of 2.2.

We determine the reducibility point in this situation, and also the lifts of all irreducible subquotients of $\mathrm{Ind}_{\widetilde{P}}^{\widetilde{Sp(n+j)}}(\rho \otimes \sigma)$ and $\mathrm{Ind}_{P}^{O(2(r+j)+1)}(\rho \otimes \tau)$. This is accomplished in Theorem 3.5 (non–exceptional case), Theorem 4.1 (exceptional case–reducibility) and Propositions 4.2, 4.3, and Theorem 4.4 (exceptional case–theta lifts).

For the reader's convenience, we give some of the main theorems here. First, we recall the following non–exceptional case (see Theorem 3.5):

Theorem. *Let* $m_r = \frac{1}{2}\dim V_r$, *where* V_r *is a quadratic space on which* $O(2r+1)$ *acts. Let* P_j *be a maximal standard parabolic subgroup of* $O(2(r+j)+1)$ *(i.e., containing the upper triangular Borel subgroup of* $O(2(r+j)+1)$*) which* *has a Levi subgroup isomorphic to* $GL(j,F)\times O(2r+1)$. *We define a parabolic subgroup* $\widetilde{P_j}$ *of* $\widetilde{Sp(n+j)}$ *analogously.*

Let ρ *be an irreducible, cuspidal, genuine representation of* $\widetilde{GL(j,F)}$, *where* $\rho \notin \{\chi_{V,\psi}|\cdot|^{\pm(n-m_r)}, \chi_{V,\psi}|\cdot|^{\pm(m_r-n-1)}\}$. *Then, the representation* $\mathrm{Ind}_{\widetilde{P_j}}^{\widetilde{Sp(n+j)}}(\rho\otimes\sigma)$ *reduces if and only if the representation* $\mathrm{Ind}_{P_j}^{O(2(r+j)+1)}(\chi_{V,\psi}^{-1}\rho\otimes\tau)$ *reduces. In the case of irreducibility, we have*

$$\Theta(\mathrm{Ind}_{\widetilde{P_j}}^{\widetilde{Sp(n+j)}}(\rho\otimes\sigma), r+j) = \mathrm{Ind}_{P_j}^{O(2(r+j)+1)}(\chi_{V,\psi}^{-1}\rho\otimes\tau).$$

If the representation $\mathrm{Ind}_{\widetilde{P_j}}^{\widetilde{Sp(n+j)}}(\rho\otimes\sigma)$ *reduces, then it has two irreducible subquotients, say* π_1 *and* π_2, *such that the following holds:*

$$0 \longrightarrow \pi_1 \longrightarrow \mathrm{Ind}_{\widetilde{P_j}}^{\widetilde{Sp(n+j)}}(\rho\otimes\sigma) \longrightarrow \pi_2 \longrightarrow 0.$$

Then, $\Theta(\pi_i, r+j) \neq 0$, *is irreducible for* $i=1,2$, *and the following holds:*

$$0 \longrightarrow \Theta(\pi_1, r+j) \longrightarrow \mathrm{Ind}_{P_j}^{O(2(r+j)+1)}(\chi_{V,\psi}^{-1}\rho\otimes\tau) \longrightarrow \Theta(\pi_2, r+j) \longrightarrow 0.$$

Just described non–exceptional case is a fairly straightforward generalization of [17]; the exceptional case is rather different than the appropriate case in [17] and it requires some arguments that are specific for the dual pair $\widetilde{Sp(n)}\times O(2r+1)$. Most of the paper is about that case. We just recall the following (see Theorem 4.1):

Theorem. *The representation* $\mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^s\otimes\sigma)$ *reduces for a unique* $s\geq 0$ *(which is* $|m_r-n-1|$*). This means that*

$\mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{m_r-n}\otimes\sigma)$ *is irreducible unless* $m_r-n=-(m_r-n-1)$, *i.e.,* $m_r-n=\frac{1}{2}$.

In Section 5 we give some examples of reducibility. Section 5.1 describes the Siegel case $j=n$ and Section 5.2 describes the first non–Siegel case i.e., when $j=n-1$.

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2 Preliminaries

2.1 Symplectic and orthogonal groups

Let F be a non–Archimedean field of characteristic different from 2. For $n \in \mathbb{Z}_{\geq 0}$, let W_n be a symplectic vector space of dimension $2n$. We fix a complete polarization as follows

$$W_n = W_n' \oplus W_n'',\ W_n' = \mathrm{span}_F\{e_1, \dots e_n\},\ W_n'' = \mathrm{span}_F\{e_1', \dots e_n'\},$$

where $e_i, e_i',\ i = 1, \dots, n$ are basis vectors of W_n and the skew–symmetric form on W_n is described by the relations

$$\langle e_i, e_j \rangle = 0,\ i, j = 1, 2, \dots, n,\ \langle e_i, e_j' \rangle = \delta_{ij}.$$

The group $Sp(W_n)$ fixes this form. Let P_j denote a maximal parabolic subgroup of $Sp(n) = Sp(W_n)$ stabilizing the isotropic space $W_n'^j = \mathrm{span}_F\{e_1, \dots e_j\}$; then there is a Levi decomposition $P_j = M_j N_j$ where $M_j = GL(W_n'^j)$. By adding, in each step, a hyperbolic plane to the previous symplectic vector space, we obtain a tower of symplectic spaces and corresponding symplectic groups.

Now we describe the orthogonal groups we consider. Let V_0 be an anisotropic quadratic space over F of odd dimension; then $\dim V_0 \in \{1, 3\}$. For description of the invariants of this quadratic space, including the quadratic character χ_{V_0} describing the quadratic form on V_0, we refer to ([9]). In each step, as for the symplectic situation, we add a hyperbolic plane and obtain an enlarged quadratic space and, consequently, a tower of quadratic spaces and a tower of corresponding orthogonal groups. In the case in which r hyperbolic planes are added to the anisotropic space, a corresponding orthogonal group will be denoted $O(V_r)$, where $V_r = V_r' + V_0 + V_r''$ and V_r' and V_r'' are defined analogously as in the symplectic space. Again, P_j will be a maximal parabolic subgroup stabilizing $\mathrm{span}_F\{e_1, \dots e_j\}$.

2.2 The metaplectic group

The metaplectic group $\widetilde{Sp(n)}$ (or $Mp(n)$) is given as the central extension

$$1 \longrightarrow \mu_2 \xhookrightarrow{i} \widetilde{Sp(n)} \xrightarrow{p} Sp(n) \longrightarrow 1, \tag{1}$$

where $\mu_2 = \{1, -1\}$ and the cocyle involved is Rao's cocycle ([20]). For the more thorough description of the structural theory of the metaplectic group we refer to [9],[20],[7],[8]. Specifically, for every subgroup G of $Sp(n)$ we denote by $\widetilde{G}$ its preimage in $\widetilde{Sp(n)}$. In this way, the standard parabolic subgroups of $\widetilde{Sp(n)}$ are defined. Then, we have $\widetilde{P_j} = \widetilde{M_j}N'_j$, where N'_j is the image in $\widetilde{Sp(n)}$ of the unique monomorphism from N_j (the unipotent radical of P_j) to $\widetilde{Sp(n)}$ ([15], Chapter 2, II.9). We emphasise that $\widetilde{M_j}$ is not a product of GL factors and a metaplectic group of smaller rank, but there is an epimorphism (this is the case of maximal parabolic subgroup)

$$\phi : \widetilde{GL(j,F)} \times \widetilde{Sp(n-j)} \to \widetilde{M_j}.$$

Here, we can view $\widetilde{GL(j,F)}$ as a two fold cover of $GL(j,F)$ in it's own right, where the multiplication is given by

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 (\det g_1, \det g_2)_F),$$

where $(\cdot, \cdot)_F$ denotes the Hilbert symbol (of course, this cocoycle for $\widetilde{GL(j,F)}$ is just the restriction of the Rao's cocoycle to $GL(j,F) \times GL(j,F)$). Then, $\phi((g, \epsilon_1), (h, \epsilon)) = (\mathrm{diag}(g,h), \epsilon_1 \epsilon (x(h), \det g)_F)$. The function $x(h)$ is defined in [20] or [9], p. 19.

In this way, an irreducible representation π of $\widetilde{M_j}$ can be considered as a representation $\rho \otimes \sigma$ of $\widetilde{GL(j,F)} \times \widetilde{Sp(n)}$, where ρ and σ are irreducible representations, provided they are both trivial or both non–trivial when restricted to μ_2.

The pair $(Sp(n), O(V_r))$ constitutes a dual pair in $Sp(n \cdot \mathrm{dim} V_r)$ ([9],[10]). Since $\dim(V_r)$ is odd, the group $Sp(n)$ does not split in $\widetilde{Sp}(n \cdot \mathrm{dim} V_r)$, so the theta correspondence relates the representations of $\widetilde{Sp(n)}$ and of $O(V_r)$, or more general, the representations of the metaplectic groups (as two–fold coverings of symplectic groups attached to the symplectic towers) with the representations of the orthogonal groups attached to the orthogonal tower (Section 5 of [9]).

From now on, we fix an additive, non–trivial character ψ of F related to theta correspondence ([9],[10]), and a character $\chi_{V,\psi}$ on $\widetilde{GL(n,F)}$ given by $\chi_{V,\psi}(g,\epsilon) = \chi_V(\det g)\epsilon\gamma(\det g, \frac{1}{2}\psi)^{-1}$. Here γ denotes the Weil index ([9], p. 13, p. 17) and χ_V is a quadratic character related to the orthogonal tower. We denote by $\alpha = \chi_{V,\psi}^2$. α is a quadratic character on $GL(n,F)$ given by $\alpha(g) = (\det g, -1)_F$ ([9], p. 17)

3 The first reducibility result

We emphasise that the results in these section are valid for every F of characteristic different from 2; i.e., we do not need the validity of Howe's conjecture.

To prove the main reducibility result (Theorem 3.5), which also describes the structure of the lift of the subquotients of the induced representation, we need the following lemmas, which are simple extension of the results known for linear groups to the case of metaplectic group.

Recall that $\widetilde{M_j}$ is a Levi subgroup of a maximal parabolic subgroup of $\widetilde{Sp(n)}$. As such, it has a character $\nu = |\det|_F$ coming from the usual character of that form on $GL(j,F)$. We call a representation π of some covering group (in our case, of the metaplectic, or of the covering of general linear group, or of the Levi subgroup of the metaplectic group) genuine if it is non–trivial on μ_2.

Lemma 3.1. *Let π be an irreducible genuine cuspidal representation of $\widetilde{M_j}$, and let V be a smooth representation of $\widetilde{M_j}$. Then, there exist two subrepresentations of V, say $V(\pi)$ and $V(\pi)^\perp$, such that we have*

$$V = V(\pi) \oplus V(\pi)^\perp,$$

and all the subquotients of $V(\pi)$ are isomorphic to $\pi\nu^s$, for some $s \in \mathbb{C}$ and $V(\pi)^\perp$ does not have an irreducible subquotient isomorphic to some $\pi\nu^s$; $s \in \mathbb{C}$.

