The subrepresentation theorem for automorphic representations

Marcela Hanzer

Abstract

We prove that every irreducible subrepresentation in the space of automorphic forms on $G(\mathbb{A})$, where $G$ is a split reductive group over a number field $k$, and $\mathbb{A}$ is the related ring of adeles, is a subrepresentation of the representation induced from a cuspidal automorphic representation of a Levi subgroup.

1 Introduction

In this note we prove the global (automorphic) version (over a number field $k$) of Casselman’s subrepresentation theorem. We explain it in more details: in the local theory (i.e., considering admissible representations of reductive groups over local fields) there is Harish-Chandra’s subquotient theorem, and then there is also Casselman’s subrepresentation theorem; both of them state that every irreducible representation (in the appropriate category) of this given reductive group is a subquotient or (in the case of Casselman’s theorem) a subrepresentation of a representation induced from the “simpler” one (of the appropriate subgroup). The global analog of the Harish–Chandra subquotient theorem would be Langlands’ theorem which describes a general automorphic representation as a subquotient of a representation induced from a cuspidal representation of a Levi subgroup.

We prove the following global version of Casselman’s subrepresentation theorem.

Theorem. Let $G$ be a connected reductive group split over $k$. Let $(\Pi, V)$ be an $(\mathfrak{g}_\infty, K_\infty) \times \prod_{v<\infty} G(k_v)$ irreducible subspace of automorphic forms in $A(G(k \setminus G(\mathbb{A})))$. Then, there exists a parabolic subgroup $P = MU$ of $G$, an
irreducible automorphic cuspidal representation \( \pi_0 \) of \( M \) (thus appearing in the space of cuspidal automorphic forms on \( M \)) such that, as abstract global representations, we have

\[
\Pi \hookrightarrow \text{ind}^{G(A)}_{P(A)} \pi_0,
\]

where we consider the normalized parabolic induction (so we extend \( \pi_0 \) trivially on \( U(A) \)) and we take \( K \)-finite vectors.

We explain all the notation in the Preliminaries section.

We are sure that the experts in the field are aware of the above claim, but we were not able to find the reference for this statement, which is somewhat more precise than the aforementioned Langlands’ result in his Corvalis lecture ([1]). The proof is a pretty straightforward application of the Langlands proof in his Corvalis lecture, with the decomposition results (on the spaces of automorphic forms) obtained (along with much stronger results) in [3]. We hope that this result will be very helpful for the explicit calculations with automorphic forms, since it is explicitly applicable to the discrete (and \( K \)-finite) part of automorphic \( L^2 \) situation.

We want to thank Neven Grbac for the helpful discussions about automorphic representations, and to Goran Muić and Marko Tadić for their interest in our work.

2 Preliminaries

Let \( k \) be a number field. Let \( G \) be a connected reductive group split over \( k \), and \( G_{\infty} = \prod_v G(k_v) \), where the product is over archimedean places of \( k \). Let \( \mathcal{U} \) be the enveloping algebra of the complexified Lie algebra \( \mathfrak{g} \) of \( G_{\infty} \) (and \( \mathfrak{g}_{\infty} \) is the Lie algebra of \( G_{\infty} \)). We denote by \( \mathfrak{z} \) the center of \( \mathcal{U} \). For a standard Levi subgroup \( M \) of \( G \), we denote by \( \mathfrak{z}^M \) the analogue of \( \mathfrak{z} \) for group \( M \). We follow the notation of the first chapter of [3]. We denote by \( Z_M \) the center of \( M \). We denote by \( K_v \) a maximal compact subgroup of \( G(k_v) \), where \( K_v = G(O_{k_v}) \) if \( v < \infty \), and \( O_{k_v} \) is the ring of integers in \( k_v \). We denote \( K_{\infty} = \prod_{v|\infty} K_v \) and \( K = \prod_v K_v \). We fix a Borel subgroup \( B \) of \( G \), and standard parabolic subgroups with respect to this Borel subgroup. We denote by \( T \) a maximal (\( k \)-split) torus of \( G \), and by \( \Delta \) the set of simple roots of \( G \) with respect to \( T \) (and \( B \)). We know that each standard parabolic subgroup corresponds to a subset \( \theta \) of \( \Delta \). We denote this by putting \( P = P_\theta \).
We use the following definition of an automorphic form: Let \( P = MU \) be a standard parabolic subgroup of \( G \) and \( \phi : U(\mathbb{A})M(k) \setminus G(\mathbb{A}) \to \mathbb{C} \) a function. We say that \( \phi \) is automorphic if it satisfies the following conditions:

1. \( \phi \) has moderate growth (cf. [3], I.2.3)
2. \( \phi \) is smooth (cf. [3], I.2.5)
3. \( \phi \) is \( K \)-finite
4. \( \phi \) is \( \mathfrak{z} \)-finite.

