The generalized injectivity conjecture for the classical $p$–adic groups

Marcela Hanzer

Abstract

We prove that the conjecture posed by Casselman and Shahidi which says that a generic subquotient of a generic standard representation of a connected quasi-split reductive $p$–adic group is a subrepresentation ("generalized injectivity conjecture"), is valid in the case of classical groups.

1 Introduction

In their paper ([4]) Casselman and Shahidi posed some very important problems in the local representation theory (and gave answers to some of them). We describe problem which was posed there, and in this paper we give the answer in the case of classical $p$–adic groups.

Let $G$ be a quasi–split connected reductive group over a local field $F$ of characteristic zero. Assume that we have a standard parabolic subgroup $P$ with Levi decomposition $P = MN$ and an irreducible, tempered, generic representation $\sigma$ of $M$. Using notation from the introduction of ([4]), where $\nu$ denotes an element in the complex dual of the real Lie algebra of the split component of $M$, taken to be in the positive Weyl chamber, let $I(\nu, \sigma)$ be the corresponding standard representation. Then it is also generic (i.e. has a Whittaker model). Two conjectures are posed:

1) the corresponding Langlands’ quotient $J(\nu, \sigma)$ is generic if and only if $I(\nu, \sigma) = J(\nu, \sigma)$, and

2) each irreducible generic subquotient of $I(\nu, \sigma)$ is a subrepresentation.
The affirmative answer to the first question (called the standard module conjecture) was given by Vogan ([25]) for the real groups, and Casselman and Shahidi settled this question for the p–adic groups, but when the inducing data is supercuspidal. The question for the classical p–adic groups was resolved by Muić ([13]).

In this paper, we are dealing with the second question (generalized injectivity conjecture) for the classical p–adic groups.

Both of these questions, besides being interesting in their own right, have important global applications. We just note the following global application of injectivity (as explained in [4]; we use the same notation). Assume \( P \) is maximal and let \( A(s, w, \sigma) \) denote a standard intertwining operator. Shahidi defined the normalized intertwining operators \( A(s, w, \sigma) \) as follows ([20]):

\[
A(s, w, \sigma) = \epsilon(s, \sigma, r_w) \frac{L(1+s, \sigma, r_w)}{L(s, \sigma, r_w)} A(s, w, \sigma).
\]

For the calculation of the poles of the normalized intertwining operators (this has very important global consequences), the holomorphy of the expression

\[
\prod_{i=1}^{m} L(is, \tilde{\sigma}, r_i)^{-1} A(s\tilde{\alpha}, \sigma, w_0)
\]

plays a significant role. Under additional assumption (about holomorphy of \( L \)–functions appearing in the expression above for \( \text{Re}(s) > 0 \) and \( \sigma \) tempered), Casselman and Shahidi ([4], Theorem 5.1) use injectivity properties for the generalized rank one case to prove the holomorphy of the previous relation.

Because of that, in this paper we prove the following:

**Theorem.** Let \( G_n \) be quasi–split, connected, classical group (as explained in Preliminaries) defined over p–adic field \( F \) (of characteristic zero), \( \tau \) an irreducible, generic tempered representation of \( G_n \), and \( \delta \) essentially square–integrable representation of \( GL(k, F) \) with \( e(\delta) > 0 \). Then, a generic subquotient of the standard representation

\[
\delta \rtimes \tau
\]

of group \( G_{n+k} \) is a subrepresentation.

The case of general standard representation \( \delta_1 \times \cdots \times \delta_k \times \tau \) of a classical group follows without difficulties from the theorem above on the generalized
rank one case, essentially using induction over \( k \) and the ideas already used in this paper. We plan to write down the details elsewhere.

The main result of this paper follows from the standard module conjecture for classical groups which is now a theorem due to Muić. For the proof we do not need new analytical arguments (besides ones, on \( L \)-functions, used by Muić for his results). Other ingredients involve combinatorial arguments, basic properties of intertwining operators and known injectivity properties for the general linear groups ([26]). We also use the classification of discrete series of classical groups by Möglin and Tadić, mainly because the use of triples, by which they classify discrete series, makes our arguments more uniform, although we could do it without this classification (essentially, generic discrete series for some classical groups were already described without this classification—[11], also [21]). But, the notions of Jordan block and of \( \varepsilon \) function simplify the technical details involved. We also emphasise that, although Möglin–Tadić classification is complete only under a natural assumption (so called the basic assumption), in the case of generic square–integrable representations, the classification is complete.

Now we describe the content of the paper. In the section Preliminaries we review the basic notation from the representation theory of general linear groups that we need. We review classical groups we study, and we also recollect main features of Möglin–Tadić classification of discrete series of classical groups (a brief overview can also be found in [14]). In the third section we prove that an \( \varepsilon \)-function, attached to a discrete series representation \( \sigma \) of a classical group by this classification, if \( \sigma \) is generic, has a special form. This is technical result, which simplifies a lot following arguments and calculations. In the fourth section, we examine when, in the generalized rank one standard representation \( \delta \psi \sigma \), where \( \delta \) and \( \sigma \) are discrete series representation of general linear and classical group, respectively, discrete series generic subquotient occurs. We also prove that if it occurs, it is a subrepresentation. We then prove then the same thing holds for a generic tempered subquotient, and at last, that a non–tempered generic subquotient is also a subrepresentation. In the fifth section we prove the analogous results for a standard generalized rank one representation (i.e. we drop the assumption that \( \sigma \) is discrete series, and assume that \( \sigma \) is tempered). In the sixth section we prove that the generalized injectivity holds also for special even–orthogonal groups, by first proving that it holds for non–connected full even orthogonal groups (where Möglin–Tadić classification holds). To do that, we prove a number of very simple claims involving representations of the full even–orthogonal
The author wishes to thank G. Muić for useful discussions about the split special even–orthogonal groups, and to F. Shahidi who suggested studying the injectivity problem.

2 Preliminaries

Let \( \mathbb{Z}, \mathbb{N}, \mathbb{R}, \mathbb{C} \) denote the ring of rational integers, positive rational integers, the field of real numbers, the field of complex numbers. From now on, \( F \) denotes a non-archimedean local field with the normalized absolute value \( | \cdot |_F \), of a characteristic zero.

We first recall some notation and facts related to the representations of the general linear groups. \( \nu \) denotes an unramified character of \( GL(n, F) \) defined by \( \nu(g) = |\det(g)|_F \). If \( \rho \) is an irreducible cuspidal representation of a general linear group \( GL(n, F) \), \( m \) a non-negative integer and \( \alpha \) a real number, by a segment \( [\nu^{\alpha, \rho}, \ldots, \nu^{\alpha+m\rho}] \) we mean an (ordered) set \( \{\nu^{\alpha, \rho}, \ldots, \nu^{\alpha+m\rho}\} \). To each such segment we can attach an essentially square–integrable representation, denoted by \( \delta([\nu^{\alpha, \rho}, \ldots, \nu^{\alpha+m\rho}]) \); this representation is a unique subrepresentation of the induced representation \( \text{Ind}_{GL((m+1)n,F)}^{GL((m+1)n,F)}(\nu^{\alpha+m\rho} \otimes \cdots \otimes \nu^{\alpha\rho}) \).

In the case of general linear group, such a parabolically induced representation we will denote by \( \nu^{\alpha+m\rho} \times \cdots \times \nu^{\alpha\rho} \). For a basic facts about the representations of the general linear groups over \( F \), we refer to ([2], [26]). We freely use the notation of Zelevinsky ([26]). For every irreducible, essentially square–integrable representation \( \delta \) of \( GL(n_{\delta}, F) \) there exists a unique real number \( e(\delta) \) such that \( \delta \nu^{-e(\delta)} \) is unitarizable. We can now write down the Langlands’ classification for the general linear group: let \( \delta_1, \ldots, \delta_k \) be unitarizable square–integrable representations of \( GL(n_{i}, F) \), \( i = 1, \ldots, k \) and \( s_1, \ldots, s_k \) real numbers satisfying \( s_1 \geq s_2 \geq \cdots \geq s_k \). Then, the representation \( \delta_1 \nu^{s_1} \times \cdots \times \delta_k \nu^{s_k} \) has a unique irreducible quotient, the Langlands’ quotient, which we denote by \( L(\delta_1 \nu^{s_1}, \ldots, \delta_k \nu^{s_k}) \).

We specify which (quasi–split) groups we are studying: assume that we have a vector space

\[
V_n = e_1 F \oplus \cdots \oplus e_n F \oplus V_0 \oplus e_{n+1} F \oplus \cdots \oplus e_{2n} F.
\]

1) the symplectic groups \( G_n \) are groups of isometries of the skew–symmetric form on \( V_n \) (with \( V_0 = 0 \)) defined by \( \langle e_i, e_{2n+1-j} \rangle = \delta_{ij}, i, j = 1, \ldots, n \).
2) the special orthogonal groups $G_n$ are isometries preserving a symmetric form on $V_n$ again defined by $\langle e_i, e_{2n+1-j} \rangle = \delta_{ij}, i, j = 1, \ldots, n$. $V_0$ is anisotropic space, orthogonal to the hyperbolic planes in the decomposition (1). $V_0 = 0$, or $F$ or $E$ (a quadratic extension of $F$ with the quadratic form $q(x) = N_{E/F}(x)$).

3) the unitary groups $G_n$: for this case, we have to change $F$'s in (1) by $E$'s (again a quadratic extension), and let $\tau$ be a nontrivial element of $Gal(E/F)$. Then, we define a $\tau$--hermitian form on $V_n$ (prescribing the same action on the basis as in the above cases). Unitary groups are groups of isometries of this $\tau$--hermitian form (with $\dim V_0 \leq 1$, because they are quasi–split).

Remark. The proofs in the subsequent sections (3, 4, 5) are actually suited for the special odd–orthogonal, symplectic, and non-split, but quasi–split special even orthogonal groups because they assume algebraic connectedness of the groups, uniform description of standard parabolic subgroups, and root system of type $B_n$ or $C_n$. In the sixth section we address the case of split special even–orthogonal group, and of (quasi–split) unitary groups.

From now on, till the sixth section, we assume that the groups we are studying are special odd–orthogonal, symplectic, and non-split, but quasi–split special even orthogonal groups.

The subscript “$n$” denotes the split rank of the group $G_n$. We fix a minimal $F$–parabolic subgroup in the classical group $G_n$ consisting of the upper–triangular matrices in the usual matrix realization of the classical groups. Then it is known that the standard $F$–parabolic subgroups are quasi–upper-triangular, and their standard Levi factors have the following form: $M \simeq GL(n_1, F) \times \cdots \times GL(n_k, F) \times G_m$. Then, if $\sigma_i$ is a representation of $GL(n_i, F)$ and $\tau$ a representation of $G_m$, a normalized parabolically induced representation $Ind^G_M(\sigma_1 \otimes \cdots \otimes \sigma_k \otimes \tau)$ will be denoted by $\sigma_1 \times \cdots \times \sigma_k \rtimes \tau$. If $\tau$ is an irreducible tempered representation of the group $G_m$, $\sigma_i$'s and $s_1, s_2, \ldots s_k$ satisfy the conditions as above (as in Langlands’ classification for the general linear groups), and, additionally, $s_k > 0$, then, analogously, the Langland’s quotient of the representation $\sigma_1^{n_1} \times \sigma_2^{n_2} \times \cdots \times \sigma_k^{n_k} \rtimes \tau$ is denoted by $L(\sigma_1^{n_1}, \sigma_2^{n_2}, \ldots, \sigma_k^{n_k} ; \tau)$.

We briefly recollect of Mœglin–Tadić classification of discrete series for classical groups. We fix a certain tower of classical groups (symplectic, special odd orthogonal or unitary); the classification is also valid for full even–orthogonal groups, but we will use them only in the 6th section. Every
discrete series representation is uniquely described by three invariants: a partial supercuspidal support, Jordan block and \(\varepsilon\) function.

A partial supercuspidal support of a discrete series representation \(\sigma\) of \(G_n\) is an irreducible supercuspidal representation \(\sigma_{cusp}\) of some \(G_m\) such that there exists an irreducible admissible representation \(\pi\) of some \(GL(m, F)\) (this defines \(m\)) such that \(\sigma\) is a subrepresentation of \(\pi \times \sigma_{cusp}\).

The set \(\text{Jord}(\sigma)\) is defined as a set of all pairs \((a, \rho)\) where \(\rho \equiv \tilde{\rho}\) is an irreducible supercuspidal representation of some \(GL(m, F)\) and \(a > 0\) is an integer such that both of the following two properties are satisfied:

(i) \(a\) is even if and only if \(L(s, \rho, r)\) has a pole at \(s = 0\). The local \(L\)–function \(L(s, \rho, r)\) is the one defined by Shahidi (for example ([20])), where \(r = \Lambda^2C^{m,\rho}\) is the exterior square of the standard representation if \(G_n\) is symplectic or even orthogonal group, and \(r = \text{Sym}^2C^{m,\rho}\) if \(G_n\) is odd orthogonal. For unitary groups, the appropriate \(L\)–functions are discussed in ([8] and [9], Section 15).

(ii) the induced representation
d\((\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho) \times \sigma\)
is irreducible.