Proof. This claim is slightly weaker than the Bernstein center decomposition. If $\widetilde{M_j}$ is Levi subgroup of $\widetilde{Sp(n)}$, then there is an epimorphism from $\widetilde{GL(j,F)} \times \widetilde{Sp(n-j)}$ to $\widetilde{M_j}$. Now, it is not difficult to see that, using the notation from [4], the group $\widetilde{M_j}^\circ$ (the intersection of all the kernels of the unramified characters of $\widetilde{M_j}^\circ$) corresponds to $GL(j,F)^\circ$, so the unramified characters on $\widetilde{M_j}$ indeed look as described in the statement of the lemma. We note that the center $Z(\widetilde{M_j})$ equals $\widetilde{Z(M_j)}$. This means that because $Z(M_j)M_j^\circ$ is of finite index in M_j, $Z(\widetilde{M_j})\widetilde{M_j^\circ}$ is of finite index in $\widetilde{M_j}$. Because of this, when we restrict an irreducible cuspidal representation (π, V) of $\widetilde{M_j}$ to $Z(\widetilde{M_j})\widetilde{M_j^\circ}$ we get a finite direct sum of irreducible cuspidal representations of $\widetilde{M_j^\circ}$ (this notion makes sense since $\widetilde{M_j^\circ}$ contains all the unipotent radical of it's parabolic subgroups, and also, there is a splitting of unipotent radicals of M_j in $\widetilde{M_j}$), in the same way as in [4], p. 43, Proposition 25. Every irreducible cuspidal representation of $\widetilde{M_j^\circ}$ is compact (Harish–Chandra's theorem, p. 36). Indeed, we can repeat the arguments of that theorem since (the M_j°–version of) Cartan decomposition $M_j^\circ = K\Lambda^{+\circ}K$ holds (p. 36 there); we also have $\widetilde{M_j^\circ} = \widetilde{K}\Lambda^{+\circ}\widetilde{K}$, where $\Lambda^{+\circ}$ is embedded (this does not have to be a homomorphism) in $\widetilde{M_j^\circ}$ as $\lambda \mapsto (\lambda, 1)$. Since we deal with the cover of a maximal compact subgroup, we do not need that it splits in $\widetilde{Sp(n)}$, and so this works for any residual characteristic. We continue to use the notation from p. 35 and 36 of [4]. It is now enough to show that matrix coefficients, i.e., the functions $\pi(a(\lambda))\xi$ have compact support in $\Lambda^{+\circ}$. Here we may take that ξ is K'–invariant, where K' is a sufficiently small congruence subgroup which splits in $\widetilde{Sp(n)}$. Now we obtain the upper and the lower bound of $\pi(a(\lambda))\xi$ in the same way as on p. 35 and 36 (we actually get a finite support on $\Lambda^{+\circ}$). Then, with all the ingredients at our hand, we can apply Proposition 26 of [4], since it relays on the fact that compact representations split the category of smooth representations, which was proved in the fifth section of the first chapter of [4], in a greater generality (than for just reductive algebraic groups), so it holds for $\widetilde{M_j}^\circ$.

□

Lemma 3.2. *Let $\widetilde{G}$ be $\widetilde{Sp(n)}$ or $O(V_r)$. Let $\widetilde{P} = \widetilde{M}N'$ be a standard parabolic subgroup of $\widetilde{G}$ and let $\widetilde{\overline{P}} = \widetilde{M}\overline{N'}$ be the opposite parabolic subgroup. Assume π is a smooth representation of $\widetilde{M}$ and Π is a smooth representation of $\widetilde{G}$. Then, the following holds*

$$\mathrm{Hom}_{\widetilde{G}}(\mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}}(\pi), \Pi) \cong \mathrm{Hom}_{\widetilde{M}}(\pi, R_{\widetilde{\overline{P}}}(\Pi)).$$

Proof. First, note that the opposite unipotent subgroup $\overline{N}$ also lifts in the metaplectic group ([15], p. 43). Then, following the original Bernstein argument ([5]; we use that, topologically, $\widetilde{P} \backslash \widetilde{G} \cong P \backslash G$) the claim follows (in the case of metaplectic group). The case of non–connected $O(V_r)$ is similar ([17]). There is an alternative proof of this fact (for reductive algebraic groups) due to Bushnell ([6]). □

Remark. We refer to the isomorphism of the previous Lemma as "the second Frobenius reciprocity." Keeping the same notation as in the above Lemma, it is obvious that it can also be expressed in the following way:

$$\mathrm{Hom}_{\widetilde{G}}(\mathrm{Ind}_{\widetilde{P}}^{\widetilde{G}}(\pi), \Pi) \cong \mathrm{Hom}_{\widetilde{M}}(\pi, (R_{\widetilde{P}}(\check{\Pi}))\check{}).$$

For any positive integer n and positive integer r, let $(Sp(n), O(V_r))$ be a reductive dual pair in $Sp(n \cdot \mathrm{dim} V_r)$; let $n' = n \cdot \mathrm{dim} V_r$ (with $\mathrm{dim} V_r$ odd). Let $\omega_{n',\psi}$ be the Weil representation of $\widetilde{Sp(n')}$ depending on the non–trivial additive character ψ ([9],[10]), and let $\omega_{n,r} = \omega_{n,r}^{\psi}$ be the pull–back of that representation to the pair $(\widetilde{Sp(n)}, O(V_r))$. Let $\chi_{V,\psi}$ be as defined in the previous section. For an irreducible, genuine, smooth representation π_1 of $\widetilde{Sp(n_1)}$, let $\Theta(\pi_1, l)$ be a smooth representation of $O(V_l)$, given as the full lift of π_1 to the l–level of the orthogonal tower, i.e., the biggest quotient of $\omega_{n_1,l}$ on which $\widetilde{Sp(n_1)}$ acts as a multiple of π_1. It is of the form $\pi_1 \otimes \Theta(\pi_1, l)$, as a representation of $\widetilde{Sp(n_1)} \times O(V_l)$ ([9], p. 33, [15], p. 45).

We fix some notation throughout this section. Let σ be an irreducible, cuspidal, smooth and genuine representation of $\widetilde{Sp(W_n)} = \widetilde{Sp(n)}$, and let $\Theta(\sigma, r)$ be the first (full) nontrivial lift of σ in the orthogonal tower. Then, $\Theta(\sigma, r)$ is an irreducible cuspidal representation of $O(V_r)$ and we will denote it by τ. Let ρ denote a genuine irreducible cuspidal representation of $\widetilde{GL(j, F)}$.

The proof of Theorem 3.5 relays on the careful analysis of the Jacquet modules of the oscillatory representations, due to Kudla ([10]).

Because of the completeness of the argument, we write down Kudla's filtration (we also want to emphasise a slight difference between our version and Kudla's original expression for the filtration, due to the difference between the choice of the isotropic spaces invariant under the action of the parabolic subgroup). From now on, we fix a non–trivial additive character ψ of F. Also, from now on,

$$m_r = \frac{1}{2} \dim V_r.$$

Proposition 3.3. *[10] Let $\omega_{n+j,r+j}$ be the oscillatory representation of $\widetilde{Sp(n+j)} \times O(V_{r+j})$ corresponding to the character ψ. Then,*

1. The Jacquet module (with respect to the parabolic subgroup P_j of $O(V_{r+j})$) $R_{P_j}(\omega_{n+j,r+j})$ has the following $M_j \times \widetilde{Sp(n+j)}$– invariant filtration by $I_{jk},\ 0 \le k \le j$:

$$I_{jk} \cong \mathrm{Ind}_{P_{jk}\times \widetilde{P_k}\times O(V_r)}^{M_j\times \widetilde{Sp(n+j)}}(\gamma_{jk}\Sigma'_k \otimes \omega_{n+j-k,r}). \tag{2}$$

Here, P_{jk} is a standard parabolic subgroup of $GL(j,F)$ corresponding to the partition $(j-k,k)$, $\widetilde{P_k}$ is a maximal Levi of $\widetilde{Sp(n+j)}$, Σ'_k is a twist of a usual representation of $GL(k,F)\times GL(k,F)$ on Schwartz space $C_c^\infty(GL(k,F))$ and is given by

$$\Sigma'_k(g_1,g_2)f(g) = \nu^{-(m_r+\frac{k-1}{2})}(g_1)\nu^{m_r+\frac{k-1}{2}}(g_2)f(g_1^{-1}gg_2),$$

and γ_{jk} is a character on $GL_{j-k}\times \widetilde{GL(k,F)}$ given by

$$\gamma_{jk}(g_1,g_2) = \nu^{-(m_r-n-\frac{j-k+1}{2})}(g_1)\chi_{V,\psi}(g_2).$$

Specifically, a quotient I_{j0} equals $\nu^{-(m_r-n-\frac{j+1}{2})}\otimes\omega_{n+j,r}$ and a subrepresentation I_{jj} equals $\mathrm{Ind}_{GL(j,F)\times\widetilde{P_j}\times O(V_r)}^{M_j\times\widetilde{Sp(n+j)}}(\chi_{V,\psi}\Sigma'_j\otimes\omega_{n,r})$.

2. The Jacquet module (with respect to the parabolic subgroup $\widetilde{P_j}$ of $\widetilde{Sp(n+j)}$) $R_{\widetilde{P_j}}(\omega_{n+j,r+j})$ has the following $\widetilde{M_j}\times O(V_{r+j})$–invariant filtration by $J_{jk},\ 0\le k\le j$:

$$J_{jk} \cong \mathrm{Ind}_{\widetilde{P_{jk}}\times P_k\times\widetilde{Sp(n)}}^{\widetilde{M_j}\times O(V_{r+j})}(\beta_{jk}\Sigma'_k\otimes\omega_{n,r+j-k}). \tag{3}$$

Here $\widetilde{P_{jk}}$ is a standard parabolic subgroup of $\widetilde{M_j}$ corresponding to the partition $(j-k,k)$, β_{jk} is a character on $\widetilde{GL(j-k)}\times\widetilde{GL(k)}$ given by $\beta_{jk}((g_1,g_2) = (\chi_{V,\psi}\nu^{m_r-n+\frac{j-k-1}{2}})(g_1)\chi_{V,\psi}(g_2)$. The representation Σ'_k is as, before, a representation of $GL(k,F)\times GL(k,F)$ on Schwartz space $C_c^\infty(GL(k,F))$ given by

$$\Sigma'_k(g_1,g_2)f(g) = \nu^{m_r+j-\frac{k+1}{2}}(g_1)\nu^{-(m_r+j-\frac{k+1}{2})}(g_2)f(g_1^{-1}gg_2).$$

Specifically, the quotient J_{j0} of the filtration is isomorphic to $\chi_{V,\psi}\nu^{m_r-n+\frac{j-1}{2}}\otimes\omega_{n,r+j}$ and the subrepresentation J_{jj} is isomorphic to $\mathrm{Ind}_{\widetilde{GL(j)}\times P_j\times\widetilde{Sp(n)}}^{\widetilde{M_j}\times O(V_{r+j})}(\chi_{V,\psi}\Sigma'_j\otimes\omega_{n,r})$.

The following proposition describes certain isotypic components in the filtration above and is crucial for the proof of the Theorem 3.5 (for the basic facts about isotypic components, we refer to [15], p. 45, 46, 47). In general, if π is an irreducible smooth representation of some group G_1, and Π a smooth representation of $G_1 \times G_2$, then the isotypic component (a smooth representation of G_2) of π in Π is denoted by $\Theta(\pi, \Pi)$ (if it is understood what G_1 and G_2 are).