Note that the space \( A(U(\mathbb{A})M(k) \setminus G(\mathbb{A})) \) of all automorphic forms as above can be related to the usual situation with the automorphic forms on \( M(k) \setminus M(\mathbb{A}) \) by attaching to each \( k \in K \) and \( \phi \) as above a function \( \phi_k : M(k) \setminus M(\mathbb{A}) \to \mathbb{C} \) defined by \( \phi_k(m) = \delta_{P}^{-\frac{1}{2}}(m)\phi(mk) \) by noting that \( \phi \) is automorphic if and only if it is smooth, \( K \)-finite, and for all \( k \in K \), \( \phi_k \) is an automorphic form on \( M(k) \setminus M(\mathbb{A}) \). We denote by \( A_0(U(\mathbb{A})M(k) \setminus G(\mathbb{A})) \) the cuspidal part of the space \( A(U(\mathbb{A})M(k) \setminus G(\mathbb{A})) \); i.e., the space of all automorphic forms \( \phi \) from \( A(U(\mathbb{A})M(k) \setminus G(\mathbb{A})) \) with the property that for every standard parabolic subgroup \( P' = M'U' \) such that \( B \subset P' \subset P \) we have \( \phi_{P'} = 0 \) (the constant term along \( P' \), defined by \( \phi_{P'}(g) = \int_{U'(k)U'(\mathbb{A})} \phi(ug)du \)).

Note that the space \( A(U(\mathbb{A})M(k) \setminus G(\mathbb{A})) \) is a module for the action of \( (g_\infty, K_\infty) \times \prod_{v < \infty} G(k_v) \), i.e., for the global idempotent Hecke algebra \( \mathcal{H} = \mathcal{H}_\infty \otimes \mathcal{H}_f \), where \( \mathcal{H}_\infty \) is related to \( U \) and finite measures on \( K_\infty \), and \( \mathcal{H}_f = \otimes'_{v < \infty} \mathcal{H}_v \), where \( \mathcal{H}_v, v < \infty \) is Hecke algebra of compactly supported, locally constant functions on \( G(k_v) \) (cf. [1], section 4). Note that \( A_0(U(\mathbb{A})M(k) \setminus G(\mathbb{A})) \) is a submodule of \( A(U(\mathbb{A})M(k) \setminus G(\mathbb{A})) \) with this action. Note that the constant term (with respect to some standard parabolic subgroup \( P = MU \)) is an intertwining operator between \( A(G(k) \setminus G(\mathbb{A})) \) and \( A(U(\mathbb{A})M(k) \setminus G(\mathbb{A})) \) (cf. [3], I.2.6).

Let \( \xi \) be a character of \( Z_M(k) \setminus Z_M(\mathbb{A}) \), and let \( \pi \) be an irreducible submodule of \( A(M(k) \setminus M(\mathbb{A})) \), for a standard Levi subgroup \( M \) of \( G \). We denote by \( A(M(k) \setminus M(\mathbb{A}))_\pi \) the isotypic submodule attached to \( \pi \) (in the theorem below we deal with cuspidal \( \pi \), so the relevant subquotients are indeed subspaces). We denote

\[
A(U(\mathbb{A})M(k) \setminus G(\mathbb{A}))_\xi = \{ \phi \in A(U(\mathbb{A})M(k) \setminus G(\mathbb{A})) : \\
\forall z \in Z_M(\mathbb{A}), g \in G(\mathbb{A}) : \phi(zg) = \delta_{P}^{\frac{1}{2}}(z)\xi(z)\phi(g) \},
\]

3
\[ A(U(\mathbb{A})M(k) \setminus G(\mathbb{A}))_\pi = \{ \phi \in A(U(\mathbb{A})M(k) \setminus G(\mathbb{A})) : \forall k \in K, \phi_k \in A(M(k) \setminus M(\mathbb{A}))_\pi \}. \]

**Proposition 2.1.** Let \( \xi \) be a character of \( Z_M(k) \setminus Z_M(\mathbb{A}) \) and let \( \Pi_0(M)_\xi \) denote the set of isomorphism classes of irreducible representations of \( M(\mathbb{A}) \) occurring as submodules in \( A_0(M(k) \setminus M(\mathbb{A}))_{\xi} \). We have the following decomposition
\[ A_0(U(\mathbb{A})M(k) \setminus G(\mathbb{A}))_{\xi} = \bigoplus_{\pi \in \Pi_0(M)_\xi} A_0(U(\mathbb{A})M(k) \setminus G(\mathbb{A}))_\pi. \]