For an irreducible, cuspidal representation \(\rho \equiv \tilde{\rho}\) of \(GL(m, F)\) we denote by \(\text{Jord}_\rho(\sigma) = \{a : (a, \rho) \in \text{Jord}(\sigma)\}\). If \(\text{Jord}_\rho(\sigma)\) is non–empty, and \(a \in \text{Jord}_\rho(\sigma)\), we denote \(a_- = \max\{b \in \text{Jord}_\rho(\sigma) : b < a\}\) if it exists. The \(\varepsilon\) function is defined on a subset of \(\text{Jord}(\sigma) \cup \text{Jord}(\sigma) \times \text{Jord}(\sigma)\), and attains values 1 and \(-1\). We will not describe all the properties of the \(\varepsilon\)–function; they can be found in [9],[8]. It is defined on \(((a, \rho_1), (b, \rho_2))\) if \(\rho_1 \equiv \rho_2\). If it is defined on the elements of \(\text{Jord}_\rho\), the value on the pair is product of the values on the elements of the pair. There is also a transitivity property, so it is actually enough to know the value of \(\varepsilon\) (if it is defined on elements, not only on pairs) on the minimal element of \(\text{Jord}_\rho(\sigma)\) and on the consecutive pairs \(\varepsilon((a_-, \rho), (a, \rho))\).

The \(\varepsilon\) function describes where the corresponding discrete series appears as a subquotient; namely for \(a_-\), \(a \in \text{Jord}_\rho(\sigma)\) \(\varepsilon((a_-, \rho), (a, \rho)) = 1\) if there exists an irreducible representation \(\pi\) of an appropriate classical group such that

\[
\sigma \hookrightarrow \delta((\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho) \times \pi).
\]
It is intuitively clear that for a generic discrete series \( \varepsilon \) function on the consecutive pairs should be equal to 1 ("many" times, since the fixed generic discrete series must appear as a (unique generic) subquotient in lots of induced representations like the one in the previous relation). We will formally prove that in the third section. The \( \varepsilon \) function has a conjectural description in terms of Langlands' correspondence—it should distinguish discrete series in the same \( L \)-packet.

These invariants completely describe the representation \( \sigma \).

On the other hand, to each "admissible triple" ([9],[8]) we can attach a discrete series representation. The admissible triple consists of an irreducible supercuspidal representation \( \sigma' \) of a classical group, a finite (maybe empty) set \( \text{Jord} \) consisting of pairs \((a, \rho)\) which satisfy the condition (i) above, and of function \( \varepsilon \) defined on the subset of \( \text{Jord} \cup \text{Jord} \times \text{Jord} \) with values in \{1, -1\}, all subjected to some conditions.

This correspondence between discrete series representations of all \( G_n \)'s and all the admissible triples is bijective and presents Mœglin–Tadić classification. The "admissibility of triples" on one side of this bijection corresponds to the forming of discrete series, starting from so called strongly positive discrete series (which include cuspidal representations, generalized Steinberg representations etc.). So, if \( a, b \) are natural integers of the appropriate parity (property (i) above) and \([a, b] \cap \text{Jord}_{\rho}(\sigma) = \emptyset\), then the representation \( \delta([\nu^{-\frac{b-1}{2}}, \nu^{\frac{b+1}{2}}]) \times \sigma \) has two discrete series, say \( \sigma' \) and \( \sigma'' \) and \( \text{Jord}(\sigma') = \text{Jord}(\sigma'') = \text{Jord}(\sigma) \cup \{(a, \rho), (b, \rho)\} \). A function \( \varepsilon_{\sigma} \) has two possible extensions on \( \text{Jord}(\sigma') \), corresponding to \( \sigma' \) and \( \sigma'' \), satisfying
\[
\varepsilon_{\sigma'}((a, \rho), (b, \rho)) = \varepsilon_{\sigma''}((a, \rho), (b, \rho)) = 1.
\]
Here we denote \( \varepsilon \) function with the subscript— the corresponding discrete series. If it is clear which discrete series this \( \varepsilon \)-function refers to, we drop it. Sometimes we write \( \varepsilon_{\rho} \) when it is clear what is a discrete series in question, and then this is a function on \( \text{Jord}_{\rho}(\sigma) \times \text{Jord}_{\rho}(\sigma) \). In this way, starting from the strongly positive discrete series representations, by appropriately adding two elements in the Jordan block (as above), we get all the discrete series representations of all the classical groups in one tower.

In the opposite direction, starting from a discrete series \( \sigma \), by appropriately removing two by two neighboring elements on which \( \varepsilon \)-function attains value 1 from the Jordan block of \( \sigma \), we obtain a strongly positive representation \( \sigma^+ \). This \( \sigma^+ \) is not unique (for given \( \sigma \)).

Throughout the paper, we use results of Jantzen ([6]) on discrete series.
of classical groups and their cuspidal supports. Let $\rho_1, \ldots, \rho_k$ denote unitary supercuspidal representations of $GL(n_i, F)$, $i = 1, \ldots, k$, with $\rho_i \not\cong \rho_j, \tilde{\rho}_j$ for $i \neq j$. Let $S(\rho_i) = \{\nu^\alpha \rho_i, \nu^\alpha \tilde{\rho}_i : \alpha \in \mathbb{R}\}$. If a discrete series $\sigma$ has a supercuspidal support in $\bigcup_{i=1}^k S(\rho_i) \cup \sigma_{cusp}$, then, for every $j = 1, \ldots, k$ there exists an irreducible representation $\sigma_j$ (completely determined by $\sigma$) and a representation $\pi_j$ such that $\sigma \hookrightarrow \pi_j \rtimes \sigma_j$, where the cuspidal support of $\sigma_j$ consists only of representations of $S(\rho_j)$ (and $\sigma_{cusp}$) and $\pi_j$ does not have elements of $S(\rho_j)$ in the cuspidal support. Jantzen proved that $\sigma$ is square–integrable if and only if $\sigma_j$ is square–integrable, for all $j = 1, \ldots, k$.

Having this in mind, it is obvious that the reducibility of the representation $\delta([\nu^{-l}\rho_1, \nu^l\rho_1]) \rtimes \sigma$ depends only on reducibility of $\delta([\nu^{-l}\rho_1, \nu^l\rho_1]) \rtimes \sigma_1$. Because of that, we will often write ”the representation $\sigma$ is ($\rho$)–strongly positive” meaning that the component $\sigma_\rho$ is strongly positive (or sometimes, in this situation, we will just say “strongly positive”, without mentioning $\rho$). It will also be clear from the context when cuspidality of $\sigma$ actually means the cuspidality of $\sigma_\rho$.

We fix a non–degenerate character $\chi$ of a unipotent radical of a minimal standard parabolic subgroup of $G_n$, defined over $F$ (as in Preliminaries of [11], or [17]). In the sections 3, 4, 5 when we say “generic” it means $\chi$–generic representations. Since the groups we study are quasi–split and connected, the standard ($\chi$)–generic modules have exactly one ($\chi$)–generic irreducible subquotient.

3 The $\varepsilon$–function of the generic discrete series

Proposition 3.1. Let $\sigma$ be an irreducible generic, discrete series representation of the group $G_n$, and $\rho$ a self–contragredient, irreducible cuspidal representation of $GL(m, F)$. If $a \in \text{Jord}_\rho(\sigma)$ is such that $a_-$ is defined, then $\varepsilon_\sigma(\rho, a)\varepsilon_\sigma(\rho, a_-)^{-1} = 1$.

Proof. We prove this statement using induction over the number of additions of pairs of elements in the Jord$_\rho$ (this describes the way the discrete series are formed, starting from the strongly positive discrete series). This means that the basis of our inductive procedure are the strongly positive discrete series $\sigma$. Assume that $\sigma_{cusp}$ is a partial cuspidal support of $\sigma$. This means that $\sigma_{cusp}$ is generic, and the representation $\nu^\alpha \rho \rtimes \sigma_{cusp}, \alpha \geq 0$ reduces for $\alpha \in$
\{0, \frac{1}{2}, 1\}. Then \(|\text{Jord}_p(\sigma_{\text{cusp}})| = 0\) or 1, so that \(|\text{Jord}_p(\sigma)| = |\text{Jord}_p(\sigma_{\text{cusp}})|\) or \(|\text{Jord}_p(\sigma)| = |\text{Jord}_p(\sigma_{\text{cusp}})| + 1\). The latter possibility occurs when \(\alpha = \frac{1}{2}\) and then \(|\text{Jord}_p(\sigma)| = 1\). So, in the case of strongly positive \(\sigma\), the statement is trivially true, since then \(\text{Jord}_p(\sigma)\) has, at most, one element. Assume now that the statement of the proposition is valid for all the generic discrete series obtained by adding \(k\) or less times two elements in the Jordan block of the strongly positive discrete series. Let \(\sigma\) be a generic discrete series obtained by adding \(k\) elements in a Jordan block of a strongly positive (i.e., \(\rho\)-strongly positive generic discrete series). Now we add two more elements, call it \(a_−\) and \(a\) in \(\text{Jord}_p(\sigma)\) (by the assumption, \(|\text{Jord}_p(\sigma)| \in \{2k, 2k + 1\}\)). The elements \(a_−\) and \(a\) are of the appropriate parity, and \([a_−, a] \cap \text{Jord}_p(\sigma) = \emptyset\).

Then, the representation

\[
\delta([\nu^{\frac{-a-1}{2}} \rho, \nu^{\frac{-a-1}{2}} \rho]) \times \sigma
\]

has two square integrable subrepresentations. We denote them by \(\sigma_1\) and \(\sigma'_1\). By the results of Muić ([13],[12]) we know that if a standard generic module contains a discrete series subquotient, then all the generic irreducible subquotients are in discrete series. Let \(\sigma_0\) be a generic subquotient of \(\sigma\).

By this result either \(\sigma_1\) or \(\sigma'_1\) is \(\sigma_0\). We now study several possible positions of \(a_−\) and \(a\) with respect to the elements of \(\text{Jord}_p(\sigma)\).

**The first case**

There exist \(b_−, b \in \text{Jord}_p(\sigma)\) such that

\[b_− < a_− < a < b, \text{Jord}_p(\sigma) \cap b_−, b]^c = \emptyset.\]

Assume that \(\sigma_1\) is the discrete series with \(\varepsilon_{\sigma_1}(\rho, b_−)\varepsilon_{\sigma_1}(\rho, a_−)^{-1} = 1\). We will prove that \(\sigma_1 = \sigma_0\). Let \(\sigma''\) be a square–integrable representation obtained by removing elements \(b_−\) and \(b\) from \(\text{Jord}_p(\sigma)\). This means

\[
\sigma \hookrightarrow \delta([\nu^{\frac{-b-1}{2}} \rho, \nu^{\frac{b+1}{2}} \rho]) \times \sigma''
\]

and \(\sigma''\) is generic. We have the following sequence of homomorphisms:

\[
\sigma_0 \hookrightarrow \delta([\nu^{\frac{-a-1}{2}} \rho, \nu^{\frac{a+1}{2}} \rho]) \times \delta([\nu^{\frac{-b-1}{2}} \rho, \nu^{\frac{b+1}{2}} \rho]) \times \sigma'' \\
\rightarrow \delta([\nu^{\frac{-b-1}{2}} \rho, \nu^{\frac{b+1}{2}} \rho]) \times \delta([\nu^{\frac{-a-1}{2}} \rho, \nu^{\frac{a+1}{2}} \rho]) \times \sigma''.
\]
The second homomorphism is induced by the obvious one on the general linear group. Since the kernel of this homomorphism is

\[ L(\delta([\nu^{-\frac{a-1}{2}} \rho, \nu^{-\frac{b-1}{2}} \rho]), \delta([\nu^{-\frac{a-1}{2}} \rho, \nu^{-\frac{b-1}{2}} \rho])) \times \sigma'' , \]

which is degenerate representation, we must have \( \sigma_0 \mapsto \delta([\nu^{-\frac{b-1}{2}} \rho, \nu^{-\frac{b-1}{2}} \rho]) \times \delta([\nu^{-\frac{a-1}{2}} \rho, \nu^{-\frac{a-1}{2}} \rho]) \times \sigma'' , \)

moreover

\[ \sigma_0 \mapsto \delta([\nu^{-\frac{a-1}{2}} \rho, \nu^{-\frac{b-1}{2}} \rho]) \times \delta([\nu^{-\frac{b-1}{2}} \rho, \nu^{-\frac{a-1}{2}} \rho]) \times \sigma'' , \]

since \( \delta([\nu^{-\frac{a-1}{2}} \rho, \nu^{-\frac{b+1}{2}} \rho]) \times \delta([\nu^{-\frac{b-1}{2}} \rho, \nu^{-\frac{a+1}{2}} \rho]) \) is a unique generic subrepresentation of \( \delta([\nu^{-\frac{b-1}{2}} \rho, \nu^{-\frac{b+1}{2}} \rho]) \times \delta([\nu^{-\frac{a-1}{2}} \rho, \nu^{-\frac{a+1}{2}} \rho]) \). We further have

\[ \sigma_0 \leq \delta([\nu^{-\frac{b-1}{2}} \rho, \nu^{-\frac{a-1}{2}} \rho]) \times \delta([\nu^{-\frac{a+1}{2}} \rho, \nu^{-\frac{b+1}{2}} \rho]) \times \delta([\nu^{-\frac{a-1}{2}} \rho, \nu^{-\frac{b-1}{2}} \rho]) \times \sigma''. \]

(3)

We can also change \( \delta([\nu^{-\frac{a+1}{2}} \rho, \nu^{-\frac{a-1}{2}} \rho]) \) from the above relation into \( \delta([\nu^{-\frac{a+1}{2}} \rho, \nu^{-\frac{a-1}{2}} \rho]) \). On the other hand, by our choice of \( \sigma_1 \), we have

\[ \sigma_1 \mapsto \delta([\nu^{-\frac{b-1}{2}} \rho, \nu^{-\frac{a-1}{2}} \rho]) \times \delta([\nu^{-\frac{a+1}{2}} \rho, \nu^{-\frac{b+1}{2}} \rho]) \times \sigma''. \]

By (3), \( \sigma_0 \) is a generic subquotient of

\[ \sigma_0 \leq \delta([\nu^{-\frac{b-1}{2}} \rho, \nu^{-\frac{a-1}{2}} \rho]) \times \delta([\nu^{-\frac{a+1}{2}} \rho, \nu^{-\frac{b+1}{2}} \rho]) \times \sigma''. \]

(4)

Let \( \sigma_2 \) be a generic square–integrable subrepresentation of \( \delta([\nu^{-\frac{a+1}{2}} \rho, \nu^{-\frac{a-1}{2}} \rho]) \times \sigma'' \). We have \( \varepsilon_{\sigma_1}((a, \rho), (b, \rho)) = 1 \) (this property satisfy both subrepresentations). By relation (4), this forces

\[ \sigma_0 \leq \delta([\nu^{-\frac{b-1}{2}} \rho, \nu^{-\frac{a-1}{2}} \rho]) \times \sigma_2 , \]

and we have \( \varepsilon_{\sigma_0}((a, \rho), (b, \rho)) = \varepsilon_{\sigma_1}((a_-, \rho), (b_-, \rho)) = 1 \), and this agrees only with \( \varepsilon \)–function of \( \sigma_1 \), so \( \sigma_0 = \sigma_1 \).