Proposition 3.4. *1. Assume that $j > 1$ and $s \in \mathbb{C}$. Then*

$$\mathrm{Hom}_{\widetilde{GL(j,F)} \times \widetilde{Sp(n)}}(R_{\widetilde{P_j}}(\omega_{n+j,r+j})/J_{jj}, \rho\nu^s \otimes \sigma) = 0$$

and

$$\mathrm{Hom}_{GL(j,F) \times O(V_r)}(R_{P_j}(\omega_{n+j,r+j})/I_{jj}, \chi_{V,\psi}^{-1}\rho\nu^s \otimes \tau) = 0.$$

2. For cuspidal representation $\rho \otimes \sigma$ (j can be equal to 1) we have

$$\Theta(\rho \otimes \sigma, J_{jj}) \cong \mathrm{Ind}_{P_j}^{O(V_{r+j})}(\chi_{V,\psi}\check{\rho} \otimes \tau),$$

and

$$\Theta(\chi_{V,\psi}^{-1}\rho \otimes \tau, I_{jj}) \cong \mathrm{Ind}_{\widetilde{P_j}}^{\widetilde{Sp(n+j)}}(\alpha\check{\rho} \otimes \sigma).$$

3. If $\rho \neq \chi_{V,\psi}|\cdot|^{m_r-n}$, then $\Theta(\rho \otimes \sigma, R_{\widetilde{P_j}}(\omega_{n+j,r+j})) \cong \mathrm{Ind}_{P_j}^{O(V_{r+j})}(\chi_{V,\psi}\check{\rho} \otimes \tau)$, and
if $\rho \neq \chi_{V,\psi}|\cdot|^{n-m_r+1}$, then $\Theta(\chi_{V,\psi}^{-1}\rho \otimes \tau, R_{P_j}(\omega_{n+j,r+j})) \cong \mathrm{Ind}_{\widetilde{P_j}}^{\widetilde{Sp(n+j)}}(\alpha\check{\rho} \otimes \sigma)$.

Proof. 1. For $0 < k < j$, the $\widetilde{GL(j,F)}$–part of the induced representation J_{jk} is induced from the representation of $\widetilde{GL(k,F)} \times \widetilde{GL(j-k,F)}$ and cannot have a cuspidal component. For $k = 0$, the $\widetilde{GL(j,F)}$–part is just $\chi_{V,\psi}\nu^{m_r-n+\frac{j-1}{2}}$ and we use the assumption that $j > 1$.

2. Again, let us just comment on the first case. Having in mind that the isotypic component of any irreducible representation π of $GL(j,F)$ in the "non–twisted" representation of $GL(j,F) \times \widetilde{GL(j,F)}$ appearing in the Jacquet module filtration is $\check{\pi}$, there is an obvious $\widetilde{GL(j,F)} \times P_j \times \widetilde{Sp(n)}$–invariant epimorphism

$$\chi_{V,\psi}\Sigma_j' \otimes \omega_{n,r} \to \rho \otimes \chi_{V,\psi}\check{\rho} \otimes \sigma \otimes \tau.$$

We immediately get an $\widetilde{M_j} \times O(V_{r+j})$–invariant epimorphism

$$J_{jj} \to \rho \otimes \sigma \otimes \mathrm{Ind}_{P_j}^{O(V_{r+j})}(\chi_{V,\psi}\check{\rho} \otimes \tau),$$

so we conclude that $\mathrm{Ind}_{P_j}^{O(V_{r+j})}(\chi_{V,\psi}\check{\rho} \otimes \tau)$ is a quotient of $\Theta(\rho \otimes \sigma, J_{jj})$. We prove that $\Theta(\rho \otimes \sigma, J_{jj})$ is also a quotient of $\mathrm{Ind}_{P_j}^{O(V_{r+j})}(\chi_{V,\psi}\check{\rho} \otimes \tau)$. Now, by Lemma 3.2, we have

$$\mathrm{Hom}_{\widetilde{M_j} \times O(V_{r+j})}(J_{jj}, \rho \otimes \sigma \otimes \Theta(\rho \otimes \sigma, J_{jj})) \cong$$
$$\cong \mathrm{Hom}_{\widetilde{M_j} \times GL(j,F) \times O(V_r)}(\chi_{V,\psi}\Sigma'_j \otimes \omega_{n,r}, \rho \otimes \sigma \otimes R_{\overline{P_j}}(\Theta(\rho \otimes \sigma, J_{jj}))).$$

For every intertwining map T from the first space, let T_0 be the corresponding intertwining map from the second space. Let ϕ be a natural epimorphism of $\widetilde{M_j} \times O(V_{r+j})$–modules belonging to the first space. Having in mind that all the relevant isotypic components are irreducible, we get that the image of ϕ_0 is isomorphic to $\rho \otimes \sigma \otimes \chi_{V,\psi}\check{\rho} \otimes \tau$. Now, we write down $\phi_0 = \phi'' \circ \phi'$, where ϕ' is just the projection with respect the kernel of ϕ_0, and ϕ'' is the isomorphism from that quotient to the image of ϕ_0. Let ϕ_1 be an operator belonging to

$$\mathrm{Hom}_{\widetilde{M_j} \times O(V_{r+j})}(\mathrm{Ind}(\chi_{V,\psi}\Sigma'_j \otimes \omega_{n,r}/\mathrm{Ker}\,\phi_0), \rho \otimes \sigma \otimes \Theta(\rho \otimes \sigma, J_{jj})),$$

such that $(\phi_1)_0 = \phi''$. Then, $(\phi_1 \circ \mathrm{Ind}(\phi'))_0 = \phi_0$, which forces $\phi_1 \circ \mathrm{Ind}(\phi') = \phi$. Since the image of $\mathrm{Ind}(\phi')$ a quotient of $\rho \otimes \sigma \otimes \mathrm{Ind}_{P_j}^{O(V_{r+j})}(\chi_{V,\psi}\check{\rho} \otimes \tau)$, so is the image of ϕ, i.e., $\rho \otimes \sigma \otimes \Theta(\rho \otimes \sigma, J_{jj})$ is a quotient of $\rho \otimes \sigma \otimes \mathrm{Ind}_{P_j}^{O(V_{r+j})}(\chi_{V,\psi}\check{\rho} \otimes \tau)$.

3. We explain in more details only the first part of the statement; the second is quite analogous. As it is obvious from the Statement 2 of this proposition, we must prove that, essentially, isotypic component corresponding to $\rho \otimes \sigma$ in the whole Jacquet module $R_{\widetilde{P_j}}(\omega_{n+j,r+j})$ actually depends only on the J_{jj}–part in the filtration of that module.

We use the part of the Bernstein decomposition from Lemma 3.1 for the representation $R_{\widetilde{P_j}}(\omega_{n+j,r+j})$ (and the notation is the same as there). If $j > 1$, from the first part of this proposition it follows that

$$R_{\widetilde{P_j}}(\omega_{n+j,r+j}/J_{jj})(\rho \otimes \sigma) = 0.$$

Since J_{jj} is a subrepresentation, we have $R_{\widetilde{P_j}}(\omega_{n+j,r+j})(\rho \otimes \sigma) = J_{jj}(\rho \otimes \sigma)$, so that

$$\mathrm{Hom}_{\widetilde{M_j}}(R_{\widetilde{P_j}}(\omega_{n+j,r+j}), \rho \otimes \sigma) \cong \mathrm{Hom}_{\widetilde{M_j}}(J_{jj}, \rho \otimes \sigma) \tag{4}$$

(as the restriction gives rise to an isomorphism, which is also $O(V_{r+j})$–equivariant). But, since there is a usual relation between taking a (smooth) part of the isotypic component of a representation and the homomorphism functor ([18]), we have the following

$$\Theta(R_{\widetilde{P_j}}(\omega_{n+j,r+j}), \rho \otimes \sigma)\check{} \cong \mathrm{Hom}_{\widetilde{M_j}}(R_{\widetilde{P_j}}(\omega_{n+j,r+j}), \rho \otimes \sigma)_\infty.$$

Now, the relation (4) completes the proof of the claim 3 in the case $j > 1$.

If $j = 1$, the filtration of $R_{\widetilde{P_1}}(\omega_{n+1,r+1})$ is of length two, and, in this case, $J_{10} = \chi_{V,\psi}|\cdot|^{m_r-n} \otimes \omega_{n,r+1}$ (we emphasise that $\rho \neq \chi_{V,\psi}|\cdot|^{m_r-n}$). On the other hand, J_{11} has a quotient $\rho \otimes \sigma \otimes \Theta(\rho \otimes \sigma, J_{11})$. Using the decomposition along the generalized central characters, we see that $\rho \otimes \sigma \otimes \Theta(\rho \otimes \sigma, J_{11}) \oplus J_{10}$ is a quotient of $R_{\widetilde{P_1}}(\omega_{n+1,r+1})$ (the sum is direct precisely when $\rho \neq \chi_{V,\psi}|\cdot|^{m_r-n}$), and we again obtain that (4) holds.

□

Theorem 3.5. *Let $m_r = \frac{1}{2}\dim V_r$, where V_r is a quadratic space on which $O(V_r)$ acts. Let P_j be a maximal standard parabolic subgroup of $O(V_{r+j})$ which has a Levi subgroup isomorphic to $GL(j, F) \times O(V_r)$, $\widetilde{P_j}$ is a standard parabolic subgroup of $\widetilde{Sp(n+j)}$ defined analogously. Let ρ be an irreducible, cuspidal, genuine representation of $\widetilde{GL(j, F)}$, where $\rho \notin \{\chi_{V,\psi}|\cdot|^{\pm(n-m_r)}, \chi_{V,\psi}|\cdot|^{\pm(m_r-n-1)}\}$. Then, the representation $\mathrm{Ind}_{\widetilde{P_j}}^{\widetilde{Sp(n+j)}}(\rho \otimes \sigma)$ reduces if and only if the representation $\mathrm{Ind}_{P_j}^{O(V_{r+j})}(\chi_{V,\psi}^{-1}\rho \otimes \tau)$ reduces. In the case of irreducibility, we have*

$$\Theta(\mathrm{Ind}_{\widetilde{P_j}}^{\widetilde{Sp(n+j)}}(\rho \otimes \sigma), r + j) = \mathrm{Ind}_{P_j}^{O(V_{r+j})}(\chi_{V,\psi}^{-1}\rho \otimes \tau).$$

If the representation $\mathrm{Ind}_{\widetilde{P_j}}^{\widetilde{Sp(n+j)}}(\rho \otimes \sigma)$ reduces, then it has two irreducible subquotients, say π_1 and π_2, such that the following holds:

$$0 \longrightarrow \pi_1 \longrightarrow \mathrm{Ind}_{\widetilde{P_j}}^{\widetilde{Sp(n+j)}}(\rho \otimes \sigma) \longrightarrow \pi_2 \longrightarrow 0.$$

Then, $\Theta(\pi_i, r + j) \neq 0$, is irreducible for $i = 1, 2$, and the following holds:

$$0 \longrightarrow \Theta(\pi_1, r + j) \longrightarrow \mathrm{Ind}_{P_j}^{O(V_{r+j})}(\chi_{V,\psi}^{-1}\rho \otimes \tau) \longrightarrow \Theta(\pi_2, r + j) \longrightarrow 0.$$

Proof. The main tool in the proof is Proposition 3.4. Now, as soon as this is established for the representations of the metaplectic group, we can proceed with the proof similarly as in the case of the dual pairs consisting of the symplectic and even–orthogonal group ([17]). □

4 The exceptional case

We continue with the notation from the previous section. We now discuss the case $\rho \in \{\chi_{V,\psi}|\cdot|^{\pm(m_r-n)}, \chi_{V,\psi}|\cdot|^{\pm(m_r-n-1)}\}$. The discussion is more subtle than in the case of "the split dual pair" (i.e., symplectic, even–orthogonal group [17]) due to the fact that the result about the unique reducibility point in the case of the parabolic induction from a maximal parabolic subgroup and cuspidal data ([23]) is not available for the metaplectic group.