**Proof.** This is explained in [3], p. 44 \( \square \)

**Remark.** By the proof of Lemma I.3.2 of [3], \( \mathfrak{z}^M \) acts on \( A(U(\mathbb{A})M(k) \setminus G(\mathbb{A})) \) by left translations; every automorphic form there is \( \mathfrak{z}^M \)-finite; analogously every element of that space is \( Z_M \)-finite, again here \( Z_M \) acts by left translations (because we examine \( K \)-finite automorphic forms). Also, it is easy to see that \( A_0(U(\mathbb{A})M(k) \setminus G(\mathbb{A})) \) is \( Z_M \)-invariant subspace with this \( Z_M \)-action.

### 3 The theorem

We prove the following theorem:

**Theorem 3.1.** Let \((\Pi, V)\) be an \( ((g_\infty, K_\infty) \times \prod_{\nu < \infty} G(k_\nu) =) \) irreducible subspace of automorphic forms in \( A(G(k) \setminus G(\mathbb{A})) \). Then, there exists a parabolic subgroup \( P = MU \) of \( G \), an irreducible automorphic cuspidal representation \( \pi_0 \) of \( M \) (thus appearing in the space of cuspidal automorphic forms on \( M \)) such that, as abstract global representations, we have
\[ \Pi \hookrightarrow \Ind_{P(\mathbb{A})}^{G(\mathbb{A})} \pi_0, \]
where we consider the normalized parabolic induction (so we extend \( \pi_0 \) trivially on \( U(\mathbb{A}) \)) and we take \( K \)-finite vectors.

The proof of this theorem follows directly from the next theorem, so our embedding from the theorem above is realized through the calculation of the constant term.

---

4
Theorem 3.2. Let \((\Pi, V)\) be an \((g_\infty, K_\infty) \times \prod_{v < \infty} G(k_v)\) irreducible subspace of automorphic forms inside \(A(G(k) \setminus G(\mathbb{A}))\) such that some constant term of a function from \(V\) does not vanish along a parabolic subgroup \(P_0\) of \(G\); assume that \(\theta\) is minimal (set of simple roots) with this property. Then, there exists an irreducible automorphic representation \(\pi_0\) of \(M_\theta(\mathbb{A})\) (appearing in \(A_0(M_\theta(k) \setminus M_\theta(\mathbb{A}))\)) such that the space of constant terms of \(V\) along \(P_0\), denoted by \(V_0\), belongs (up to a left translation by an element from \(Z_{M_\theta}\)) to the space \(A_0(U_\theta(\mathbb{A})M_\theta(k) \setminus G(k_v)(\mathbb{A}))\pi_0\) of cuspidal automorphic forms.

Proof. Let \(f \in V\). By definition, the constant term \(f_{P_0}(g) = \int_{U_{P_0}(k) \setminus U_\theta(\mathbb{A})} f(ug)du\) belongs to \(A(U_\theta(\mathbb{A})M_\theta(k) \setminus G(\mathbb{A}))\), more precisely, to the cuspidal part of this space (because of minimality of \(\theta\); [3], I.2.6, I.2.18). By the remark above the theorem, \(Z_{M_\theta(\mathbb{A})}\) acts on \(A_0(U_\theta(\mathbb{A})M_\theta(k) \setminus G(\mathbb{A}))\) by left translations, and every function from this space is \(Z_{M_\theta(\mathbb{A})}\)–finite. For every \(z \in Z_{M_\theta(\mathbb{A})}\), let \(V_0^z = l(z)V_0\) (the action by left translations). We know that taking the constant term is intertwining operator, so \(V_0\) (and \(V_0^z\)) is (as an abstract \((g_\infty, K_\infty) \times \prod_{v < \infty} G(k_v)\)–representation) irreducible and isomorphic to \(V\). Let \(W = \sum_{z \in Z_{M_\theta(\mathbb{A})}} V_0^z\).