The second case

We assume now that \( a < b_- \), where \( b_- = \min\{b : b \in \text{Jord}_\rho(\sigma)\} \). We additionally assume that \( \text{Jord}_\rho(\sigma) \) does not consist only of \( b_- \); we denote the
smallest element of $\text{Jord}_\rho(\sigma) \setminus \{b_\cdot\}$ by $b$. Analogously as in the previous case, we have

$$\sigma_1, \sigma'_1 \hookrightarrow \delta([\nu^{-\frac{a_\cdot-1}{2}} \rho, \nu^{\frac{a_\cdot-1}{2}} \rho]) \rtimes \sigma.$$  

Here again $\sigma_1$ denotes a discrete series representation such that its $\varepsilon$–function has value one on each pair of the consecutive elements in $\text{Jord}_\rho(\sigma)$. Let $\sigma''$ be a generic discrete series obtained by removing $b_\cdot$ and $b$ from $\text{Jord}_\rho(\sigma)$; this representation satisfies the induction hypothesis on the $\varepsilon$–function. Then we have

$$\sigma \hookrightarrow \delta([\nu^{-\frac{b_\cdot-1}{2}} \rho, \nu^{\frac{b_\cdot-1}{2}} \rho]) \rtimes \sigma''.$$  

The Jordan block of $\sigma''$ satisfies the induction hypothesis. So, we have

$$\sigma_0 \hookrightarrow \delta([\nu^{-\frac{a_\cdot-1}{2}} \rho, \nu^{\frac{a_\cdot-1}{2}} \rho]) \times \delta([\nu^{-\frac{b_\cdot-1}{2}} \rho, \nu^{\frac{b_\cdot-1}{2}} \rho]) \rtimes \sigma''.$$  

This forces

$$\sigma_0 \leq \delta([\nu^{-\frac{a_\cdot-1}{2}} \rho, \nu^{\frac{a_\cdot-1}{2}} \rho]) \times \delta([\nu^{-\frac{b_\cdot-1}{2}} \rho, \nu^{\frac{b_\cdot-1}{2}} \rho]) \rtimes \sigma''$$  

since the $GL$–part of the above induced representation contains

$$\delta([\nu^{-\frac{a_\cdot-1}{2}} \rho, \nu^{\frac{a_\cdot-1}{2}} \rho]) \times \delta([\nu^{-\frac{b_\cdot-1}{2}} \rho, \nu^{\frac{b_\cdot-1}{2}} \rho])$$  

as a subquotient. The representation $\delta([\nu^{-\frac{a_\cdot-1}{2}} \rho, \nu^{\frac{a_\cdot-1}{2}} \rho]) \rtimes \sigma''$ has a unique square–integrable generic subrepresentation, say $\sigma_2$, whose Jordan block satisfies the induction hypothesis on $\varepsilon$–function (number of elements in the Jordan block!) This means $\sigma_0 \leq \delta([\nu^{-\frac{a_\cdot-1}{2}} \rho, \nu^{\frac{a_\cdot-1}{2}} \rho]) \rtimes \sigma_2$. But, when we examine the only possible common square–integrable subquotient of $\delta([\nu^{-\frac{a_\cdot-1}{2}} \rho, \nu^{\frac{a_\cdot-1}{2}} \rho]) \rtimes \sigma_2$ and of $\delta([\nu^{-\frac{a_\cdot-1}{2}} \rho, \nu^{\frac{a_\cdot-1}{2}} \rho]) \rtimes \sigma$, we see that $\sigma_0 = \sigma_1$. We treat the case in which $a_\cdot > \max\{b : b \in \text{Jord}_\rho(\sigma)\}$ (and in which $\text{Jord}_\rho(\sigma)$ has at least two elements) totally analogously.

The third case

We now deal with the case $|\text{Jord}_\rho(\sigma)| \leq 1$. In the case $\text{Jord}_\rho(\sigma) = \emptyset$, the claim is trivially true. If $\text{Jord}_\rho(\sigma) = \{b\}$, then for the representation $\sigma$ we have the following

$$\sigma \hookrightarrow \delta([\nu^{\alpha} \rho, \nu^{\frac{b_\cdot-1}{2}} \rho]) \rtimes \sigma',$$

where $\sigma'$ is a generic $(\rho)$–strongly positive discrete series with $\text{Jord}_\rho(\sigma') = \emptyset$, and $\alpha \in \{\frac{1}{2}, 1\}$. Let us assume, additionally, that $a < b$. The case $b < a_\cdot$ is treated quite analogously. Let $\sigma_1$, $\sigma'_1$ and $\sigma'$ be as in the previous
cases. The representation \( \delta(\nu^\alpha \rho, \nu^{a_1+1} \rho) \times \sigma' \) has a unique square–integrable subquotient; it is necessarily generic; we denote it by \( \sigma'' \). In an appropriate Grothendieck group we have the following:

\[
\sigma_0 \leq \delta(\nu^{a_1} \rho, \nu^{a_1+1} \rho) \times \delta(\nu^\alpha \rho, \nu^{b_1+1} \rho) \times \sigma' \leq \\
\delta(\nu^{a_1-1} \rho, \nu^{a_1+1} \rho) \times \delta(\nu^\alpha \rho, \nu^{b_1+1} \rho) \times \delta(\nu^\alpha \rho, \nu^{a_1+1} \rho) \times \sigma'.
\]

This means that we must have

\[
\sigma_0 \leq \delta(\nu^{a_1-1} \rho, \nu^{b_1+1} \rho) \times \delta(\nu^\alpha \rho, \nu^{a_1+1} \rho) \times \sigma''',
\]
i.e. using the hereditary properties of generic representations ([18])

\[
\sigma_0 \leq \delta(\nu^{a_1-1} \rho, \nu^{b_1+1} \rho) \times \sigma''.
\]

The induced representation on the right–hand side has two discrete series representations. By comparison of \( \varepsilon \)–functions of the discrete series subrepresentations of \( \delta(\nu^{a_1-1} \rho, \nu^{b_1+1} \rho) \times \sigma'' \) and of \( \delta(\nu^{a_1-1} \rho, \nu^{b_1+1} \rho) \times \sigma \), we see that \( \sigma_0 = \sigma_1 \). \( \square \)

**Remark.** We can even prove more, i.e. if the \( \varepsilon \) function is defined on the members of \( \text{Jord}_\rho(\sigma) \) (not only on pairs), for a generic \( \sigma \), then \( \varepsilon \) attains the value 1 on every member of \( \text{Jord}_\rho(\sigma) \). To see that, firstly assume that \( \text{Jord}_\rho(\sigma) \) consists of even numbers, and let \( 2a_1 + 1 = \min\{2a + 1; 2a + 1 \in \text{Jord}_\rho(\sigma)\} \).

Then, by Proposition 3.1, the value \( \varepsilon(2a_1 + 1) \) completely determines it’s value on the whole \( \text{Jord}_\rho(\sigma) \). If \( \text{Jord}_\rho(\sigma) = \{2a_1 + 1\} \), the representation \( \sigma \) is \( \rho \)–strongly positive, and we must have \( \varepsilon(2a_1 + 1) = 1 \), by the definition of the strongly positive representations (e.g. discussion preceding Theorem 1.1 of [14]). So, let \( 2a_2 + 1 \in \text{Jord}_\rho(\sigma) \) with the property \( (2a_2+1)_- = 2a_1 + 1 \). Let \( \sigma' \) be a generic discrete series with the property \( \sigma \leq \delta(\nu^{a_1} \rho, \nu^{a_2} \rho) \times \sigma' \). Now, by embedding \( \sigma' \) in an appropriate induced representation, with \( (\sigma')^+ \) on the classical part, with \( \text{Jord}_\rho((\sigma')^+) = \emptyset \) or max(\( \text{Jord}_\rho(\sigma) \)) and by repeating the analysis similar to the one in the previous proposition, we obtain that there exists an generic discrete series \( \sigma_2 \) such that \( \sigma \leq \delta(\nu^{a_2} \rho, \nu^{a_1} \rho) \times \sigma_2 \), so that \( \text{Jord}_\rho(\sigma) = \text{Jord}_\rho(\sigma_2) \cup \{2a_1 + 1\} \). We can add elements in the Jordan block in any (permissible –in view of \( \varepsilon \) function) way we want, so there exists a generic discrete series \( \sigma'' \) such that

\[
\sigma \leq \delta(\nu^{a_2} \rho, \nu^{a_1} \rho) \times \cdots \times \delta(\nu^\frac{1}{2} \rho, \nu^{a_1} \rho) \times \sigma'' \cong \\
\cong \delta(\nu^\frac{1}{2} \rho, \nu^{a_1} \rho) \times \delta(\nu^{a_2} \rho, \nu^{a_1} \rho) \times \cdots \times \sigma''.
\]
From this follows that

$$\sigma \hookrightarrow \delta([\nu^{\frac{1}{2}} \rho, \nu^{\alpha_1} \rho]) \ltimes \sigma_2$$

and, by the definition of $\varepsilon$–function ([14], discussion preceding Theorem 1.1), $\varepsilon_\sigma(a_1, \rho) = 1$.

In the case when the elements of $\text{Jord}_\rho(\sigma)$ are odd, the $\varepsilon$ function is defined on the elements of that set iff $\text{Jord}_\rho(\sigma_{\text{cusp}}) = \emptyset$ (i.e. $\rho \ltimes \sigma_{\text{cusp}}$ reduces). In this case it is not obvious how to distinguish two square–integrable representations obtained when two elements are added in the Jordan block (Mœglin resolved that using intertwining operators). In more words, the representation $\delta([\nu^{-l_1} \rho, \nu^{l_2} \rho]) \ltimes \sigma_{\text{cusp}}$ (for $l_1, l_2 \in \mathbb{Z}_+$) has two square–integrable representations $\sigma_1$ and $\sigma_2$. Let say that $\varepsilon_{\sigma_1}(\rho, 2l_1 + 1) = \varepsilon_{\sigma_1}(\rho, 2l_2 + 1) = 1$ and $\varepsilon_{\sigma_2}(\rho, 2l_1 + 1) = \varepsilon_{\sigma_2}(\rho, 2l_2 + 1) = -1$. On the other hand

$$\sigma_1, \sigma_2 \hookrightarrow \delta([\nu^{l_1+1} \rho, \nu^{l_2} \rho]) \ltimes \delta([\nu^{-l_1} \rho, \nu^{l_1} \rho]) \ltimes \sigma_{\text{cusp}}.$$

Let $\delta([\nu^{-l_1} \rho, \nu^{l_1} \rho]) \ltimes \sigma_{\text{cusp}} = T_1 \oplus T_2$; then $\sigma_i$ is a unique subrepresentation of $\delta([\nu^{l_1+1} \rho, \nu^{l_2} \rho]) \ltimes T_i$, $i = 1, 2$. The convention is that $\sigma_1$ is picked in the above way if $T_1$ is generic (only one between $T_1$ and $T_2$ is). Now the claim follows in this case two. (This will not hold when we generalize our considerations to the non-connected case of $O(2n, F)$, where can happen that the Whittaker model is not unique, i.e. both $T_1$ and $T_2$ can be generic (with respect to the same character)).

4 The representation $\delta([\nu^{-l_1} \rho, \nu^{l_2} \rho]) \ltimes \sigma$

In this section we study the generalized principal series representation

$$\delta([\nu^{-l_1} \rho, \nu^{l_2} \rho]) \ltimes \sigma,$$

where $\rho$ is an irreducible, cuspidal representation of $GL(m, F)$ and $\sigma$ is a generic discrete series of $G_n$.

We now give two results which will enable us to put some restrictions on the induced representation we study.

**Theorem 4.1** ([23],[24]). Assume that $\rho \ncong \rho$ or $2l_1 + 1 \notin \mathbb{Z}$. Then, the representation

$$\delta([\nu^{-l_1} \rho, \nu^{l_2} \rho]) \ltimes \sigma$$

is irreducible.
Theorem 4.2. Assume that $\rho \cong \tilde{\rho}$ and $2l_1 + 1 \in \mathbb{Z}$.

(i) If $\text{Jord}_\rho(\sigma) \neq \emptyset$, but $2l_1 - a \notin 2\mathbb{Z}$, $a \in \text{Jord}_\rho(\sigma)$, then $\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$ is irreducible.