We retain the notation from the previous section. For $\rho \notin \{\chi_{V,\psi}|\cdot|^{\pm(m_r-n)}, \chi_{V,\psi}|\cdot|^{\pm(m_r-n-1)}\}$, the uniqueness of the reducibility point for the representation (we introduce a shorter notation)

$$\rho\nu^s \rtimes \sigma := \mathrm{Ind}_{\widetilde{M_j}}^{\widetilde{Sp(n+j)}}(\rho\nu^s \otimes \sigma)$$

follows from Theorem 3.5 and the uniqueness of the reducibility point $s = s_0 \geq 0$ for the representation $\mathrm{Ind}_{M_j}^{O(V_{r+j})}(\rho\nu^s \otimes \tau)$ ([23]). In these exceptional cases we study in this section, we will determine the reducibility point and the structure of the lift of all the subquotients using again theta correspondence.

We recall that σ and τ are irreducible cuspidal representations of $\widetilde{Sp(n)}$ and $O(V_r)$, respectively, such that $\Theta(\sigma, r) = \tau$.

From ([15], p. 69 Théorème principal) we know that

$$\Theta(\sigma, r+1) \hookrightarrow \mathrm{Ind}_{P_1}^{O(V_{r+1})}(|\cdot|^{n-m_r} \otimes \tau),\ R_{P_1}(\Theta(\sigma, r+1)) = |\cdot|^{n-m_r} \otimes \tau.$$

We conclude that the representation $\mathrm{Ind}_{P_1}^{O(V_{r+1})}(|\cdot|^{n-m_r} \otimes \tau)$ is reducible. Also, note that $m_r \in \frac{1}{2} + \mathbb{Z}$, so $n - m_r \in \frac{1}{2} + \mathbb{Z}$. In the same way we have

$$\Theta(\tau, n+1)) \hookrightarrow \mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{m_r-n-1} \otimes \sigma),$$
$$R_{\widetilde{P_1}}(\Theta(\tau, n+1)) = \chi_{V,\psi}|\cdot|^{m_r-n-1} \otimes \sigma,$$

and the representation $\mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{m_r-n-1} \otimes \sigma)$ is reducible. So, Theorem 3.5 guarantees that the only point of reducibility of the representation $\mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{s} \otimes \sigma)$, $s \in \mathbb{R}$ is $s = \pm(m_r - n - 1)$ provided we show that the representations we obtain for $s = \pm(n - m_r)$ are irreducible.

Remark. In the situation where $m_r - n = \frac{1}{2}$ we have $n - m_r = m_r - n - 1$; we know then that the representation $\mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{-\frac{1}{2}} \otimes \sigma)$ is reducible and that $s = \pm\frac{1}{2}$ is the only point of reducibility. For $n = 0$, this covers the case of reducibility in $\widetilde{SL(2, F)}$, because, if we formally take sgn to be a nontrivial–character of $\mu_2 \cong \widetilde{Sp(0, F)}$, it lifts, in a split orthogonal tower to $r = 0$–level to a trivial representation of $\mu_2 = O(V_0)$, so $m_r = \frac{1}{2}$ and $n - m_r = -\frac{1}{2}$ is satisfied.

We now describe how to bypass the exceptional cases from Theorem 3.5.

Theorem 4.1. *The representation* $\mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^s \otimes \sigma)$ *reduces for a unique* $s \geq 0$ *(which is* $|m_r - n - 1|$*). In particular, this implies*

$\mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{m_r-n} \otimes \sigma)$ *is irreducible unless* $m_r - n = -(m_r - n - 1)$*, i.e.,* $m_r - n = \frac{1}{2}$*.*

Proof. As observed in the discussion prior to the remark above, we only have to check that $\mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{m_r-n} \otimes \sigma)$ is irreducible (unless $m_r - n = \frac{1}{2}$). Here we use the notion of pairs of orthogonal towers with the same quadratic character $\chi_V = \chi_{V'}$, so that $\chi_{V,\psi} = \chi_{V',\psi}$ ([9], Chapter V). So, if our original tower has one–dimensional anisotropic space V_0, then, the "dual" tower has a three–dimensional anisotropic space V_0' at its bottom and vice versa. Let r' denote the level to which the representation σ lifts in this second orthogonal tower (the first occurence), and let $\Theta(\sigma, r') = \tau'$ (a cuspidal representation). Since Dichotomy Conjecture holds for cuspidal representations ([11]), we have $r + r' = 2n$. But, if we calculate $m_{r'} - n$, we get that $m_r - n \notin \{m_{r'} - n, -(m_{r'} - n)\}$, so we are not in the problematic situation in the second tower, meaning that if $m_r - n \notin \{m_{r'} - n - 1, -(m_{r'} - n - 1)\}$ the representation $\mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{m_r-n} \otimes \sigma)$ is irreducible (since then we can apply Theorem 3.5 on the representations of the second tower and $\mathrm{Ind}_{P_1}^{O(V_{r'+1})}(|\cdot|^{m_r-n} \otimes \tau')$ is irreducible). If $m_r - n = m_{r'} - n - 1$ (the possibility $m_r - n = -(m_{r'} - n - 1)$ leads to contradiction with $r + r' = 2n$) we get $m_r - n = \frac{1}{2}$ and this is already covered. □

We now describe the lifts of the irreducible subquotients of the representation $\mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^s \otimes \sigma)$, $s \in \{\pm(m_r - n), \pm(m_r - n - 1)\}$.

Proposition 4.2. *Assume that* $m_r - n \neq \frac{1}{2}$*. Then,*

$\Theta(\mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{m_r-n} \otimes \sigma), r+1)$ *has a unique irreducible quotient, isomorphic to* $\Theta(\sigma, r+1)$. *Moreover,*

$$\Theta(\Theta(\sigma, r+1), n+1) = \mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{m_r-n} \otimes \sigma).$$

If we denote by π_1 *the other irreducible subquotient of* $\mathrm{Ind}_{P_1}^{O(V_{r+1})}(|\cdot|^{n-m_r} \otimes \tau)$, *then* $\Theta(\pi_1, n+1) = 0$.

Proof. We denote $\pi = \mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{n-m_r} \otimes \sigma)$. We can apply the third part of Proposition 3.4 to see that

$$\Theta(\chi_{V,\psi}|\cdot|^{n-m_r} \otimes \sigma, R_{\widetilde{P_1}}(\omega_{n+1,r+1})) = \mathrm{Ind}_{P_1}^{O(V_{r+1})}(|\cdot|^{m_r-n} \otimes \tau). \tag{5}$$

Now, we can apply the Frobenius reciprocity

$$\mathrm{Hom}_{\widetilde{Sp(n+1)}}(\omega_{n+1,r+1}, \pi) \cong \mathrm{Hom}_{\widetilde{M_1}}(R_{\widetilde{P_1}}(\omega_{n+1,r+1}), \chi_{V,\psi}|\cdot|^{n-m_r} \otimes \sigma).$$

Observing that the Frobenius isomorphism above is also an isomorphism of $O(V_{r+1})$–modules, and then taking the smooth part of it, it gives us the isomorphism between the contragredients of the corresponding isotypic components:

$$\Theta(\pi, r+1)\check{} \cong \mathrm{Ind}_{P_1}^{O(V_{r+1})}(|\cdot|^{m_r-n} \otimes \tau)\check{},$$

and the first part of the claim follows, since $\mathrm{Ind}_{P_1}^{O(V_{r+1})}(|\cdot|^{m_r-n} \otimes \tau)$ has a unique quotient, namely $\Theta(\sigma, r+1)$.

To prove the second claim, we proceed as follows: Let ξ be some irreducible representation of $O(V_{r+1})$. Then, the Frobenius reciprocity gives

$$\mathrm{Hom}_{\widetilde{Sp(n+1)} \times O(V_{r+1})}(\omega_{n+1,r+1}, \pi \otimes \xi) \cong$$
$$\mathrm{Hom}_{\widetilde{M_1} \times O(V_{r+1})}(R_{\widetilde{P_1}}(\omega_{n+1,r+1}), \chi_{V,\psi}|\cdot|^{n-m_r} \otimes \sigma \otimes \xi),$$

and, by the third part of Proposition 3.4, the last part is isomorphic to $\mathrm{Hom}_{O(V_{r+1})}(|\cdot|^{m_r-n} \otimes \tau), \xi)$, and this is non–zero only if $\xi \cong \Theta(\sigma, r+1)$. So, we conclude that π is a quotient of $\Theta(\Theta(\sigma, r+1), n+1)$. On the other hand, we have an epimorphism

$$\omega_{n+1,r+1} \to \Theta(\sigma, r+1) \otimes \Theta(\Theta(\sigma, r+1), n+1),$$

which leads to the epimorphism

$$R_{P_1}(\omega_{n+1,r+1}) \to |\cdot|^{n-m_r} \otimes \tau \otimes \Theta(\Theta(\sigma, r+1), n+1).$$

Now we again apply the third part of Proposition 3.4 (but the different part of the statement from the one used just above) to get that $\Theta(\Theta(\sigma, r+1), n+1)$ is a quotient of π, so, at the end, $\pi \cong \Theta(\Theta(\sigma, r+1), n+1)$.

To prove the last part of this proposition, we note that, if $\Theta(\pi_1, n+1) \neq 0$, an irreducible quotient of that full lift would have to have a cuspidal support consisting of σ and $|\cdot|^{\pm(n-m_r)}$ ([9], p. 55). Then, there would exist an epimorphism $\omega_{n+1,r+1} \to \pi \otimes \pi_1$, but this is impossible by the previous discussion. This guarantees $\Theta(\pi_1, n+1) = 0$. □

Proposition 4.3. *Assume that* $m_r - n \neq \frac{1}{2}$. *Then, the induced representation* $\mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{m_r-n-1} \otimes \sigma)$ *has two irreducible subquotients,* $\Theta(\tau, n+1)$ *and, say* π_2*, which lift as follows* $\Theta(\Theta(\tau, n+1), r+1) = \mathrm{Ind}_{P_1}^{O(V_{r+1})}(|\cdot|^{m_r-n-1} \otimes \tau)$*, while* $\Theta(\pi_2, r+1) = 0$. *Moreover, the lift* $\Theta(\mathrm{Ind}_{P_1}^{O(V_{r+1})}(|\cdot|^{m_r-n-1} \otimes \tau), n+1)$ *has the unique irreducible quotient isomorphic to* $\Theta(\tau, n+1)$.