We prove that there exists \(F \in W, \ F \neq 0\) such that \(\dim_{\mathbb{C}}\{l(z)F : z \in Z_{M_\theta(\mathbb{A})}\} = 1\). Firstly, let \(F \neq 0\) be an element from \(W\) such that the dimension of the space \(Y := \{l(z)F : z \in Z_{M_\theta(\mathbb{A})}\}\) is minimal. We claim that this dimension is one. Indeed, let us assume that this dimension (of \(Y\)) is greater than one. If, for every \(a \in Z_{M_\theta(\mathbb{A})}\) acting on \(Y\), the whole space \(Y\) is an eigenspace for certain eigenvalue, it would mean that \(l(a)\), for every \(a\), acts as a scalar operator on \(Y\), and then every one-dimensional subspace, (also the one spanned by \(F\) would be one-dimensional; a contradiction. So, there exists \(a \in Z_{M_\theta(\mathbb{A})}\) with a non-zero eigenspace, attached to an eigenvalue \(\alpha \neq 0\), strictly smaller than \(Y\). Let \(Y_1 := \{l(a) - \alpha\}Y\) and \(F_1 := (l(a) - \alpha)F \in Y_1\). \(F_1\) is obviously non-zero; otherwise \(l(b)\) would be an eigenvector of \(l(a)\) for eigenvalue \(\alpha\) for every \(b \in Z_{M_\theta(\mathbb{A})}\), so that the whole \(Y\) is an eigenspace for \(\alpha\); a contradiction. Now, we easily see that the span of the set \(\{l(b)F_1 : b \in Z_{M_\theta(\mathbb{A})}\}\) is inside \(Y_1\), which leads to contradiction with our choice of \(F\).

So, we conclude that there exists a character \(\xi\) of \(Z_M(k) \setminus Z_M(\mathbb{A})\) such that

\[
l(z)F(g) = \delta_{P_0}^z(z)\xi(z)F(g), \forall g \in G(\mathbb{A}), \forall z \in Z_{M_\theta(\mathbb{A})}.
\]  

(1)

Now, let \(W_0\) denote a \((g_\infty, K_\infty) \times \prod_{v < \infty} G(k_v)\)–subspace of \(W\) generated by \(F\). For every vector from this space, (1) holds. Now, since \(W = \)
\[ \sum_{a \in Z_{M_{\theta}(\mathbb{A})}} V_0^a, \] where \( V_0^a \) are irreducible subspaces, \( W \) is also a direct sum of irreducible subspaces (for example, [2], chapter XVII), and every \( (\mathfrak{g}_\infty, K_\infty) \times \prod_{v < \infty} G(k_v) \) submodule of \( W \) has a direct summand. From this directly follows that \( W_0 \) has an irreducible submodule; indeed if \( W = \bigoplus_{z \in I} V_0^z \), for some \( I \subset Z_{M_{\theta}(\mathbb{A})} \), then some projection attached to this decomposition \( p_z : W \to V_0^z \) is non-zero on \( W_0 \). Now \( \text{Ker } p_z \cap W_0 \) has a direct summand \( W_1 \) in \( W \), and it is easy to see that \( W_1 \cap W_0 \) is an irreducible submodule of \( W_0 \). This means that we have found an irreducible subspace of \( W \) (so necessarily isomorphic to \( V \) i.e., to \( V_0 \)) where the relation (1) holds. This realization of \( V \) inside \( A_0(U_{\theta}(\mathbb{A})M_{\theta}(k) \setminus G(\mathbb{A}))_{\xi} \) is thus obtained through taking of (maybe translated) constant term along \( P_\theta \). According to Proposition 2.1 we have

\[ A_0(U_{\theta}(\mathbb{A})M_{\theta}(k) \setminus G(\mathbb{A}))_{\xi} = \bigoplus_{\pi \in \Pi_{0}(M_{\theta})} A_0(U_{\theta}(\mathbb{A})M_{\theta}(k) \setminus G(\mathbb{A}))_{\pi}, \]

and, combining our embedding with the appropriate projection, we have obtained an embedding

\[ \Pi \hookrightarrow A_0(U_{\theta}(\mathbb{A})M_{\theta}(k) \setminus G(\mathbb{A}))_{\pi_0}, \]

for some automorphic (cuspidal) representation \( \pi_0 \) of \( M_{\theta}(\mathbb{A}) \); this space can be identified with \( \text{ind}_{K_{M_{\theta}(\mathbb{A})} \cap M_{\theta}(\mathbb{A})}^{K_{M_{\theta}(\mathbb{A})}} A_0(M_{\theta}(k) \setminus M_{\theta}(\mathbb{A}))_{\pi_0} \).

Note that the space \( A_0(M_{\theta}(k) \setminus M_{\theta}(\mathbb{A}))_{\pi_0} \) is semisimple (Gelfand, Piatetski-Shapiro, cf.[1], section 4), which gives us the proof of Theorem 3.1.

**References**