(ii) Assume $\text{Jord}_\rho(\sigma) = \emptyset$. Then $\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$ is reducible if and only if $l_1 \geq -\frac{1}{2}$, and $2l_1 + 1$ is even if and only if $L(s,\rho, r)$ has a pole at $s = 0$. If it is reducible, then in appropriate Grothendieck group

$$\delta \rtimes \sigma = \begin{cases} 
\sigma_1 + L(\delta; \sigma), & \text{if } l_1 = -\frac{1}{2}, \\
\sigma_1 + \sigma_2 + L(\delta; \sigma), & \text{if } l_1 \geq 0
\end{cases}$$

where $\sigma_i, i = 1, 2$ is a discrete series such that $\text{Jord}(\sigma_i)$ is equal to

$$\begin{cases} 
\text{Jord}(\sigma) \cup \{(\rho, 2l_2 + 1)\}, & \text{if } l_1 = -\frac{1}{2}, \\
\text{Jord}(\sigma) \cup \{(\rho, 2l_1 + 1), (\rho, 2l_2 + 1)\}, & \text{if } l_1 \geq 0.
\end{cases}$$

and their $\varepsilon_{\sigma_i}, i = 1, 2$ are determined as follows. If $l_1 \geq 0$, there are two possible extensions of $\varepsilon_\sigma$ such that $\varepsilon_{\sigma_i}((\rho, 2l_1 + 1)\varepsilon_{\sigma_i}(\rho, 2l_2 + 1)^{-1} = 1, i = 1, 2$. Moreover $\sigma_1$ and $\sigma_2$ are not isomorphic. If $l_1 = -\frac{1}{2}$ then $\varepsilon_{\sigma_1}(\rho, 2l_2 + 1) = 1$.

Proof. This is Theorem 2.3 of ([14]). \qed

In situation described in the previous theorem, case (ii), we already noted that the generic subquotient is square–integrable ([9],[13],[12]) and that it is a subrepresentation of the standard representation.

These two theorems enable us two impose the following conditions (which will be assumed throughout the whole section). We assume

$$\begin{cases} 
\text{Jord}_\rho(\sigma) \neq \emptyset, \\
l_1 - a \in \mathbb{Z}, \forall 2a + 1 \in \text{Jord}_\rho(\sigma).
\end{cases}$$

(5)

4.1 Discrete series subquotients

Let $2l_1 + 1$, $2l_2 + 1$ be of an appropriate parity (from now on, this means that the second condition of (5), i.e. the condition (i) in the definition of Jord is
satisfied). We study all the possible appearances of the discrete series generic subquotients in the composition series of the representation

$$\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma,$$

where, as before, $\rho$ is self-contragredient cuspidal representation of some $GL(m, F)$ and $\sigma$ is a generic discrete series representation of $G_n$, and the requirements of (5) are fulfilled. We first note the necessary conditions for the appearance of the discrete series subquotient in (6), in terms of the Jordan blocks and Supp(Jord($\sigma$)) ([9], the second section). Supp(Jord($\sigma$)) is defined as a multiset

$$\text{Supp}(\text{Jord}(\sigma)) = \sum_{(\rho, a) \in \text{Jord}(\sigma)} [\nu^{-\frac{a-1}{2}}\rho, \nu^{-\frac{a-1}{2}}\rho].$$

It is evident how to reconstruct Jord($\sigma$) from Supp(Jord($\sigma$)). For (6) to have a discrete series subquotient, a necessary condition is that a cuspidal support of that representation gives rise to valid Supp(Jord($\sigma_1$)) for some discrete series representation $\sigma_1$. By ([12]) we know that if (6) has a square–integrable subquotient, then the generic subquotient is also square–integrable.

### 4.1.1 The case $l_1 \geq 0$

If we additionally require that $[2l_1+1, 2l_2+1] \cap \text{Jord}_\rho(\sigma) = \emptyset$, we obtain a trivial case and we know that there is a discrete series generic subrepresentation of that induced representation. So, further on, we assume that $[2l_1+1, 2l_2+1] \cap \text{Jord}_\rho(\sigma) \neq \emptyset$, and let $[2l_1+1, 2l_2+1] \cap \text{Jord}_\rho(\sigma) = \{2a_1+1, \ldots, 2a_m+1\}$.

From now on, $m$ denotes the number of elements of $[2l_1+1, 2l_2+1] \cap \text{Jord}_\rho(\sigma)$. We will separate our study into several cases, depending on the parity of $m$.

We assume $a_1 < a_2 < \ldots < a_m$. Assume that $\sigma_1 \leq \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$, for some square–integrable $\sigma_1$. Then

$$\text{Supp}(\text{Jord}(\sigma_1)) = \text{Supp}(\text{Jord}(\sigma)) + [-l_1, l_1] + [-l_2, l_2].$$

We know that the Jord($\sigma_1$) has to be multiplicity free, so the only condition we can extract from the previous relation is $2l_1+1$, $2l_2+1 \notin \text{Jord}_\rho(\sigma)$. This turns out to be, is also a sufficient condition.

**Proposition 4.3.** We assume that (5) holds. Assume, additionally, that $l_1 \geq 0$ and $2l_1+1, 2l_2+1 \notin \text{Jord}_\rho(\sigma)$. Then, the generic subquotient of the representation $\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$ is a discrete series representation.
Proof. \( m \) is even

Now, by Proposition 3.1, we know that the value of \( \varepsilon \)-function on each pair of elements of \( \text{Jord}_\rho(\sigma) \) is equal to one. From this follows that

\[
\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \sigma \rightarrow \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho])
\times \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-a_{m-1}}\rho, \nu^{a_m}\rho]) \times \sigma', \tag{7}
\]

for some generic discrete series representation \( \sigma' \). For this generic discrete series we have

\[
\text{Jord}(\sigma') = \text{Jord}(\sigma) \setminus \{2a_1 + 1, \ldots, 2a_m + 1\}.
\]

On the other hand, let \( \sigma'' \) be a unique generic discrete series obtained by consecutive adding of elements \( \{2l_1 + 1, 2a_1 + 1, \ldots, 2a_m + 1\} \) in \( \text{Jord}(\sigma') \). Then, the following holds:

\[
\sigma'' \rightarrow \delta([\nu^{-l_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-a_2}\rho, \nu^{a_3}\rho]) \times \cdots \times \delta([\nu^{-a_{m-2}}\rho, \nu^{a_{m-1}}\rho]) \times \delta([\nu^{-a_{m-1}}\rho, \nu^{l_2}\rho]) \times \sigma'. \tag{8}
\]

When we compare the cuspidal support of the induced representations on the right-hand sides of (7) and (8), we see that the cuspidal support is the same. So, we may form a representation, say \( \pi \), induced from the supercuspidal representation on the Levi subgroup such that \( \sigma'' \) is a unique irreducible generic subquotient in its composition series. This means that \( \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \sigma \leq \pi \), so \( \pi'' \leq \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \sigma \).

\( m \) is odd

Assume now that the number of elements of \( \text{Jord}_\rho(\sigma) \) between 2\( l_1 + 1 \) and 2\( l_2 + 1 (= m) \) is odd, but there exists an element of \( \text{Jord}_\rho(\sigma) \) smaller than 2\( l_1 + 1 \) or greater than 2\( l_2 + 1 \). Assume that 2\( b + 1 \) is the greatest element of \( \text{Jord}_\rho(\sigma) \) smaller than 2\( l_1 + 1 \); the other case is treated similarly. Then

\[
\sigma \rightarrow \delta([\nu^{-b}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-a_2}\rho, \nu^{a_3}\rho]) \times \cdots \times \delta([\nu^{-a_{m-1}}\rho, \nu^{a_m}\rho]) \times \sigma',
\]

for a generic discrete series \( \sigma' \).

On the other hand, the representation

\[
\delta([\nu^{-b}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-a_{m-2}}\rho, \nu^{a_{m-1}}\rho]) \times \delta([\nu^{-a_m}\rho, \nu^{l_2}\rho]) \times \sigma'
\]

16
has a unique generic discrete series subrepresentation; call it \( \sigma'' \) (since
\[
\text{Jord}_\rho(\sigma') \cap [2b + 1, 2l_2 + 1] = \emptyset.
\]
But this representation has the same cuspidal support as \( \delta([\nu^{-l_1} \rho, \nu^{l_2} \rho]) \times \sigma \). This forces \( \sigma'' \leq \delta([\nu^{-l_1} \rho, \nu^{l_2} \rho]) \times \sigma \).

Now, we assume that \( m \) is odd, and that there is no element in \( \text{Jord}_\rho(\sigma) \) greater than \( 2l_2 + 1 \) or smaller than \( 2l_1 + 1 \). This means that \( \text{Jord}_\rho(\sigma) \) has odd number of elements. Then, there exists a unique strongly positive generic discrete series \( \sigma^+ \) with \( \text{Jord}_\rho(\sigma^+) = \{2a_1 + 1\} \) such that \( \sigma \hookrightarrow \delta([\nu^{-a_2} \rho, \nu^{a_3} \rho]) \times \cdots \times \delta([\nu^{-a_m-1} \rho, \nu^{a_m} \rho]) \times \sigma^+ \). There is thus, \( \sigma_{\text{cusp}} \) ((\( \rho \)–cuspidal) generic representation such that \( \sigma^+ \hookrightarrow \delta([\nu^\rho \rho, \nu^{\alpha_1} \rho]) \times \sigma\text{_{cusp}} \). Also, there is a \( \rho \) strongly positive generic \( \sigma^+_1 \) such that \( \sigma^+_1 \hookrightarrow \delta([\nu^\rho \rho, \nu^{\alpha_2} \rho]) \times \sigma_{\text{cusp}} \). The cuspidal support of the representation \( \delta([\nu^{-l_1} \rho, \nu^{l_2} \rho]) \times \sigma \) equals the cuspidal support of the representation
\[
\delta([\nu^{-l_1} \rho, \nu^{\alpha_1} \rho]) \times \delta([\nu^{-a_2} \rho, \nu^{a_3} \rho]) \times \cdots \times \delta([\nu^{-a_m-1} \rho, \nu^{a_m} \rho]) \times \sigma^+_1.
\]
The cuspidal support is equal to the sum of the cuspidal support of \( \sigma_{\text{cusp}} \) and
\[
\sum_{i=1}^m [-a_i, a_i] + [-l_1, l_1] + [-l_2, l_2] \quad (\text{if } \alpha = \frac{1}{2}, \text{ and without one copy of } \rho \text{ in this sum if } \alpha = 1).
\]
But this representation, because of it’s structure, obtained by adding two by two more elements in the Jordan block of \( \sigma^+_1 \) has a unique square–integrable generic subquotient, say \( \sigma'' \). Then \( \sigma'' \leq \delta([\nu^{-l_1} \rho, \nu^{l_2} \rho]) \times \sigma \).

We have now covered all the cases and proved the proposition. We note that Muć obtained a similar result (on appearance of the discrete series subquotient in a standard representation) using \( L \)-functions ([12]).

We now prove that a generic discrete series appearing in as a generic subquotient in the previous proposition (we emphasise that (5) and \( 2l_1 + 1, 2l_2 + 1 \notin \text{Jord}_\rho(\sigma) \) holds) is, actually, a subrepresentation of \( \delta([\nu^{-l_1} \rho, \nu^{l_2} \rho]) \times \sigma \).

**Proposition 4.4.** Assume that (5) holds and that \( 2l_1 + 1, 2l_2 + 1 \notin \text{Jord}_\rho(\sigma) \) with \( l_1 \geq 0 \). Then, the generic discrete series \( \sigma_1 \leq \delta([\nu^{-l_1} \rho, \nu^{l_2} \rho]) \times \sigma \) is a subrepresentation.

**Proof.** Firstly, assume that \( m \) is even. Using a result on \( \varepsilon \)-function for the generic discrete series, we know that there exists a generic discrete series representation \( \sigma' \) such that
\[
\sigma \hookrightarrow \delta([\nu^{-a_1} \rho, \nu^{a_2} \rho]) \times \cdots \times \delta([\nu^{-a_m-1} \rho, \nu^{a_m} \rho]) \times \sigma'.
\]
On the other hand, we know that there is a square–integrable generic subquotient \( \sigma_1 \leq \delta([\nu^{-a_1} \rho, \nu^{a_2} \rho]) \rtimes \sigma \). For \( \sigma_1 \) we have \( \text{Jord}(\sigma_1) = \text{Jord}(\sigma) \cup \{(\rho, 2l_1 + 1), (\rho, 2l_2 + 1)\} \). We conclude, by the standard addition of elements in the Jordan block, that the following holds:

\[
\sigma_1 \hookrightarrow \delta([\nu^{-a_1} \rho, \nu^{a_2} \rho]) \times \cdots \times \delta([\nu^{-a_{m-1}} \rho, \nu^{a_m} \rho]) \times \delta([\nu^{l_1} \rho, \nu^{l_2} \rho]) \rtimes \sigma'.
\]

There exists an intertwining operator acting on the induced representation on the right–hand side of the above relation, induced from the intertwining operator \( T \) acting on the general linear group:

\[
\delta([\nu^{-a_1} \rho, \nu^{a_2} \rho]) \times \cdots \times \delta([\nu^{-a_{m-1}} \rho, \nu^{a_m} \rho]) \times \delta([\nu^{l_1} \rho, \nu^{l_2} \rho]) \rtimes \sigma' \rightarrow \\
\delta([\nu^{-a_1} \rho, \nu^{a_2} \rho]) \times \cdots \times \delta([\nu^{-a_{m-1}} \rho, \nu^{a_m} \rho]) \times \delta([\nu^{-a_{m-1}} \rho, \nu^{a_m} \rho]) \rtimes \sigma'.
\]

The kernel of the homomorphism \( T \) is

\[
\delta([\nu^{-a_1} \rho, \nu^{a_2} \rho]) \times \cdots \times L(\delta([\nu^{l_1} \rho, \nu^{l_2} \rho]), \delta([\nu^{-a_{m-1}} \rho, \nu^{a_m} \rho])) \rtimes \sigma'.
\]

Since not all of the inducing representations are generic, we have \( \sigma_1 \not\in \text{Ker}(T) \), so

\[
\sigma_1 \hookrightarrow \delta([\nu^{-a_1} \rho, \nu^{a_2} \rho]) \times \cdots \times \delta([\nu^{-a_{m-1}} \rho, \nu^{a_m} \rho]) \times \delta([\nu^{l_1} \rho, \nu^{l_2} \rho]) \rtimes \sigma'.
\]

We can reason analogously further (since \( l_1 < a_1 < a_2 < \cdots a_{m-1} < a_m < l_2 \)) and we have

\[
\sigma_1 \hookrightarrow \delta([\nu^{-l_1} \rho, \nu^{l_2} \rho]) \times \delta([\nu^{-a_1} \rho, \nu^{a_2} \rho]) \times \cdots \times \delta([\nu^{-a_{m-1}} \rho, \nu^{a_m} \rho]) \rtimes \sigma'.
\]

We denote the induced representation on the right–hand side of the above relation by \( \pi \); then, the following holds:

\[
\delta([\nu^{-l_1} \rho, \nu^{l_2} \rho]) \rtimes \sigma \hookrightarrow \pi.
\]

This forces \( \sigma_1 \hookrightarrow \delta([\nu^{-l_1} \rho, \nu^{l_2} \rho]) \rtimes \sigma \) (otherwise, the multiplicity of \( \sigma_1 \) in the composition series of the representation \( \pi \) would be greater than one, which is not possible).