Proof. The situation is totally symmetric to Proposition 4.2. □

It remains to discuss the most difficult case $m_r - n = \frac{1}{2}$.

Theorem 4.4. *Assume that* $m_r - n = \frac{1}{2}$.

(i) The representations $\mathrm{Ind}_{P_1}^{O(V_{r+1})}(|\cdot|^{\frac{1}{2}} \otimes \tau)$ *and* $\mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{\frac{1}{2}} \otimes \sigma)$ *reduce, and*

$$\Theta(\tau, n+1) \hookrightarrow \mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{-\frac{1}{2}} \otimes \sigma), \ \Theta(\sigma, r+1) \hookrightarrow \mathrm{Ind}_{P_1}^{O(V_{r+1})}(|\cdot|^{-\frac{1}{2}} \otimes \tau).$$

Moreover, we have the following:

$$\Theta(\Theta(\tau, n+1), r+1) = \mathrm{Ind}_{P_1}^{O(V_{r+1})}(|\cdot|^{\frac{1}{2}} \otimes \tau)$$

and

$$\Theta(\Theta(\sigma, r+1), n+1) = \mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{\frac{1}{2}} \otimes \sigma).$$

(ii) Let π_1 *(*π_2*, respectively) be the other irreducible subquotient of the representation* $\mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{\frac{1}{2}} \otimes \sigma)$ *(*$\mathrm{Ind}_{P_1}^{O(V_{r+1}}(|\cdot|^{\frac{1}{2}} \otimes \tau)$*, respectively). Then, one of the following holds:*

– $\Theta(\pi_1, r+1) = 0$ *and* $\Theta(\pi_1, r+2) \neq 0$, *moreover, every irreducible quotient of* $\Theta(\pi_1, r+2)$ *is a tempered subrepresentation of* $\delta([\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}]) \rtimes \tau$,

– $\Theta(\pi_1, r+1) \neq 0$, *then every irreducible quotient of* $\Theta(\pi_1, r+1)$ *is* π_2. *Every irreducible quotient of* $\Theta(\pi_1, r+2)$ *is the unique common tempered subquotient of* $\delta([\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}]) \rtimes \tau$ *and* $\nu^{\frac{1}{2}} \rtimes L(\nu^{\frac{1}{2}}; \tau)$.

Here $\delta([\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}])$ *denotes the unique irreducible (and necessarily square–integrable) subrepresentation of* $\nu^{\frac{1}{2}} \times \nu^{-\frac{1}{2}}$ *(we use the standard notation for the parabolic induction for the general linear groups, [26]). The representation* $L(\nu^{\frac{1}{2}}; \tau)$ *denotes the Langlands quotient of the representation* $\nu^{\frac{1}{2}} \rtimes \tau$.

Proof. We now prove (i). The reducibility of $\nu^{-\frac{1}{2}} \rtimes \tau$ and of $\chi_{V,\psi}\nu^{-\frac{1}{2}} \rtimes \sigma$ and the fact that $\Theta(\sigma, r+1)$ and $\Theta(\tau, n+1)$ are the subrepresentations of these representations, follow from [15], p. 69 Théorème principal. We can apply the third part of Proposition 3.4 to obtain

$$\Theta(|\cdot|^{-\frac{1}{2}} \otimes \tau, R_{P_1}(\omega_{n+1,r+1})) \cong \mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{\frac{1}{2}} \otimes \sigma) \tag{6}$$

and

$$\Theta(\chi_{V,\psi}|\cdot|^{-\frac{1}{2}} \otimes \sigma, R_{\widetilde{P_1}}(\omega_{n+1,r+1})) \cong \mathrm{Ind}_{P_1}^{O(V_{r+1})}(|\cdot|^{\frac{1}{2}} \otimes \tau). \tag{7}$$

Using this and Frobenius reciprocity, we get

$$\begin{aligned} &\mathrm{Hom}_{\widetilde{Sp(n+1)} \times O(V_{r+1})}(\omega_{n+1,r+1}, \mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{-\frac{1}{2}} \otimes \sigma) \otimes \Theta(\sigma, r+1)) \cong \\ &\cong \mathrm{Hom}_{\widetilde{M_1} \times O(V_{r+1})}(R_{\widetilde{P_1}}(\omega_{n+1,r+1}), \chi_{V,\psi}|\cdot|^{-\frac{1}{2}} \otimes \sigma \otimes \Theta(\sigma, r+1)) \cong \\ &\cong \mathrm{Hom}_{O(V_{r+1})}(\mathrm{Ind}_{P_1}^{O(V_{r+1})}(|\cdot|^{\frac{1}{2}} \otimes \tau), \Theta(\sigma, r+1)) \neq 0. \end{aligned}$$

This means that $\Theta(\Theta(\sigma, r+1), n+1) \neq 0$, so we have an epimorphism

$$\omega_{n+1,r+1} \to \Theta(\Theta(\sigma, r+1), n+1) \otimes \Theta(\sigma, r+1),$$

and, taking Jacquet modules in the orthogonal side, we have the following epimorphism

$$R_{P_1}(\omega_{n+1,r+1}) \to \Theta(\Theta(\sigma, r+1), n+1) \otimes |\cdot|^{-\frac{1}{2}} \otimes \tau,$$

so, by relation (6), we conclude that $\Theta(\Theta(\sigma, r+1), n+1)$ is a quotient of $\mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{\frac{1}{2}} \otimes \sigma)$.

On the other hand, using Kudla's filtration (Proposition 3.3), and an explicit description of quotient J_{j0} we obtain the following chain of epimorphisms:

$$R_{\widetilde{P_1}}(\omega_{n+1,r+1}) \to \chi_{V,\psi}|\cdot|^{\frac{1}{2}} \otimes \omega_{n,r+1} \to \chi_{V,\psi}|\cdot|^{\frac{1}{2}} \otimes \sigma \otimes \Theta(\sigma, r+1),$$

so that

$$\mathrm{Hom}_{\widetilde{M_1} \otimes O(V_{r+1})}(R_{\widetilde{P_1}}(\omega_{n+1,r+1}), \chi_{V,\Psi}|\cdot|^{\frac{1}{2}} \otimes \sigma \otimes \Theta(\sigma, r+1)) \cong$$

$$\cong \mathrm{Hom}_{\widetilde{Sp(n+1)} \times O(V_{r+1})}(\omega_{n+1,r+1}, \mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{\frac{1}{2}} \otimes \sigma) \otimes \Theta(\sigma, r+1)) \neq 0.$$

Take a non–zero intertwining operator from the last space, say T. Then, the image of this operator is isomorphic to $\Pi \otimes \Theta(\sigma, r+1)$, where Π is a subrepresentation of $\mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{\frac{1}{2}} \otimes \sigma)$ ([15], p. 45 Lemme III.3). On the other hand, this Π has to be a quotient of $\Theta(\Theta(\sigma, r+1), n+1)$. From our previous reasoning about $\Theta(\Theta(\sigma, r+1), n+1)$ the only possibilities are that $\Theta(\Theta(\sigma, r+1), n+1)$ is $\Theta(\tau, n+1)$ or $\mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{\frac{1}{2}} \otimes \sigma)$. In the first case, Π is a quotient of $\Theta(\tau, n+1)$, so $\Pi = \Theta(\tau, n+1)$, but this cannot be a subrepresentation of $\mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{\frac{1}{2}} \otimes \sigma)$. We, then, must have

$$\Theta(\Theta(\sigma, r+1), n+1) \cong \mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{\frac{1}{2}} \otimes \sigma)$$

and $\Pi = \mathrm{Ind}_{\widetilde{P_1}}^{\widetilde{Sp(n+1)}}(\chi_{V,\psi}|\cdot|^{\frac{1}{2}} \otimes \sigma)$.

Analogously, one gets

$$\Theta(\Theta(\tau, n+1), r+1) \cong \mathrm{Ind}_{P_1}^{O(V_{r+1})}(|\cdot|^{\frac{1}{2}} \otimes \tau).$$

We now prove (ii). Firstly, we prove that $\Theta(\pi_1, r) = 0$. If $\Theta(\pi_1, r) \neq 0$, then, examining a cuspidal support of every quotient of this representation ([9], p. 55), we see that it would have to be equal to τ. But, then $\Theta(\tau, n) = \sigma$ and $\Theta(\tau, n+1) = \pi_1$; the last relation contradicts the results of the first part of this theorem. Analogously, we get that $\Theta(\pi_2, n) = 0$.

To proceed further, we prove claim (8)(see below). We use the idea of descending in the orthogonal tower, starting from some stable range appearance place, downwards to prove that the lift does not vanish even lower in the tower. This idea was already present in ([16]).

Assume that $\Theta(\pi_1, l) \neq 0$, for some l large enough. Then, there exists an epimorphism $\omega_{n+1,l} \to \pi_1 \otimes \Theta(\pi_1, l)$. Using Kudla's filtration (Proposition 3.3), we get epimorphisms

$$R_{P_1}(\omega_{n+1,l+1}) \to \nu^{-(m_l-n-1)} \otimes \omega_{n+1,l} \to \nu^{-(m_l-n-1)} \otimes \pi_1 \otimes \Theta(\pi_1, l).$$

Using Frobenius reciprocity, from the relation above we get that there exists a non–zero $Sp(n+1) \times O(V_{l+1})$–intertwining from $\omega_{n+1,l+1}$ to $\pi_1 \otimes \mathrm{Ind}_{P_1}^{O(V_{L+1})}(\nu^{-(m_l-n-1)} \otimes \Theta(\pi_1, l)))$. This immediately gives us a non–trivial intertwining between $\Theta(\pi_1, l+1)$ and $\mathrm{Ind}_{P_1}^{O(V_{L+1})}(\nu^{-(m_l-n-1)} \otimes \Theta(\pi_1, l))$, and, consequently, between $R_{P_1}(\Theta(\pi_1, l+1))$ and $\nu^{-(m_l-n-1)} \otimes \Theta(\pi_1, l)$. If we denote by $\pi(\chi)$ an isotypic part of part of some smooth representation π corresponding to a generalized central character χ, we can write down our conclusion as $R_{P_1}(\Theta(\pi_1, l+1))(\nu^{-(m_l-n-1)}) \neq 0$. This proves one direction of the following claim:

Assume that $m_l - n - 1 \neq -\frac{1}{2}$. Then,

$$\Theta(\pi_1, l) \neq 0 \Leftrightarrow R_{P_1}(\Theta(\pi_1, l+1))(\nu^{-(m_l-n-1)}) \neq 0. \tag{8}$$

Now, we prove the other direction, so we assume that $R_{P_1}(\Theta(\pi_1, l+1))(\nu^{-(m_l-n-1)}) \neq 0$. This also means that $\Theta(\pi_1, l+1) \neq 0$, so there is an epimorphism $\omega_{n+1,r+1} \to \pi_1 \otimes \Theta(\pi_1, l+1)$, and, when we apply Jacquet module, $R_{P_1}(\omega_{n+1,r+1}) \to \pi_1 \otimes R_{P_1}(\Theta(\pi_1, l+1)) \to \pi_1 \otimes \nu^{-(m_l-n-1)} \otimes \tau_1$ for some representation τ_1 of $O(V_r)$ (a non–zero map). We use here that the Jacquet module of $\Theta(\pi_1, l+1)$ is of finite length. Now, if we assume that this map, restricted to a subrepresentation I_{11} of $R_{P_1}(\omega_{n+1,r+1})$ (Proposition 3.3) is zero, we get an existence of a non–zero mapping from $I_{10} \cong \nu^{-(m_l-n-1)} \otimes \omega_{n+1,l}$ to $\pi_1 \otimes \nu^{-(m_l-n-1)} \otimes \tau_1$. This means that $\Theta(\pi_1, l) \neq 0$. If we assume the opposite, i.e., the restriction of the above mapping to I_{11} is non zero, applying the second Frobenius map, we get a non–zero intertwining map

$$\chi_{V,\psi}\Sigma_1' \otimes \omega_{n,l} \to \nu^{-(m_l-n-1)} \otimes \tau_1 \otimes R_{P_1}(\check{\pi_1})\check{},$$

and, from this follows that $\pi_1 \hookrightarrow \chi_{V,\psi}\nu^{-(m_l-n-1)} \rtimes \sigma$. Of course, if $m_l - n - 1 \neq -\frac{1}{2}$, we get that this is impossible, and the claim (8) is proved.