Assume that \( m \) is odd, and that there exists \( 2b + 1 \in \text{Jord}_{\rho}(\sigma) \) such that either \( (2a_1 + 1)_- = 2b + 1 \) either \( (2b + 1)_- = 2a_m + 1 \) (in \( \text{Jord}_{\rho}(\sigma) \)). Then, if \( \sigma_1 \) is a generic discrete series subquotient of \( \delta([\nu^{-l_1} \rho, \nu^{l_2} \rho]) \rtimes \sigma \), it is a subrepresentation.
Assume firstly that \((2a_1 + 1)_+ = 2b + 1\). Let \(\sigma''\) be a generic discrete series obtained by removing the elements \(2b + 1, 2a_1 + 1, \ldots, 2a_m + 1\) from \(\text{Jord}_\rho(\sigma)\). This means
\[
\sigma \hookrightarrow \delta([\nu^{-b}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-a_2}\rho, \nu^{a_3}\rho]) \times \cdots \times \delta([\nu^{-a_{m-1}}\rho, \nu^{a_m}\rho]) \rtimes \sigma''. \tag{9}
\]
On the other hand, from Proposition 3.1 we know the \(\varepsilon\) function of \(\sigma_1\). This means that
\[
\sigma_1 \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-a_2}\rho, \nu^{a_3}\rho]) \times \cdots \times \delta([\nu^{-a_{m-1}}\rho, \nu^{a_m}\rho]) \times \delta([\nu^{-b}\rho, \nu^{l_2}\rho]) \rtimes \sigma''. \tag{10}
\]
We apply the intertwining operators induced form the general linear group, and then, using the same reasoning as before (when considering the non–genericity of the kernel of these operators), we obtain that
\[
\sigma_1 \hookrightarrow \delta([\nu^{-b}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-a_2}\rho, \nu^{a_3}\rho]) \times \cdots \times \delta([\nu^{-a_{m-1}}\rho, \nu^{a_m}\rho]) \times \sigma''. \tag{11}
\]
Now we return to relation (9). Using Lemma 3.2 of ([9]), we know that there exists an irreducible generic subquotient \(\pi\) of
\[
\delta([\nu^{-a_2}\rho, \nu^{a_3}\rho]) \times \cdots \times \delta([\nu^{-a_{m-1}}\rho, \nu^{a_m}\rho]) \rtimes \sigma'' \tag{12}
\]
such that \(\sigma \hookrightarrow \delta([\nu^{-b}\rho, \nu^{a_1}\rho]) \rtimes \pi\). But, since (12) has a square–integrable subquotient, \(\pi\) is a unique square–integrable discrete series. From (11) we get that
\[
\sigma_1 \leq \delta([\nu^{-b}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{a_1}\rho]) \rtimes \pi.
\]
But from (11), the fact that
\[
\delta([\nu^{-b}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{a_1}\rho]) \rtimes \pi \hookrightarrow \delta([\nu^{-b}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{a_1}\rho]) \rtimes \sigma''
\]
we get that
\[
\sigma_1 \hookrightarrow \delta([\nu^{-b}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{a_1}\rho]) \rtimes \pi.
\]
Now we have
\[
\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \sigma \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-b}\rho, \nu^{a_1}\rho]) \times \pi \hookrightarrow
\]
\[
\hookrightarrow \delta([\nu^{-b}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{a_1}\rho]) \rtimes \pi. \tag{13}
\]
Since both $\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \sigma$ and $\sigma_1$ are subrepresentations of the representation appearing on the right-hand side of the relation (13), we have $\sigma_1 \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \sigma$. We treat the case where $(2b + 1)_- = 2a_m + 1$ analogously.

Assume that $m$ is odd and, additionally, assume that for every $2a_i + 1 \in \text{Jord}_\rho(\sigma)$, $i = 1, 2, \ldots, a_m$ we have $2l_1 + 1 < 2a_i + 1 < 2l_2 + 1$. Let $\sigma_1$ be a square–integrable generic subquotient of $\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \sigma$.

There exists a strongly positive generic discrete series $\sigma^+_1$ such that $\text{Jord}_\rho(\sigma^+_1) = \{2l_2 + 1\}$, a strongly positive discrete series $\sigma^+$ such that $\text{Jord}_\rho(\sigma^+) = \{2a_1 + 1\}$. Moreover, we have

$$
\sigma_1 \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-a_2}\rho, \nu^{a_3}\rho]) \times \cdots \times \delta([\nu^{-a_{m-1}}\rho, \nu^{a_m}\rho]) \times \sigma^+_1,
$$

$$
\sigma \hookrightarrow \delta([\nu^{-a_2}\rho, \nu^{a_3}\rho]) \times \cdots \times \delta([\nu^{-a_{m-1}}\rho, \nu^{a_m}\rho]) \times \sigma^+.
$$

By the work of Muić ([14]) and Moeglin and Tadić ([9]), we know that $\sigma^+_1 \hookrightarrow \delta([\nu^{a_1+1}\rho, \nu^{l_2}\rho]) \times \sigma^+$. As before, using the intertwining operators, we get that

$$
\sigma_1 \hookrightarrow \delta([\nu^{a_1+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-a_2}\rho, \nu^{a_3}\rho]) \times \cdots \times \delta([\nu^{-a_{m-1}}\rho, \nu^{a_m}\rho]) \times \sigma^+.
$$

By the work of Muić ([14]) and Moeglin and Tadić ([9]), we know that $\sigma^+_1 \hookrightarrow \delta([\nu^{a_1+1}\rho, \nu^{l_2}\rho]) \times \sigma^+$. As before, using the intertwining operators, we get that

$$
\sigma_1 \hookrightarrow \delta([\nu^{a_1+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-a_2}\rho, \nu^{a_3}\rho]) \times \cdots \times \delta([\nu^{-a_{m-1}}\rho, \nu^{a_m}\rho]) \times \sigma^+.
$$

On the other hand, by known $GL$–theory and above relation for $\sigma$, the following holds:

$$
\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \sigma \hookrightarrow \delta([\nu^{a_1+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{-a_2}\rho, \nu^{a_3}\rho]) \times \cdots \times \delta([\nu^{-a_{m-1}}\rho, \nu^{a_m}\rho]) \times \sigma^+.
$$

So, both $\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \sigma$ and $\sigma_1$ are subrepresentations of the representation appearing on the right-hand side of (14). This forces

$$
\sigma_1 \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \sigma.
$$

\[4.1.2\] \hspace{1cm} l_1 < 0

Firstly, we treat the case $l_1 = -\frac{1}{2}$. Assume that $\sigma_1 \leq \delta([\nu^{\frac{1}{2}\rho}, \nu^{l_2}\rho]) \times \sigma$, for some square–integrable $\sigma_1$. Then

$$
\text{Supp}(\text{Jord}(\sigma_1)) = \text{Supp}(\text{Jord}(\sigma)) + [\nu^{l_2}\rho, \nu^{l_2}\rho].
$$

The necessary condition on the support is that $2l_2 + 1 \notin \text{Jord}_\rho(\sigma)$. It turns out that this is also a sufficient condition (assuming (5)).
Proposition 4.5. Assume that (5) holds. If $2l_2 + 1 \not\in \text{Jord}_\rho(\sigma)$, the representation \( \delta([\nu^{\frac{1}{2}}\rho, \nu^{l_2}]) \times \sigma \) has a square–integrable generic subquotient \( \sigma_1 \) which is a subrepresentation.

Proof. If \( \text{Jord}_\rho(\sigma) \) has even number of elements, then \( \text{Jord}_\rho(\sigma^+) \) is empty. This means that \( \nu^{\frac{1}{2}}\rho \times \sigma_{\text{cusp}} \) reduces. Let \( \text{Jord}_\rho(\sigma) = \{2a_1 + 1, 2a_2 + 1, \ldots, 2a_m + 1\} \). Let \( A = \{i : 2a_i + 1 < 2l_2 + 1\} \). If \( A \) has even number of elements, say \( \{2a_1 + 1, \ldots, 2a_k + 1\} \) (and \( m - |A| \) is then also even), then there exists a generic discrete series \( \sigma_1 \) such that the following holds:

\[
\sigma_1 \hookrightarrow \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-a_{k-1}}\rho, \nu^{a_k}\rho]) \times \delta([\nu^{-a_k+1}\rho, \nu^{a_{k+1}}\rho]) \times \\
\cdots \times \delta([\nu^{-a_m-1}\rho, \nu^{a_m}\rho]) \times \sigma^+.
\]

Here, \( \sigma^+ \) is a generic, \( \rho \)–strongly positive discrete series with \( \text{Jord}_\rho(\sigma^+) = \{2l_2 + 1\} \). This means that \( \sigma^+ \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{l_2}\rho]) \times \sigma_{\text{cusp}} \). Here, \( \sigma_{\text{cusp}} \) is \( \rho \)–cuspidal (as explained in Preliminaries). Now, when we plug this to the previous relation, we note that

\[
\delta([\nu^{-a_1}\rho, \nu^{a_{i+1}}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{l_2}\rho]) \cong \delta([\nu^{\frac{1}{2}}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-a_i}\rho, \nu^{a_{i+1}}\rho])
\]

if \( a_{i+1} > l_2 \). If \( a_{i+1} < l_2 \), by observing that the kernel of an appropriate \( GL \)–intertwining operator is not generic (the same reasoning as before), we get

\[
\sigma_1 \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-a_{m-1}}\rho, \nu^{a_m}\rho]) \times \sigma_{\text{cusp}}.
\]

Since

\[
\sigma \hookrightarrow \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-a_{m-1}}\rho, \nu^{a_m}\rho]) \times \sigma_{\text{cusp}},
\]

using the multiplicity one for the generic subquotient, we get that

\[
\sigma_1 \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{l_2}\rho]) \times \sigma.
\]

If \( |A| = k \) is odd, then we can embed \( \sigma_1 \) in the following way:

\[
\sigma_1 \hookrightarrow \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-a_{k-2}}\rho, \nu^{a_{k-1}}\rho]) \times \delta([\nu^{-a_k+1}\rho, \nu^{a_{k+2}}\rho]) \times \\
\cdots \times \delta([\nu^{-a_m-2}\rho, \nu^{a_{m-1}}\rho]) \times \delta([\nu^{-a_k}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a_m}\rho]) \times \sigma_{\text{cusp}}. \quad (15)
\]
On the other hand, we know that we have the following embedding of \( \sigma \)

\[
\sigma \hookrightarrow \delta([\nu^{-a_1} \rho, \nu^{a_2} \rho]) \times \cdots \times \delta([\nu^{-a_k-2} \rho, \nu^{a_k-1} \rho]) \times \delta([\nu^{-a_k+1} \rho, \nu^{a_k+2} \rho]) \times \cdots \times \\
\delta([\nu^{-a_m-2} \rho, \nu^{a_m-1} \rho]) \times \delta([\nu^{-a_k} \rho, \nu^{a_m} \rho]) \times \sigma_{cusp}.
\]

We apply intertwining operators to (15) and note the following: the kernel of the intertwining operator

\[
\delta([\nu^{-a_1} \rho, \nu^{a_1+1} \rho]) \times \delta([\nu^{1/2} \rho, \nu^{a_m} \rho]) \rightarrow \delta([\nu^{1/2} \rho, \nu^{a_m} \rho]) \times \delta([\nu^{-a_1} \rho, \nu^{a_1+1} \rho])
\]

for \( a_{i+1} < a_m \) is never generic. Also, for \( i \leq k - 1 \), \( \delta([\nu^{-a_i-1} \rho, \nu^{a_i} \rho]) \times \delta([\nu^{-a_k} \rho, \nu^{1/2} \rho]) \) is irreducible. The same is true for \( i \geq k + 2 \). This forces:

\[
\sigma_1 \hookrightarrow \delta([\nu^{1/2} \rho, \nu^{a_m} \rho]) \times \delta([\nu^{-a_k} \rho, \nu^{1/2} \rho]) \times \delta([\nu^{-a_1} \rho, \nu^{a_2} \rho]) \times \cdots \times \\
\delta([\nu^{-a_k-2} \rho, \nu^{a_k-1} \rho]) \times \delta([\nu^{-a_k+1} \rho, \nu^{a_k+2} \rho]) \times \cdots \times \delta([\nu^{-a_m-2} \rho, \nu^{a_m-1} \rho]) \times \sigma_{cusp}.
\]