Using claim (8), we prove that $\Theta(\pi_1, r+2) \neq 0$ in Lemma 4.6. Assuming that, we now examine two possibilities: $\Theta(\pi_1, r+1) \neq 0$ and $\Theta(\pi_1, r+1) = 0$. Firstly, assume that $\Theta(\pi_1, r+1) = 0$. Let Π be an irreducible quotient of

$\Theta(\pi_1, r+2)$, so that there exists an epimorphism $\omega_{n+1,r+2} \to \pi_1 \otimes \Pi$, and, consequently, an epimorphism $R_{\widetilde{P_1}}(\omega_{n+1,r+2}) \to R_{\widetilde{P_1}}(\pi_1) \otimes \Pi = \chi_{V,\Psi}\nu^{\frac{1}{2}} \otimes \sigma \otimes \Pi$. If the last epimorphism is equal to zero on the subrepresentation J_{11} of $R_{\widetilde{P_1}}(\omega_{n+1,r+2})$, it gives rise to an epimorphism $\chi_{V,\psi}\nu^{\frac{3}{2}} \otimes \omega_{n,r+2} \to \chi_{V,\Psi}\nu^{\frac{1}{2}} \otimes \sigma \otimes \Pi$. This is impossible, so there exists a non–zero intertwining

$$\mathrm{Ind}_{\widetilde{GL_1} \times \widetilde{Sp(n)} \times P_1}^{\widetilde{M_1} \times O(V_{r+2})} (\chi_{V,\psi}\Sigma_1' \otimes \omega_{n,r+1}) \to \chi_{V,\psi}\nu^{\frac{1}{2}} \otimes \sigma \otimes \Pi.$$

Using the second Frobenius reciprocity, as before, we get an embedding $\Pi \hookrightarrow \nu^{\frac{1}{2}} \rtimes \Theta(\sigma, r+1)$. An intertwining operator, induced from the GL–situation acts on the second representation, so that we have a composition of intertwining operators

$$\Pi \hookrightarrow \nu^{\frac{1}{2}} \rtimes \Theta(\sigma, r+1) \hookrightarrow \nu^{\frac{1}{2}} \times \nu^{-\frac{1}{2}} \rtimes \tau \to \nu^{-\frac{1}{2}} \times \nu^{\frac{1}{2}} \rtimes \tau.$$

If we assume that Π is not embedded in the kernel of the last intertwining operator (i.e., in $\delta([\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}]) \rtimes \tau$), there would exist an embedding

$$\Pi \hookrightarrow \nu^{-\frac{1}{2}} \times \nu^{\frac{1}{2}} \rtimes \tau,$$

and this would force $R_{P_1}(\Theta(\pi_1, r+2))(\nu^{-\frac{1}{2}}) \neq 0$. By plugging $l = r+1$ in the relation (8), we get that $\Theta(\pi_1, r+1) \neq 0$, contrary to our assumption. This means that $\Pi \hookrightarrow \delta([\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}]) \rtimes \tau$, and this case is covered.

Assume now that $\Theta(\pi_1, r+1) \neq 0$ and let Π be an irreducible quotient of $\Theta(\pi_1, r+1)$. Then, there exists an epimorphism of Jacquet modules $R_{\widetilde{P_1}}(\omega_{n+1,r+1}) \to \chi_{V,\psi}\nu^{\frac{1}{2}} \otimes \sigma \otimes \Pi$. Again, by examining the filtration of $R_{\widetilde{P_1}}(\omega_{n+1,r+1})$, we firstly assume that the epimorphism above is zero, when restricted to a subrepresentation J_{11} of $R_{\widetilde{P_1}}(\omega_{n+1,r+1})$. Then, there exists an epimorphism $J_{10} \cong \chi_{V,\psi}\nu^{\frac{1}{2}} \otimes \omega_{n,r+1} \to \chi_{V,\psi}\nu^{\frac{1}{2}} \otimes \sigma \otimes \Pi$. We get $\Pi \cong \Theta(\sigma, r+1)$. Now, by Kudla's filtration there exists an epimorphism $R_{P_1}(\omega_{n+1,r+2}) \to \nu^{-\frac{1}{2}} \otimes \omega_{n+1,r+1}$, and, consequently, a non–zero map $R_{P_1}(\omega_{n+1,r+2}) \to \nu^{-\frac{1}{2}} \otimes \pi_1 \otimes \Theta(\sigma, r+1) \hookrightarrow \nu^{-\frac{1}{2}} \otimes \pi_1 \otimes \nu^{-\frac{1}{2}} \rtimes \tau$. By Frobenius reciprocity, we get a non–zero ($\widetilde{Sp(n+1)} \times O(V_{r+2})$–invariant) intertwining map $\omega_{n+1,r+2} \to \pi_1 \otimes \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \rtimes \tau$. So, there exists an non–zero intertwining map $R_{P_2}(\omega_{n+1,r+2}) \to \pi_1 \otimes \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \otimes \tau$. We now use filtration of $R_{P_2}(\omega_{n+1,r+2})$ ([9]); note that it has $t = \min\{2, n+1\}$ members. Here we assume that $n \geq 1$; if not, we are in a simpler situation. We use, as always, I_{jk} to denote the members of filtration of $R_{P_2}(\omega_{n+1,r+2})$. We see that there cannot exist a non–zero intertwining map from $I_{20} \cong 1_{GL(2)} \otimes \omega_{n+1,r}$ to $\pi_1 \otimes \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \otimes \tau$. Also, there cannot exist a non–zero intertwining map from I_{22} to $\pi_1 \otimes \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \otimes \tau$ (we examine a cuspidal support of π_1). So, there must exist a non–zero intertwining

$$I_{21} \cong \mathrm{Ind}_{\widetilde{Gl(1)} \times \widetilde{Sp(n)} \times GL(1) \times GL(1)}^{\widetilde{Sp(n+1)} \times GL(2) \times O(V_r)} (\beta_{21} \Sigma_1' \otimes \omega_{n,r}) \to \pi_1 \otimes \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \otimes \tau.$$

When we examine the last relation more carefully, applying the second Frobenius reciprocity, we get a non–zero $\widetilde{Sp(n+1)}$–intertwining between $\chi_{V,\psi}\nu^{\frac{1}{2}} \rtimes \sigma$ and π_1, which is impossible.

We recall our assumption on $R_{\widetilde{P_1}}(\omega_{n+1,r+1})$; the discussion above shows that there exists a non–zero intertwining from a subrepresentation J_{11} to $\chi_{V,\psi}\nu^{\frac{1}{2}} \otimes \sigma \otimes \Pi$. After applying the second Frobenius reciprocity, we get $\Pi = \pi_2$.

To determine an irreducible quotient Π' of $\Theta(\pi_1, r+2)$ we proceed as follows: since there exists an epimorphism T from $R_{P_1}(\omega_{n+1,r+2})$ to $\pi_1 \otimes R_{P_1}(\Pi')$, we study the filtration of $R_{P_1}(\omega_{n+1,r+2})$; if $T|_{I_{11}} = 0$, employing 2^{nd} Frobenius reciprocity, as before, we get that $\nu^{-\frac{1}{2}} \otimes \pi_2 \leq R_{P_1}(\Pi')$; if $T|_{I_{11}} \neq 0$, we get that $\nu^{\frac{1}{2}} \otimes L(\nu^{\frac{1}{2}}; \tau) \leq R_{P_1}(\Pi')$.

Now we calculate $R_{P_2}(\Pi')$. The filtration of $R_{P_2}(\omega_{n+1,r+2})$ has three members; easy analysis shows that there only I_{21} can have a non–zero intertwining with $\pi_1 \otimes R_{P_2}(\Pi')$. If $\xi \otimes \tau$ is an irreducible subquotient of $R_{P_2}(\Pi')$ such that the intertwining space (we relax the notation since it is obvious which are the inducing subgroups in question)

$$\mathrm{Hom}(\mathrm{Ind}(\beta_{21}\Sigma_1' \otimes \omega_{n,r}), \pi_1 \otimes \xi \otimes \tau)$$

is non–zero, we get $\xi = \delta([\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}])$.

Now, the proof is complete as soon as we show Lemma 4.5. Namely, assuming this lemma, we see that, since $\delta([\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}]) \otimes \tau \leq R_{P_2}(\Pi')$, we must have $\Pi' = T_1$ or $\Pi' = T_2$. This also means that the possibility $\nu^{-\frac{1}{2}} \otimes \pi_2 \leq R_{P_1}(\Pi')$ does not occur, since, in any case, Π' is tempered. Now, we refer to the first part of the proof of Lemma 4.6: This first part is valid if $\Theta(\pi_1, l+1) \neq 0$, and $l \geq r+1$, and we if we put $l = r+1$ we immediately get

$$\Pi' \hookrightarrow \nu^{\frac{1}{2}} \rtimes \Theta(\sigma, r+1) = \nu^{\frac{1}{2}} \rtimes L(\nu^{\frac{1}{2}}; \tau),$$

so we see that $\Pi' = T_2$. □

To finish the proof of Theorem 4.4, we need some facts about Jacquet modules of the representation $\nu^{\frac{1}{2}} \times \nu^{\frac{1}{2}} \rtimes \tau$.

Lemma 4.5. *The representation $\nu^{\frac{1}{2}} \times \nu^{\frac{1}{2}} \rtimes \tau$ is of length four; the irreducible subquotients are T_1, T_2, $L(\nu^{\frac{1}{2}}, \nu^{\frac{1}{2}}; \tau)$, $L(\nu^{\frac{1}{2}}; \pi_2)$, where T_1 and T_2 are irreducible tempered subrepresentations of $\delta([\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}]) \rtimes \tau$. The representations $\delta([\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}]) \rtimes \tau$ and $\nu^{\frac{1}{2}} \rtimes L(\nu^{\frac{1}{2}}; \tau)$ have a unique common irreducible subquotient, we denote it by T_2. The multiplicity of $\delta([\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}]) \otimes \tau$ in $R_{P_2}(\nu^{\frac{1}{2}} \times \nu^{\frac{1}{2}} \rtimes \tau)$ is two and each $\delta([\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}]) \otimes \tau$ can only come from a Jacquet module of T_1 or T_2.*

Proof. In the appropriate Grothendieck group, we have

$$\nu^{\frac{1}{2}} \times \nu^{-\frac{1}{2}} \rtimes \tau = \delta([\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}]) \rtimes \tau + L(\nu^{\frac{1}{2}}, \nu^{-\frac{1}{2}}) \rtimes \tau =$$
$$\nu^{\frac{1}{2}} \rtimes \pi_2 + \nu^{\frac{1}{2}} \rtimes L(\nu^{\frac{1}{2}}; \tau).$$

The representation $\delta([\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}]) \rtimes \tau$ is of length two (we see that by taking a restriction to the appropriate special odd orthogonal group, and having in mind that, for an irreducible representation π of a full odd orthogonal group $O(V_r)$, $\pi_{|SO(V_r)}$ is irreducible. Also, $(\pi \rtimes \tau')_{|SO(V_{r+n})} \cong \pi \rtimes (\tau')_{|SO(V_r)}$, if τ' is a representation of $O(V_r)$ and π of $GL(n, F)$. Then, using Aubert duality ([1]) for $SO(V_{r+n})$ and the fact that, in our case, we have $O(V_{r+n}) = \{\pm 1\} \cdot SO(V_{r+n})$, we see that $L(\nu^{\frac{1}{2}}, \nu^{-\frac{1}{2}}) \rtimes \tau$ is of length two; analogously, we see that both $\nu^{\frac{1}{2}} \rtimes \pi_2$ and $\nu^{\frac{1}{2}} \rtimes L(\nu^{\frac{1}{2}}; \tau)$ are of length two; now the Langlands parameters of all the subquotients are easily determined. We also see that $\nu^{\frac{1}{2}} \rtimes \pi_2$ and $\nu^{\frac{1}{2}} \rtimes L(\nu^{\frac{1}{2}}; \tau)$ must each contain one tempered subquotient; we denote the former one by T_2. Using Tadić's formula for the Jacquet modules of the induced representations ([24]), we get

$$R_2(\nu^{\frac{1}{2}} \times \nu^{\frac{1}{2}} \rtimes \tau) = \nu^{\frac{1}{2}} \times \nu^{\frac{1}{2}} \otimes \tau + \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \otimes \tau + 2\delta([\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}]) \otimes \tau + 2L(\nu^{\frac{1}{2}}, \nu^{-\frac{1}{2}}) \otimes \tau.$$

Since $T_1, T_2 \hookrightarrow \delta([\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}]) \rtimes \tau$ and $T_2 \hookrightarrow \nu^{\frac{1}{2}} \rtimes L(\nu^{\frac{1}{2}}; \tau)$, the rest of the claims now follow.

□

To complete the proof of Theorem 4.4, we prove the following

Lemma 4.6. *Let π_1 be as before. Then, $\Theta(\pi_1, r+2) \neq 0$.*

Proof. Let $l > r$ be large enough, so $\Theta(\pi_1, l+1) \neq 0$ and the claim (8) holds (this also means $l \geq r+1$). Let Π' be an irreducible quotient of $\Theta(\pi_1, l+1)$. A non–zero $\widetilde{Sp(n+1)} \times O(V_{l+1})$–intertwining $\omega_{n+1,l+1} \to \chi_{V,\psi}\nu^{\frac{1}{2}} \rtimes \sigma \otimes \Pi'$ gives rise to a non–zero intertwining $R_{\widetilde{P_1}}(\omega_{n+1,l+1}) \to \chi_{V,\psi}\nu^{\frac{1}{2}} \otimes \sigma \otimes \Pi'$. Using again Kudla's filtration of $R_{\widetilde{P_1}}(\omega_{n+1,l+1})$, we see that either there exists a non–zero intertwining $\chi_{V,\psi}\nu^{m_l - n} \otimes \omega_{n,l+1} \to \chi_{V,\psi}\nu^{\frac{1}{2}} \otimes \sigma \otimes \Pi'$, which is impossible since $m_l - n > \frac{1}{2}$, either (this must be a case) there exists a non–zero intertwining $\mathrm{Ind}(\chi_{V,\psi}\Sigma'_1 \otimes \omega_{n,l}) \to \chi_{V,\psi}\nu^{\frac{1}{2}} \otimes \sigma \otimes \Pi'$, which, by 2nd Frobenius reciprocity, gives an intertwining $\chi_{V,\psi}\Sigma'_1 \otimes \omega_{n,l} \to \chi_{V,\psi}\nu^{\frac{1}{2}} \otimes \sigma \otimes R_{P_1}(\check{\Pi'})\check{}$. We conclude that $\nu^{-\frac{1}{2}} \rtimes \Theta(\sigma, l) \to \Pi'$, i. e.,

$$\Pi' \hookrightarrow \nu^{\frac{1}{2}} \rtimes \Theta(\sigma, l).$$

Since σ is cuspidal, we have a more precise information on $\Theta(\sigma, l)$, namely ([9]) $\Theta(\sigma, l) \hookrightarrow \nu^{n-m_l+1} \rtimes \Theta(\sigma, l-1)$. Note that $\Theta(\sigma, l-1) \neq 0$. So, if $n - m_l + 1 \notin \{\frac{3}{2}, -\frac{1}{2}\}$, by Zelevinsky results for general linear groups, we have

$$\Pi' \hookrightarrow \nu^{\frac{1}{2}} \rtimes \Theta(\sigma, l) \hookrightarrow \nu^{\frac{1}{2}} \times \nu^{n-m_l+1} \rtimes \Theta(\sigma, l-1) \cong \nu^{n-m_l+1} \times \nu^{\frac{1}{2}} \rtimes \Theta(\sigma, l-1).$$

This is satisfied if $l \geq r+2$. In that case $R_{P_1}(\Theta(\pi_1, l+1))(\nu^{n-m_l+1}) \geq R_{P_1}(\Pi')(\nu^{n-m_l+1}) \neq 0$, and, by claim (8), we have $\Theta(\pi_1, l) \neq 0$. Moreover, $\Theta(\pi_1, r+2) \neq 0$. □

Remark. At this stage, using our our approach, we were not able to understand more thoroughly when each of the possibilities in Theorem 4.4, (*ii*) occurs. But, if we assume that the Dichotomy Conjecture holds for π_1, we can easily see that, in that case, the second possibility should occur.

5 Examples

By Theorem 3.5 and Theorem 4.1, we have completely described reducibility of the representations of the metaplectic group in the generalized rank one case, in terms of reducibility of the related representations on the odd–orthogonal group, assuming that F has characteristic different from 2. Using known facts about reducibility for the orthogonal groups, we can make this more explicit; we describe some of the situations which occur in a few examples. In the following examples we use Shahidi's results on reducibility and L–functions, so this requires char $F = 0$.

We use Shahidi's machinery to calculate the reducibility point for the representations of $SO(V_r)$ induced from a maximal parabolic subgroup and the generic representations of an appropriate Levi factor. To do so, the odd orthogonal tower must be split, i.e., $\dim(V_0) = 1$. In general, if ρ is a self–contagredient irreducible cuspidal representation of $GL(j, F)$ and τ an irreducible, cuspidal, generic representation of $SO(V_r)$ in the split tower, the Plancherel measure can be expressed in terms of L–functions, and, we have, up to an ε–factor (for example, [21],[22])

$$\mu(s, \rho \otimes \tau) \approx \frac{L(1+s, \rho \times \tau)}{L(s, \rho \times \tau)} \frac{L(1-s, \rho \times \tau)}{L(-s, \rho \times \tau)} \frac{L(1+2s, \rho, Sym^2\rho_j)}{L(2s, \rho, Sym^2\rho_j)} \frac{L(1-2s, \rho, Sym^2\rho_j)}{L(-2s, \rho, Sym^2\rho_j)}. \quad (9)$$

Here $Sym^2\rho_j$ is a symmetric square representation of $GL(j, \mathbb{C})$, and τ is always self–contragredient ([15]). The zeros and poles of the Plancherel measure completely describe the reducibility point of the representation $\mathrm{Ind}_{P_j}^{SO(V_r)}(\rho\nu^s \otimes \sigma)$.

5.1 The Siegel case

We recall that we study genuine representations of $\widetilde{Sp(n)}$. Let ω_0 denote the non–trivial character of $\widetilde{Sp(0)} \cong \mu_2$. Assume ρ is a genuine, unitarizable cuspidal representation of $\widetilde{GL(j, F)}$, $j \geq 2$ with $(\chi_{V,\psi}^{-1}\rho)\check{} \cong (\chi_{V,\psi}^{-1}\rho)$. Then, $\mathrm{Ind}_{\widetilde{P_j}}^{\widetilde{Sp(j)}}(\rho \otimes \omega_0)$ is the Sigel case for our considerations. We have the following proposition:

Proposition 5.1. *Assume ρ is a genuine, unitarizable suprecuspidal representation of $\widetilde{GL(j, F)}$, $j \geq 2$ with $(\chi_{V,\psi}^{-1}\rho)\check{} \cong (\chi_{V,\psi}^{-1})$. The representation $\mathrm{Ind}_{\widetilde{P_j}}^{\widetilde{Sp(j)}}(\rho\nu^s \otimes \omega_0)$, $s \in \mathbb{R}_{\geq 0}$ reduces for $s = 0$ if $L(s, \chi_{V,\psi}^{-1}\rho, Sym^2\rho_j)$ does not have a pole for $s = 0$; otherwise, it reduces for $s = \frac{1}{2}$.*

Proof. By ([10], p. 238) we have $\omega_{m,0} \cong 1_{O(V_m)} \otimes \omega_0^{\dim(V_m)}$. Now, assume that we study theta correspondence between the representations of the metaplectic groups with the representations of groups in the split odd–orthogonal tower. In this case $\dim(V_0) = 1$, and $O(V_0) \cong \mu_2$. This means $\Theta(\omega_0, 0) = 1_{O(V_0)}$, and by Theorem 3.5, $\mathrm{Ind}_{\widetilde{P_j}}^{\widetilde{Sp(j)}}(\rho\nu^s \otimes \omega_0)$ reduces if and only if $\pi = \mathrm{Ind}_{P_j}^{O(V_j)}(\chi_{V,\psi}^{-1}\rho\nu^s \otimes 1_{O(V_0)})$ reduces. Now, we note that π reduces if and only if $\pi|_{SO(V_j)}$ reduces. Since we are in the generic case, we can apply (9) and the claim readily follows. □

Corollary 5.2. *We keep the notation of the previous proposition. Let ρ be an irreducible, genuine, cuspidal representation of $\widetilde{GL(j, F)}$ with $\chi_{V,\psi}^{-1}\rho$ self–dual.*

(i) If j is odd, the representation $\mathrm{Ind}_{\widetilde{P_j}}^{\widetilde{Sp(j)}}(\rho\nu^s \otimes \omega_0)$ reduces for $s = \frac{1}{2}$.