We denote

\[
\pi = \delta([\nu^{-a_1} \rho, \nu^{a_2} \rho]) \times \cdots \times \delta([\nu^{-a_k-2} \rho, \nu^{a_k-1} \rho]) \times \delta([\nu^{-a_k+1} \rho, \nu^{a_k+2} \rho]) \times \cdots \times \\
\delta([\nu^{-a_m-2} \rho, \nu^{a_m-1} \rho]).
\]

This means

\[
\sigma \hookrightarrow \pi \times \delta([\nu^{-a_k} \rho, \nu^{a_m} \rho]) \times \sigma_{cusp},
\]

and

\[
\sigma_1 \hookrightarrow \delta([\nu^{1/2} \rho, \nu^{a_m} \rho]) \times \delta([\nu^{-a_k} \rho, \nu^{1/2} \rho]) \times \pi \times \sigma_{cusp}.
\]

Moreover, applying intertwining operators as before (for \( \sigma_1 \)) we get

\[
\sigma \hookrightarrow \delta([\nu^{-a_k} \rho, \nu^{a_m} \rho]) \times \pi \times \sigma_{cusp}.
\]

From this relation, we obtain:

\[
\delta([\nu^{1/2} \rho, \nu^{1/2} \rho]) \times \sigma \hookrightarrow \delta([\nu^{1/2} \rho, \nu^{1/2} \rho]) \times \delta([\nu^{-a_k} \rho, \nu^{a_m} \rho]) \times \pi \times \sigma_{cusp}.
\]

But the representation on the right-hand side here is a subrepresentation of the right-hand side of (16). We thus obtain

\[
\sigma_1 \hookrightarrow \delta([\nu^{1/2} \rho, \nu^{1/2} \rho]) \times \sigma.
\]

22
If $m$ is odd, either $k$ or $m-k$ are even. We say a few words about the case of even $k$, the other case is very similar. Analogously as above, we have

$$
\sigma_1 \hookrightarrow \delta([\nu^{-a_1} \rho, \nu^{a_2} \rho]) \times \cdots \times \delta([\nu^{-a_{k-1}} \rho, \nu^{a_k} \rho]) \times \delta([\nu^{-l_2} \rho, \nu^{a_{k+1}} \rho]) \times \delta([\nu^{-a_{k+2}} \rho, \nu^{a_{k+3}} \rho]) \times \cdots \times \delta([\nu^{-a_{m-1}} \rho, \nu^{a_m} \rho]) \rtimes \sigma_{\text{cusp}},
$$

and

$$
\sigma \hookrightarrow \delta([\nu^{-a_1} \rho, \nu^{a_2} \rho]) \times \cdots \times \delta([\nu^{-a_{k-1}} \rho, \nu^{a_k} \rho]) \times \delta([\nu^{-a_{k+2}} \rho, \nu^{a_{k+3}} \rho]) \times \cdots \times \delta([\nu^{-a_{m-1}} \rho, \nu^{a_m} \rho]) \times \delta([\nu^1 \rho, \nu^{a_{k+1}} \rho]) \rtimes \sigma_{\text{cusp}}.
$$

With

$$
\pi = \delta([\nu^{-a_1} \rho, \nu^{a_2} \rho]) \times \cdots \times \delta([\nu^{-a_{k-1}} \rho, \nu^{a_k} \rho]) \times \delta([\nu^{-a_{k+2}} \rho, \nu^{a_{k+3}} \rho]) \times \cdots \times \delta([\nu^{-a_{m-1}} \rho, \nu^{a_m} \rho]) \quad (17)
$$

we have

$$
\sigma \hookrightarrow \delta([\nu^1 \rho, \nu^{a_{k+1}} \rho]) \times \pi \rtimes \sigma_{\text{cusp}}
$$

and

$$
\sigma_1 \hookrightarrow \delta([\nu^{-l_2} \rho, \nu^{a_{k+1}} \rho]) \times \pi \rtimes \sigma_{\text{cusp}} \hookrightarrow \delta([\nu^1 \rho, \nu^{a_{k+1}} \rho]) \times \delta([\nu^{-l_2} \rho, \nu^1 \rho]) \times \pi \rtimes \sigma_{\text{cusp}}.
$$

This means

$$
\delta([\nu^1 \rho, \nu^{a_2} \rho]) \rtimes \sigma \hookrightarrow \delta([\nu^1 \rho, \nu^{a_{k+1}} \rho]) \times \delta([\nu^1 \rho, \nu^{a_2} \rho]) \times \pi \rtimes \sigma_{\text{cusp}}.
$$

We want to relate the last two relations, describing the embeddings of $\sigma_1$ and of $\delta([\nu^1 \rho, \nu^{a_2} \rho]) \rtimes \sigma$. We have to check is whether $\sigma_1$ is in the kernel of the intertwining operator $T$ induced by the intertwining operator $T'$ acting on a smaller group

$$
T' : \delta([\nu^{-l_2} \rho, \nu^{-1} \rho]) \rtimes \sigma_{\text{cusp}} \rightarrow \delta([\nu^1 \rho, \nu^{a_2} \rho]) \rtimes \sigma_{\text{cusp}}.
$$

This is a non–standard intertwining operator; it exists since the length of the representation $\delta([\nu^1 \rho, \nu^{a_2} \rho]) \rtimes \sigma_{\text{cusp}}$ is two, and the quotient of the representation $\delta([\nu^{-l_2} \rho, \nu^{-1} \rho]) \rtimes \sigma_{\text{cusp}}$ is a subrepresentation of $\delta([\nu^1 \rho, \nu^{a_2} \rho]) \rtimes \sigma_{\text{cusp}}$. 23
This intertwining operator is a unique, up to a scalar. We have \( \text{Im}T' = \sigma^+ \). But if this were the case, and if \( T \) is the intertwining operator

\[
\delta([\nu^{\frac{1}{2}}\rho, \nu^{\alpha_k+1}\rho]) \times \pi \times \delta([\nu^{-l_2}\rho, \nu^{-\frac{1}{2}}\rho]) \times \sigma_{\text{cusp}} \rightarrow \\
\delta([\nu^{\frac{1}{2}}\rho, \nu^{\alpha_k+1}\rho]) \times \pi \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{l_2}\rho]) \times \sigma_{\text{cusp}}
\]

then there would exist an epimorphism

\[
\delta([\nu^{\frac{1}{2}}\rho, \nu^{\alpha_k+1}\rho]) \times \pi \times \delta([\nu^{-l_2}\rho, \nu^{-\frac{1}{2}}\rho]) \times \sigma_{\text{cusp}} \rightarrow \text{Im}(T).
\]

On the left–hand side of the above relation we have a non-generic representation, and on the right–hand side–a generic representation, and this cannot hold.

Now we treat the case \( l_1 \leq -1 \). Denote \( a = -l_1 - 1 \geq 0 \). Then, for possible square–integrable subquotient \( \sigma_1 \) of \( \delta([\nu^{\alpha_1+1}\rho, \nu^{l_2}\rho]) \times \sigma \) we have

\[
\text{Supp}(\text{Jord}(\sigma_1)) = \text{Supp}(\text{Jord}(\sigma)) + [\nu^{a+1}\rho, \nu^{l_2}\rho] + [\nu^{-l_2}\rho, \nu^{-a-1}\rho].
\]

The right–hand side of the previous relation can be Jordan support of a square–integrable representation only if \( 2a + 1 \in \text{Jord}_\rho(\sigma) \); then \( \text{Jord}(\sigma_1) = \text{Jord}(\sigma) \setminus \{(\rho, 2a + 1)\} \cup \{(\rho, 2l_2 + 1)\} \). This is also a sufficient condition.

**Proposition 4.6.** Assume that (5) holds and that, with \( a \) as above, \( 2a + 1 \in \text{Jord}_\rho(\sigma) \) and \( 2l_2 + 1 \notin \text{Jord}_\rho(\sigma) \). Then, the representation

\[
\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \sigma
\]

has a generic discrete series subrepresentation.

**Proof.** Let \( \text{Jord}_\rho(\sigma) = \{2a_1 + 1, \ldots, 2a_m + 1\} \) so that \( 2a + 1 = 2a_{i_0} + 1 \) for some \( i_0 \in \{1, 2, \ldots, m\} \) and let \( \sigma_1 \) be a generic discrete series representation which has the same partial cuspidal support as \( \sigma \) and \( \text{Jord}(\sigma_1) = (\text{Jord}(\sigma) \setminus \{2a_{i_0} + 1\}) \cup \{2l_2 + 1\} \) (\( \sigma_1 \) is uniquely determined by this data). We assume that the sequence \( a_1, \ldots, a_m \) is increasing. If \( |\text{Jord}_\rho(\sigma)| \) is even, the following holds:

\[
\sigma \leftrightarrow \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-a_{m-1}}\rho, \nu^{a_m}\rho]) \times \sigma_{\text{cusp,} \rho}
\]
and analogously for $\sigma_1$, where we drop $a_{i_0}$ and insert $l_2$ on the proper place in $a_1 < a_2 < \ldots < a_{i_0} < \ldots < a_m$ so that the new sequence would again be increasing. This means that the representation $\sigma_1$ and $\delta([\nu^{a_1} \rho, \nu^{l_2} \rho]) \rtimes \sigma$ have the same cuspidal support, so, there exists a generic (standard) representation $\pi = \rho^\alpha \times \cdots \times \rho^\alpha_k \rtimes \sigma_{\text{cusp}, \rho}$ such that

$$
\delta([\nu^{a_1} \rho, \nu^{l_2} \rho]) \rtimes \sigma \leq \pi, \quad \sigma_1 \leq \pi.
$$

To prove that $\sigma_1$ is a subrepresentation, we proceed similarly as in $l_1 = -\frac{1}{2}$ case. Let $2b_1 + 1, \ldots, 2b_k + 1$ be the elements of $\text{Jord}_\rho(\sigma)$ between $2a + 1$ and $2l_2 + 1$ ($\notin \text{Jord}_\rho(\sigma)$). We analyze the cases in which $k = 0$, $k$ is even and $k$ is odd. If $k = 0$ we proceed as follows. If $\text{Jord}_\rho(\sigma) = 2a + 1$, we are, essentially, in the strongly positive situation, where the required embedding follows from the work of Moeglin and Tadić ([9]). If $2a + 1$ is defined, there is a generic discrete series $\sigma_0$ such that

$$
\delta([\nu^{a_1} \rho, \nu^{l_2} \rho]) \rtimes \sigma \hookrightarrow \delta([\nu^{a_1} \rho, \nu^{l_2} \rho]) \rtimes \delta([\nu^{-a_1} \rho, \nu^a \rho]) \rtimes \sigma_0.
$$

The representation $\delta([\nu^{a_1} \rho, \nu^{l_2} \rho]) \rtimes \sigma_0$ is a subrepresentation of the right-hand side above, and, as $[2a_+ + 1, 2l_2 + 1] \cap \text{Jord}_\rho(\sigma_0) = \emptyset$ holds, $\sigma_1$ is it’s subrepresentation. On the other hand, if $2a + 1$ is the smallest element in $\text{Jord}_\rho(\sigma)$, let $(2b + 1)_- = 2a + 1$. Then, for appropriate $\sigma_0 :$

$$
\delta([\nu^{a_1} \rho, \nu^{l_2} \rho]) \rtimes \sigma \hookrightarrow \delta([\nu^{a_1} \rho, \nu^{l_2} \rho]) \rtimes \delta([\nu^{-a_1} \rho, \nu^b \rho]) \rtimes \sigma_0 \cong \delta([\nu^{-a_1} \rho, \nu^b \rho]) \times \delta([\nu^{a_1} \rho, \nu^{l_2} \rho]) \rtimes \sigma_0.
$$

By ([15]), since $[2a_+ + 1, 2l_2 + 1] \cap \text{Jord}_\rho(\sigma_0) = \emptyset$, the representation $\delta([\nu^{a_1} \rho, \nu^{l_2} \rho]) \rtimes \sigma_0$ is irreducible, so the right–hand side of the above relation is isomorphic to

$$
\delta([\nu^{-a_1} \rho, \nu^b \rho]) \times \delta([\nu^{-l_2} \rho, \nu^{-(a+1)} \rho]) \rtimes \sigma_0.
$$

The representation $\delta([\nu^{-l_2} \rho, \nu^b \rho]) \rtimes \sigma_0$ is a subrepresentation of this representation, and so is $\sigma_1$ and the claim is proved in this case. To prove that $\sigma_1$ is a subrepresentation in the case $k > 0$, we split the analysis to the case when $k$ is odd, and to the case $k$ is even, analogously to the case $l = -\frac{1}{2}$ so we do not repeat the arguments.
4.2 Tempered subquotients

Analogously to the notion of Supp((\text{Jord}(\sigma))) where \sigma is discrete series, we have the same notion for tempered \sigma ([9], Section 13). We now analyze when the tempered (non–square–integrable) subquotients appear in the composition series of the representation \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma.

**Proposition 4.7.** Assume that (5) holds with \(l_1 \geq 0\). Further, assume that \(2l_1 + 1 \in \text{Jord}_{\nu}(\sigma)\) or \(2l_2 + 1 \in \text{Jord}_{\rho}(\sigma)\). Then, the representation \(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma\) has a tempered (non–square integrable) generic subquotient. Moreover, this subquotient is a subrepresentation of \(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma\).

**Proof.** Now, if \(2l_1 + 1 \in \text{Jord}_{\rho}(\sigma), \ 2l_2 + 1 \notin \text{Jord}_{\rho}(\sigma)\), we proceed as follows:

\[
\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma \leq \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma.
\]

By the previous section the representation \(\delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma\) has a generic square–integrable subquotient, so the right–hand side of the previous relation has a tempered generic subquotient, say \(T\), and so does \(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma\).

We have the following sequence of homomorphisms:

\[
T \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma \rightarrow \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma,
\]

where the second homomorphism is induced by the obvious \(GL\)–intertwining.

Since \(T\) is not in the kernel of the second homomorphism (which is a degenerate representation),

\[
T \hookrightarrow \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_1}\rho]) \rtimes \sigma.
\]

The representation \(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma\) is a subrepresentation of the representation on the right–hand side, and the claim follows.

Now assume that \(2l_1 + 1 \notin \text{Jord}_{\rho}(\sigma), \ 2l_2 + 1 \in \text{Jord}_{\rho}(\sigma)\). We know that there is a tempered generic subquotient, too. From the proof of the last case in the previous subsection, we know that there exists a generic discrete series, say, \(\sigma_0\), such that \(\text{Jord}(\sigma_0) = \text{Jord}(\sigma) \setminus \{(\rho, 2l_2 + 1) \cup \{(\rho, 2l_1 + 1)\}\), moreover \(\sigma \hookrightarrow \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma_0\). Then, we have the following sequence of homomorphisms

\[
\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma_0 \cong
\]

\[
\cong \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma_0 = \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) \rtimes \sigma_0. \quad (18)
\]
The last equality is in the Grothendieck group. The representation
\[
\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \times \sigma_0 \text{ is a subrepresentation of the last representation on the right--hand side. Let } T \text{ be an irreducible tempered generic subquotient of } \\
\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \times \sigma_0, \text{ which, then, is the generic subquotient of } \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \sigma. \text{ Then, we have the following sequence of homomorphisms:}
\]
\[
T \hookrightarrow \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \times \sigma_0 \hookrightarrow \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) \times \sigma_0.
\]
We conclude by, for example, ([9], Lemma 3.2), that there is an irreducible subquotient, say \( \pi' \), of \( \delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) \) such that
\[
T \hookrightarrow \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \times \pi'.
\]
\( \pi' \) has to be generic, so \( \pi' \) is actually a tempered subquotient of \( \delta([\nu^{-l_2}\rho, \nu^{l_1}\rho]) \times \sigma_0 \), and, this is, by the previous case, a subrepresentation of \( \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \sigma_0 \). So, we have
\[
T \hookrightarrow \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \sigma_0 \cong \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{l_1+1}\rho, \nu^{l_2}\rho]) \times \sigma_0.
\]
Now, by (18), it follows that \( T \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \sigma \). The case \( 2l_1 + 1, 2l_2 + 1 \in \text{Jord}_\rho(\sigma) \) is just an easy combination of the two cases we have described so we skip further explanations. \( \square \)

Now we settle the appearances of the tempered generic subquotients for \( l_1 = -\frac{1}{2} \) and \( l_1 \leq -1 \). In the latter case, we again denote \( a = -l_1 - 1 \).

**Proposition 4.8.** Assume that (5) holds. Then, if \( l_1 = -\frac{1}{2} \) and \( 2l_2 + 1 \in \text{Jord}_\rho(\sigma) \), the representation \( \delta([\nu^{\frac{1}{2}}\rho, \nu^{l_2}\rho]) \times \sigma \) has an irreducible tempered generic subrepresentation. Also, if \( l_1 \leq -1 \), and \( 2a + 1, 2l_2 + 1 \in \text{Jord}_\rho(\sigma) \), the representation \( \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \sigma \) has an irreducible tempered generic subrepresentation.

**Proof.** Note that the reducibility result for \( \delta([\nu^{\frac{1}{2}}\rho, \nu^{l_2}\rho]) \times \sigma \) agrees with results in ([15]) since, from Proposition 3.1 \( \varepsilon(\pi, 2l_2 + 1) = 1 \). Assume first that \( \text{Jord}_\rho(\sigma) \setminus \{2l_2 + 1\} \neq \emptyset \). If there is \( 2l_3 + 1 \in \text{Jord}_\rho(\sigma) \) such that \( (2l_3 + 1)_{-} = 2l_2 + 1 \), we have the following:
\[
\delta([\nu^{\frac{1}{2}}\rho, \nu^{l_2}\rho]) \times \sigma \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_3}\rho]) \times \sigma' \cong \\
\cong \delta([\nu^{-l_2}\rho, \nu^{l_3}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{l_2}\rho]) \times \sigma'.
\]
for a generic discrete series \( \sigma' \). Since \( 2l_2 + 1 \notin \text{Jord}_\rho(\sigma') \), we can apply Proposition 4.5, so there exists a generic discrete series \( \sigma_0 \) such that \( \sigma_0 \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{l_2}\rho]) \rtimes \sigma' \). Now we can apply Proposition 4.7 to get tempered generic \( T \hookrightarrow \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_0 \), and the claim follows. The procedure is the same if there is \( 2l_1 + 1 \in \text{Jord}_\rho(\sigma) \) such that \( (2l_2 + 1)_{-} = 2l_1 + 1 \).

On the other hand, if \( \text{Jord}_\rho(\sigma) = \{ 2l_2 + 1 \} \), \( \sigma \) is \( \rho \)-strongly positive, and there exists a generic discrete series \( \sigma_{\text{cusp}, \rho} \) such that \( \sigma \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{l_2}\rho]) \rtimes \sigma_{\text{cusp}, \rho} \). Then, the following holds:

\[
\delta([\nu^{\frac{1}{2}}\rho, \nu^{l_2}\rho]) \rtimes \sigma \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{l_2}\rho]) \rtimes \sigma_{\text{cusp}, \rho} = \delta([\nu^{\frac{1}{2}}\rho, \nu^{l_2}\rho]) \rtimes \delta([\nu^{-l_2}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma_{\text{cusp}, \rho}.
\]

The representation \( \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_{\text{cusp}, \rho} \) is a subrepresentation of the last representation above, and has a tempered generic subrepresentation \( T \). So, we have \( T \hookrightarrow \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \rtimes \sigma_{\text{cusp}, \rho} \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma_{\text{cusp}, \rho} \).

Now, analogously as in Proposition 4.7, we discuss the generic subquotient of \( \delta([\nu^{-l_2}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma_{\text{cusp}, \rho} \) and the claim follows.

We treat the case \( l_1 \leq -1 \) analogously as \( l_1 = -\frac{1}{2} \), but here we only have the case \( \text{Jord}_\rho(\sigma) \setminus \{ 2l_2 + 1 \} \neq \emptyset \). The reasoning is the same. \( \square \)

### 4.3 Non–tempered generic subquotients

#### Conclusions
If (5) does not hold, either the representation \( \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma \) is irreducible, either has a known structure; in both cases generalized injectivity trivially holds. So, assume that (5) holds. We have proved that if \( l_1 \geq 0 \), the generic subquotient is tempered and generalized injectivity holds. If \( l_1 = -\frac{1}{2} \), the generic subquotient is again tempered and the generalized injectivity holds, similarly as in the case \( l_1 \leq -1 \), when \( 2a + 1 = -2l_1 - 1 \in \text{Jord}_\rho(\sigma) \).

So, the only case left to consider is \( l_1 \leq -1 \) and \( -2l_1 - 1 = 2a + 1 \notin \text{Jord}_\rho(\sigma) \). Using the reducibility results of ([15], e.g. Introduction) we see that, in this case, the only instance when the representation \( \delta([\nu^{a+1}\rho, \nu^{l_2}\rho]) \rtimes \sigma \) is reducible is when \( |2a + 1, 2l_2 + 1| \cap \text{Jord}_\rho(\sigma) \neq \emptyset \).

**Proposition 4.9.** Assume that the conditions of the discussion above hold. Let \( 2\beta_1 + 1 = \min\{2\beta + 1 : 2\beta + 1 \in \{2a + 1, 2l_2 + 1\} \cap \text{Jord}_\rho(\sigma)\} \). Then, the
representation \( \delta([\nu^{a+1} \rho, \nu^{b_1} \rho]) \times \tau \) is irreducible, generic subrepresentation of \( \delta([\nu^{a+1} \rho, \nu^{l_2} \rho]) \rtimes \sigma \). Here \( \tau \) equals a square–integrable generic subrepresentation of \( \delta([\nu^{b_1+1} \rho, \nu^{l_2} \rho]) \rtimes \sigma \) if \( 2l_2 + 1 \notin \text{Jord}_\rho(\sigma) \) or equals a generic tempered subrepresentation of \( \delta([\nu^{b_1+1} \rho, \nu^{l_2} \rho]) \rtimes \sigma \) if \( 2l_2 + 1 \in \text{Jord}_\rho(\sigma) \) (Propositions 4.6 and 4.8).

**Proof.** Assume that \( 2l_2 + 1 \notin \text{Jord}_\rho(\sigma) \). Then the following holds:

\[
\delta([\nu^{a+1} \rho, \nu^{l_2} \rho]) \rtimes \sigma \leq \delta([\nu^{a+1} \rho, \nu^{b_1} \rho]) \times \delta([\nu^{b_1+1} \rho, \nu^{l_2} \rho]) \rtimes \sigma.
\]

Since \( 2b_1 + 1 \in \text{Jord}_\rho(\sigma) \), by Proposition 4.6, the representation

\[
\delta([\nu^{b_1+1} \rho, \nu^{l_2} \rho]) \rtimes \sigma
\]

has a square–integrable generic subrepresentation, say, \( \sigma_1 \), such that \( \text{Jord}(\sigma_1) = \text{Jord}(\sigma) \setminus \{(\rho, 2b_1 + 1) \cup \{(\rho, 2l_2 + 1) \} \}. \) The representation \( \delta([\nu^{a+1} \rho, \nu^{b_1} \rho]) \rtimes \sigma_1 \) is irreducible ([15], Introduction), and is the generic subquotient of \( \delta([\nu^{a+1} \rho, \nu^{l_2} \rho]) \rtimes \sigma \). To prove that it is actually a subrepresentation, we follow the same procedure as before. We examine a sequence of homomorphisms:

\[
\delta([\nu^{a+1} \rho, \nu^{b_1} \rho]) \rtimes \sigma_1 \hookrightarrow \delta([\nu^{a+1} \rho, \nu^{b_1} \rho]) \times \delta([\nu^{b_1+1} \rho, \nu^{l_2} \rho]) \rtimes \sigma \\
\rightarrow \delta([\nu^{b_1+1} \rho, \nu^{l_2} \rho]) \times \delta([\nu^{a+1} \rho, \nu^{b_1} \rho]) \rtimes \sigma.
\]

Since the kernel of the last homomorphism (induced from the obvious one in \( \text{GL} \) setting) is not generic, we have

\[
\delta([\nu^{a+1} \rho, \nu^{b_1} \rho]) \rtimes \sigma_1 \hookrightarrow \delta([\nu^{b_1+1} \rho, \nu^{l_2} \rho]) \times \delta([\nu^{a+1} \rho, \nu^{b_1} \rho]) \rtimes \sigma.
\]

Since

\[
\delta([\nu^{a+1} \rho, \nu^{l_2} \rho]) \rtimes \sigma \hookrightarrow \delta([\nu^{b_1+1} \rho, \nu^{l_2} \rho]) \times \delta([\nu^{a+1} \rho, \nu^{b_1} \rho]) \rtimes \sigma
\]

the claim follows.

The discussion in the case \( 2l_2 + 1 \in \text{Jord}_\rho(\sigma) \) is the same, as soon as we prove that the representation \( \delta([\nu^{a+1} \rho, \nu^{b_1} \rho]) \rtimes T \) is irreducible, where \( T \) is a generic tempered subrepresentation of \( \delta([\nu^{b_1+1} \rho, \nu^{l_2} \rho]) \rtimes \sigma \). Now, we have \( T \hookrightarrow \delta([\nu^{l_2} \rho, \nu^{l_2} \rho]) \rtimes \sigma' \), where \( \sigma' \) is a generic discrete series such that \( \sigma \hookrightarrow \delta([\nu^{b_1} \rho, \nu^{l_2} \rho]) \rtimes \sigma' \). The representation \( \delta([\nu^{a+1} \rho, \nu^{b_1} \rho]) \times \delta([\nu^{l_2} \rho, \nu^{l_2} \rho]) \) is irreducible, and the representation \( \delta([\nu^{a+1} \rho, \nu^{b_1} \rho]) \rtimes \sigma' \) is irreducible (by [15]), and this is enough to conclude that \( \delta([\nu^{a+1} \rho, \nu^{b_1} \rho]) \rtimes T \) is irreducible (by [16], Lemma 2.4). \qed
5 Standard representations

We now study the generalized injectivity conjecture for the representations
\( \delta([\nu^{-1} \rho, \nu^{1/2} \rho]) \times \tau \), where \( \tau \) is a tempered (non-square-integrable), generic representation.

For a tempered representation \( \tau \) there exist square-integrable representations \( \delta_1, \delta_2, \ldots, \delta_k \) of the general linear groups and a discrete series representation \( \sigma \) of a classical group such that \( \tau \hookrightarrow \delta_1 \times \cdots \times \delta_k \times \sigma \).

**Proposition 5.1.** Assume that the representation \( \delta([\nu^{-1} \rho, \nu^{1/2} \rho]) \times \sigma \) has a tempered (generic) subquotient. Then, the representation \( \delta([\nu^{-1} \rho, \nu^{1/2} \rho]) \times \tau \) has a tempered generic subrepresentation.