(ii) If $j = 2$ the representation $\mathrm{Ind}_{\widetilde{P_j}}^{\widetilde{Sp(j)}}(\rho\nu^s \otimes \omega_0)$ reduces for $s = \frac{1}{2}$ if the central character of $\chi_{V,\psi}^{-1}\rho$ is non–trivial.

Proof. The first claim follows from Proposition 5.1 and Theorem 6.2. of ([22]). The second claim follows from the fact that $L(s, \chi_{V,\psi}^{-1}\rho, \Lambda^2\rho_2) = L(s, \omega_{\chi_{V,\psi}^{-1}\rho})$, where $\omega_{\chi_{V,\psi}^{-1}\rho}$ is a central character of $\chi_{V,\psi}^{-1}\rho$. Since we precisely know the form of L–function $L(s, \chi)$, where χ is a character, the claim follows. □

5.2 Reducibility of $\mathrm{Ind}_{\widetilde{P_j}}^{\widetilde{Sp(j+1)}}(\rho \otimes \pi)$, where π is irreducible cuspidal representation of $\widetilde{SL(2, F)}$

In this situation we use the knowledge about the liftings of cuspidal representations of $\widetilde{SL(2, F)}$ to various odd–orthogonal groups ([25]). We assume that the characteristic of F is zero.

To simplify the calculation, we assume that if $j = 1$, then

$$\rho \notin \{\chi_{V,\psi}\nu^{\pm(m_r-1)}, \chi_{V,\psi}\nu^{\pm(m_r-2)}\},$$

where $m_r(\pi) = \frac{1}{2}\dim(V_r)$, and r denotes the first occurence of non–zero lift of π in a certain odd orthogonal tower. We will fix a quadratic character χ_V, as in the introductory section, so that we have attached to it two odd–orthogonal towers, one with $\dim(V_0) = 1$ ((+)–tower) and other with $\dim(V_0) = 3$ ((−)–tower) ([9], Chapter V). Since for the cuspidal representations the conservation principle holds, ([12]), for a fixed cuspidal representation π of $\widetilde{SL(2,F)}$ having in mind our notation, we have $2m_r(\pi)^+ + 2m_r(\pi)^- = 8$.

The cuspidal representations of $\widetilde{SL(2,F)}$ lift to $2m_r(\pi)^+ \in \{1,3,5\}$ (the stable range!) (precise description of the lifts is rather subtle ([25], [13])). Now assume that for a cuspidal π we know where it lifts.

The first case:

If $2m_r(\pi)^+ = 1$ (i. e. $r = 0$) (for example, if π is an odd part of the Weil representation attached to an appropriate additive character of F([9], p. 89, 90). Then, $\Theta(\pi, 0)^+ = sgn_{O(V_0)}$ (as for an odd Weil representation). The representation $\mathrm{Ind}_{\widetilde{P_j}}^{\widetilde{Sp(j+1)}}(\rho \otimes \pi)$ reduces if $\mathrm{Ind}_{P_j}^{O(V_j)}(\chi_{V,\psi}^{-1}\rho \otimes sgn_{O(V_0)})$ reduces, and this reduces if and only if $\mathrm{Ind}_{P_j}^{SO(V_j)}(\chi_{V,\psi}^{-1}\rho \otimes 1)$ reduces, and we are again in the Siegel case.

The second case:

If $2m_r(\pi)^+ = 3$, then $\Theta(\pi, 1)$ is a cuspidal representation of $O(V_1)$. The representation $\mathrm{Ind}_{P_j}^{O(V_{j+1})}(\chi_{V,\psi}^{-1}\rho \otimes \Theta(\pi, 1))$ reduces only if it's restriction to $SO(V_{j+1})$ reduces. Now, we use $\Theta(\pi, 1)$ to denote also a restriction of this representation to $SO(V_1)$. So, if we plug $\tau = \Theta(\pi, 1)$ in (9) we can draw some conclusions, since $SO(V_1) \cong PGL_2(F)$, so that $\Theta(\pi, 1)$ is necessarily generic. The L–function $L(s, \chi_{V,\psi}^{-1}\rho \times \Theta(\pi, 1))$ is, essentially, an L–function of pairs, and it has a pole for $s = 0$ only if $\Theta(\pi, 1) \cong \chi_{V,\psi}^{-1}\rho$. Then $j = 2$ and a central character of $\chi_{V,\psi}^{-1}\rho$ is necessarily trivial. If this isomorphism occurs, the reducibility point is $s = 1$.

If, on the other hand, $L(s, \chi_{V,\psi}^{-1}\rho \times \Theta(\pi, 1))$ does not have a pole for $s = 0$ ($\Theta(\pi, 1) \not\cong \chi_{V,\psi}^{-1}\rho$; this trivially holds if $j \neq 2$), then $\mathrm{Ind}_{\widetilde{P_j}}^{\widetilde{Sp(j+1)}}(\rho \otimes \pi)$ reduces if $L(s, \chi_{V,\psi}^{-1}\rho, Sym^2\rho_j)$ does not have a pole for $s = 0$, otherwise this representation reduces for $s = \frac{1}{2}$. Note that, in these cases, there is no dependency on π.

The third case:

If $2m_r(\pi)^+ = 5$, then our knowledge on L–functions appearing in (9) is limited; also, we would like to avoid the discussion on (non)–genericity of $\Theta(\pi, 2)^+$. To accomplish that, we will try to use the fact that $2m_r^-(\pi) = 3$. The vector space V_0^- is a vector subspace of trace–zero quaternions in a non–

split quaternion algebra D over F. The group $SO(V_0)^-$ is anisotropic, and isomorphic to PD^* and an inner form of $SO(V_0)$ is the split $SO(3)$ (in the usual notation), isomorphic to $PGL(2)$. Now we want to use Jacquet–Langlands correspondence to relate representations of $SO(V_0)^-$ and split $SO(3)$, and to show that we can use this correspondence to calculate the Plancherel measure $\mu(s, \chi_{V,\psi}^{-1}\rho \otimes \Theta(\pi, 0)^-)$, so that we can calculate the reducibility point of $\mathrm{Ind}_{P_j}^{SO(V_j)^-}(\chi_{V,\psi}^{-1}\rho \otimes \Theta(\pi, 0)^-)$.

To relate $\mu(s, \chi_{V,\psi}^{-1}\rho \otimes \Theta(\pi, 0)^-)$ with $\mu(s, \chi_{V,\psi}^{-1}\rho \otimes JL(\Theta(\pi, 0)^-))$, where $JL((\Theta(\pi, 0)^-)$ denotes a Jacquet–Langlands lift of $\Theta(\pi, 0)^-$, we use (in this, non–Siegel case), an idea of ([19]), which they use in the Siegel case.

We briefly describe this idea. Let k be a number field, such that there exist two places of k, say v_1 and v_2 such that $k_{v_i} \cong F,\ i = 1, 2$. Let $\mathbf{D}$ be a quaternionic algebra over k, such that it splits for every place $k_v \neq k_{v_i},\ i = 1, 2$, and $\mathbf{D}(k_{v_i}) \cong D$. Let $\mathbf{G}$ be an orthogonal group over k, such that $\mathbf{G}(k_v) \cong SO(2j + 3)(k_v),\ v \notin \{v_1, v_2\}$, is a split group, and $\mathbf{G}(k_{v_i}) \cong SO(V_j)(F)^-,\ i = 1, 2$.

Let $\mathbf{G}'$ be an orthogonal group over k, which is an inner form of $\mathbf{G}$, but split at every place of k. Let $\mathbf{M}$ and $\mathbf{M}'$ be the appropriate Levi subgroups, so that $\mathbf{M}(k_{v_i}) \cong GL(j, k_{v_i}) \times SO(V_0)$ and $\mathbf{M}'(k_{v_i}) \cong GL(j, k_{v_i}) \times SO(3)$.

Let $\tau \cong \otimes\tau_v$ be an automorphic cuspidal representation of $\mathbf{D}^*$ with the trivial central character, such that $\tau_{v_i} \cong \Theta(\pi, 0)$ (thought of as a representation of $PD^* \cong SO(V_0)^-$). Then, there exists an automorphic cuspidal representation $\tau' \cong \otimes\tau'_v$ of $GL(2)$ such that $\tau'_v \cong \tau_v,\ v \notin \{v_1, v_2\}$, and $\tau'_{v_i} \cong JL(\Theta(\pi, 0)^-),\ i = 1, 2$. The existence of such representations can be checked by [19]. Let $\sigma \cong \otimes\sigma_v$ be an automorphic cuspidal representation of $GL(j)$ such that $\sigma_{v_i} \cong \chi_{V,\psi}^{-1}\rho,\ i = 1, 2$.

Now, using the global functional equation for the global intertwining operators, and choosing the appropriate normalizations of the Haar measures on the unipotent radicals, one can show that, on each split place, there is a cancellation of local factors coming from the local intertwining operators; the only thing which remains is

$$\mu(s, \chi_{V,\psi}^{-1}\rho \otimes \Theta(\pi, 0)^-)^2 = \mu(s, \chi_{V,\psi}^{-1}\rho \otimes JL(\Theta(\pi, 0)^-))^2.$$

The positivity on the imaginary axis of the Plancherel measure guarantees that we actually have an equality of the Plancherel measures above, not only their squares.

Now, we can calculate the poles and zeroes of the Plancherel measure on the right side above in terms of L–functions, since $JL(\Theta(\pi,0)^-)$ is a generic, square–integrable representation of $GL(2,F)$ (with the trivial central character). We have two situations to consider: if $\Theta(\pi,0)^-$ is not one–dimensional, $JL(\Theta(\pi,0)^-)$ is a cuspidal generic representation of the split $SO(3)$, and we are in the previous case. If $\Theta(\pi,0)^-$ is a character of D^* trivial on F^*, then it is given by $\chi\circ\nu$, where ν is a reduced norm on D^*, and χ is a quadratic character of F^*, and $JL(\Theta(\pi,0)^-)\cong\chi St_{GL(2,F)}\hookrightarrow\chi\nu^{\frac{1}{2}}\times\chi\nu^{-\frac{1}{2}}$. Here $St_{GL(2,F)}$ denotes the Steinberg representation of $GL(2,F)$. Then, the relation (9) still holds, but we use the multiplicativity of the γ–factors to simplify the L–functions involved. We use $\gamma(s,\chi_{V,\psi}^{-1}\rho\times\chi St_{GL(2,F)},\psi)=\gamma(s,\chi_{V,\psi}^{-1}\rho\times\chi\nu^{\frac{1}{2}},\psi)\times\gamma(s,\chi_{V,\psi}^{-1}\rho\times\chi\nu^{-\frac{1}{2}},\psi)$.

If $\chi_{V,\psi}^{-1}\rho\neq\chi$, the reducibility only depends on the poles of $L(s,\chi_{V,\psi}^{-1}\rho,Sym^2\rho_j)$ and can be described in the same way as in the previous case.

We excluded the case $\chi_{V,\psi}^{-1}\rho=\chi=1$ in the beginning. We now only have to consider the case $\chi_{V,\psi}^{-1}\rho=\chi\neq 1$. In this case, the non–negative reducibility point is $\frac{3}{2}$.

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