**Proof.** Let \( T \) be a tempered generic subrepresentation of \( \delta([\nu^{-1} \rho, \nu^{1/2} \rho]) \times \sigma \). Then, we have the following sequence of homomorphisms:

\[
T' \hookrightarrow \delta_1 \times \cdots \times \delta_k \times T \hookrightarrow \delta_1 \times \cdots \times \delta_k \times \delta([\nu^{-1} \rho, \nu^{1/2} \rho]) \times \sigma \rightarrow \\
\delta_1 \times \cdots \times \delta_{k-1} \times \delta([\nu^{-1} \rho, \nu^{1/2} \rho]) \times \delta_k \times \sigma \rightarrow \\
\delta([\nu^{-1} \rho, \nu^{1/2} \rho]) \times \delta_1 \times \cdots \times \delta_k \times \sigma.
\]

The second and all the subsequent homomorphisms are induced from the \( GL \) case. This homomorphisms are either isomorphisms (if \( \delta_i \times \delta([\nu^{-1} \rho, \nu^{1/2} \rho]) \) is irreducible, \( i \in \{1, 2, \ldots, k\} \)) either their kernels are (non-generic!) Langlands quotients if this representation is reducible. This follows from the fact that \( \delta([\nu^{-1} \rho, \nu^{1/2} \rho]) \) has a central character in the positive Weyl chamber, and \( \delta_i \) is unitarizable. In any case, the generic representation \( T' \) (which is a unique generic subrepresentation of \( \delta_1 \times \cdots \times \delta_k \times T \)) cannot embed in these kernels, so we have

\[
T' \hookrightarrow \delta([\nu^{-1} \rho, \nu^{1/2} \rho]) \times \delta_1 \times \cdots \times \delta_k \times \sigma.
\]

Also

\[
\delta([\nu^{-1} \rho, \nu^{1/2} \rho]) \times \tau \hookrightarrow \delta([\nu^{-1} \rho, \nu^{1/2} \rho]) \times \delta_1 \times \cdots \times \delta_k \times \sigma
\]

and the claim follows. \( \square \)

We are left to prove the conjecture for \( \delta([\nu^{-1} \rho, \nu^{1/2} \rho]) \times \tau \) in the case when \( \delta([\nu^{-1} \rho, \nu^{1/2} \rho]) \times \sigma \) has a non-tempered generic subquotient. This is the case when \( \delta([\nu^{-1} \rho, \nu^{1/2} \rho]) \times \sigma \) is irreducible, or is reducible and the generic quotient is \( \delta([\nu^{a+1} \rho, \nu^{b+1} \rho]) \times \pi \) (Proposition 4.9). Note that in the proof of
the previous proposition only the first embedding actually dependent on the fact that $T'$ is tempered; the important ingredient was the unitarity of the image of the first embedding; the reasoning about the kernel of the third and subsequent homomorphisms will be repeated later.

**Proposition 5.2.** We retain the notation introduced in the beginning of the section. Assume that $\delta([\nu^{-1}\rho, \nu^{l_2}\rho]) \times \sigma$ does not have a tempered generic subquotient, and assume that $\delta([\nu^{-1}\rho, \nu^{l_2}\rho]) \times \delta_i$, $i = 1, \ldots, k$ and $\delta([\nu^{-1}\rho, \nu^{l_2}\rho]) \times \delta_i^\prime$, $i = 1, \ldots, k$ is irreducible. Then, the generalized injectivity holds for the representation $\delta([\nu^{-1}\rho, \nu^{l_2}\rho]) \times \tau$.

**Proof.** First, assume that $\delta([\nu^{-1}\rho, \nu^{l_2}\rho]) \times \sigma$ is irreducible. Then, by ([15], Lemma 2.1 and Lemma 2.4) $\delta([\nu^{-1}\rho, \nu^{l_2}\rho]) \times \tau$ is irreducible and the claim follows. Now, assume that $l_1 \leq -1$, and put $a = -l_1 - 1$. If the representation $\delta([\nu^{-1}\rho, \nu^{l_2}\rho]) \times \sigma$ is reducible, then $\delta([\nu^{a+1}\rho, \nu^{\beta_i}\rho]) \times \pi$ is it’s irreducible generic subrepresentation (notation from Proposition 4.9). Then, we have

$$\delta([\nu^{a+1}\rho, \nu^{l_2}\rho]) \times \tau \hookrightarrow \delta([\nu^{a+1}\rho, \nu^{l_2}\rho]) \times \delta_1 \times \cdots \times \delta_k \times \sigma \cong \delta_1 \times \cdots \times \delta_k \times \delta([\nu^{a+1}\rho, \nu^{l_2}\rho]) \times \tau. \quad (19)$$

The representation $\delta_1 \times \cdots \times \delta_k \times \delta([\nu^{a+1}\rho, \nu^{\beta_i}\rho]) \times \pi$ is a subrepresentation of (19). On the other hand, we claim that $\delta([\nu^{a+1}\rho, \nu^{\beta_i}\rho]) \times \delta_i$, $i = 1, \ldots, k$ is irreducible. Assume that $\delta_i = \delta([\nu^{-a_i}\rho_i, \nu^{a_i}\rho_i])$ for some unitarizable, supercuspidal $\rho_i$ and $a_i \in \frac{1}{2}\mathbb{Z}$. If $\delta_i \times \delta([\nu^{a+1}\rho, \nu^{\beta_i}\rho])$ were reducible, we would have $\rho_i \cong \rho$ and $a \leq a_i$, $\beta_1 > a_i$. But since $l_2 > \beta_1$, this would imply that $\delta_i \times \delta([\nu^{a+1}\rho, \nu^{l_2}\rho])$ is reducible, contrary to our assumptions. This means that

$$\delta_1 \times \cdots \times \delta_k \times \delta([\nu^{a+1}\rho, \nu^{\beta_i}\rho]) \times \pi \cong \delta([\nu^{a+1}\rho, \nu^{\beta_i}\rho]) \times \delta_1 \times \cdots \times \delta_k \times \pi.$$  

Let $\sigma_2$ be a tempered generic subrepresentation of $\delta_1 \times \cdots \times \delta_k \times \pi$. Then, the representation $\delta([\nu^{a+1}\rho, \nu^{\beta_i}\rho]) \times \sigma_2$ is irreducible (obviously since $\delta([\nu^{a+1}\rho, \nu^{\beta_i}\rho]) \times \pi$ is irreducible, and $\delta([\nu^{a+1}\rho, \nu^{l_2}\rho]) \times \delta_i$ is also) and generic and a subrepresentation of (19). This forces

$$\delta([\nu^{a+1}\rho, \nu^{\beta_i}\rho]) \times \sigma_2 \hookrightarrow \delta([\nu^{a+1}\rho, \nu^{l_2}\rho]) \times \tau.$$  

$\square$
Proposition 5.3. Assume that $\delta([ν^{-l_1} ρ, ν^{l_2} ρ]) \times σ$ does not have a tempered generic subquotient, and assume that there exists $i_0 \in \{1, 2, \ldots, k\}$ such that

$\delta_{i_0} \times \delta([ν^{-l_1} ρ, ν^{l_2} ρ])$ is reducible. Then, the generalized injectivity conjecture holds for $δ([ν^{-l_1} ρ, ν^{l_2} ρ]) \times τ$.

Proof. Assume firstly that $δ([ν^{-l_1} ρ, ν^{l_2} ρ]) \times σ$ reduces. This means that the generic subrepresentation of that representation is $δ([ν^{a+1} ρ, ν^{l_2} ρ]) \times π$. We can group $δ_i$’s in the expression for $τ$ any way we like, since $δ_i × δ_j$ is irreducible, for every $i$ and $j$. So, let $δ_1, δ_2, \ldots, δ_l$ be such that $δ_i \times δ([ν^{a+1} ρ, ν^{l_2} ρ])$ is irreducible; the rest of them are $δ_{l+1}, \ldots, δ_k$. Let $δ_i = δ([ν^{-s_i} ρ, ν^{s_i} ρ])$, $i = l + 1, \ldots, k$. The assumption that $δ_i \times δ([ν^{a+1} ρ, ν^{l_2} ρ])$ is reducible forces $a \leq s_i < l_2$, $i = l + 1, \ldots, k$. We assume that $s_{i+1} \geq s_{i+2} \geq \ldots \geq s_k$. Now we have two cases: $s_k = a$ and $s_k > a$.

In the first case, the representation

$$δ_1 \times \cdots \times δ_l \times \cdots \times δ_k-1 \times δ([ν^{-a} ρ, ν^{l_2}]) \times σ$$

is a subrepresentation of

$$δ_1 \times \cdots \times δ_l \times \cdots \times δ_k-1 \times δ([ν^{a+1} ρ, ν^{l_2} ρ]) \times δ([ν^{-a} ρ, ν^{a}]) \times σ. \quad (20)$$

On the other hand, $δ([ν^{-a} ρ, ν^{l_2}]) \times σ$ is reducible, and has a tempered generic subrepresentation. This means that $(20)$ has an irreducible tempered generic subrepresentation, and moreover, by the examining the kernels of the intertwining operators (the same procedure like in the previous section) we know that this generic subquotient is also a subrepresentation of

$$δ([ν^{a+1} ρ, ν^{l_2} ρ]) \times δ_1 \times \cdots \times δ_k \times σ,$$

what is also true for $δ([ν^{a+1} ρ, ν^{l_2} ρ]) \times τ$, and the claim follows.

Now we assume that $s_k > a$. We consider several cases.

The first case is $2l_2 + 1$, $2s_k + 1 \notin \text{Jord}_μ(σ)$. Let $σ_k$ be a generic discrete series such that $σ_k \leftarrow δ([ν^{-s_k} ρ, ν^{l_2} ρ]) \times σ$. By Proposition 4.9, we know that there exist a tempered representation $T_2$ such that the representation $δ([ν^{a+1} ρ, ν^{l_2} ρ]) \times T_2$ is irreducible and

$$δ([ν^{a+1} ρ, ν^{l_2} ρ]) \times T_2 \leftarrow δ([ν^{a+1} ρ, ν^{s_k} ρ]) \times σ_k \leftarrow δ([ν^{a+1} ρ, ν^{l_2} ρ]) \times σ.$$
We conclude that

\[ \delta_1 \times \cdots \times \delta_{k-1} \times \delta([\nu^{a+1}\rho, \nu^{\beta_i}\rho]) \times T_2 \leftarrow \]

\[ \leftarrow \delta_1 \times \cdots \times \delta_{k-1} \times \delta([\nu^{a+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-s_k}\rho, \nu^{s_k}\rho]) \times \sigma. \]  

(21)

The representation \(\delta([\nu^{a+1}\rho, \nu^{\beta_i}\rho]) \times \delta_i, \ i = l + 1, \ldots, k - 1\) is irreducible.

In more words, since \(0 < \beta_1 \leq s_k \leq s_i, \ i = l + 1, \ldots, k - 1\) we have \([\nu^{a+1}\rho, \nu^{\beta_i}\rho] \subset [\nu^{-s_k}\rho, \nu^{s_k}\rho]\) (if \(\beta_1 \geq s_k\) the representation \(\delta([\nu^{a+1}\rho, \nu^{s_k}\rho]) \times \sigma_k\) is irreducible by ([15]) and then we have a simpler situation: we only note that \(\delta([\nu^{a+1}\rho, \nu^{s_k}\rho]) \times \delta_i, \ i = l + 1, \ldots, k - 1\) is irreducible). Also, the irreducibility of \(\delta([\nu^{a+1}\rho, \nu^{s_k}\rho]) \times \delta_i, \ i = 1, \ldots, l\) follows from the irreducibility of \(\delta_i \times \delta([\nu^{a+1}\rho, \nu^{l_2}\rho]), \ i = 1, \ldots, l\).

In this way, we get that \(\delta([\nu^{a+1}\rho, \nu^{\beta_i}\rho]) \times \delta_1 \times \cdots \times \delta_{k-1} \times T_2\) is a subrepresentation of (21). If \(T_3\) is a generic tempered subrepresentation of \(\delta_1 \times \cdots \times \delta_{k-1} \times T_2\), the representation \(\delta([\nu^{a+1}\rho, \nu^{\beta_i}\rho]) \times T_3\) is an irreducible (factorization of the long intertwining operator) subrepresentation of the representation (21). On the other hand, applying the same reasoning about the kernels of the intertwining operators, we conclude that

\[ \delta([\nu^{a+1}\rho, \nu^{\beta_i}\rho]) \times T_3 \leftarrow \delta([\nu^{a+1}\rho, \nu^{l_2}\rho]) \times \delta_1 \times \cdots \delta_k \times \sigma, \]

and the claim follows.

Now, we just comment on the rest of the cases, since the general frame of the proof is the same: assume that \(2l_2 + 1 \in \text{Jord}_\rho(\sigma), \ 2s_k + 1 \notin \text{Jord}_\rho(\sigma)\).

Let \(T_1\) an irreducible generic tempered subquotient of \(\delta([\nu^{-s_k}\rho, \nu^{l_2}\rho]) \times \sigma\) (instead of \(\sigma_k\) from the previous case). We have

\[ \delta([\nu^{a+1}\rho, \nu^{s_k}\rho]) \times T_1 \leftarrow \delta([\nu^{a+1}\rho, \nu^{s_k}\rho]) \times \delta([\nu^{-s_k}\rho, \nu^{l_2}\rho]) \times \sigma \leftarrow \]

\[ \leftarrow \delta([\nu^{a+1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-s_k}\rho, \nu^{s_k}\rho]) \times \sigma. \]

Let \(\pi_0\) be a generic subquotient of \(\delta([\nu^{a+1}\rho, \nu^{s_k}\rho]) \times T_1\). We know that \(T_1 \leftarrow \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \times \sigma', \) where \(\sigma'\) is a generic discrete series with \(\text{Jord}_\rho(\sigma) \setminus \{2l_2 + 1\} \cup \{2s_k + 1\}\). So, we have the following

\[ \delta([\nu^{a+1}\rho, \nu^{s_k}\rho]) \times T_1 \leftarrow \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{a+1}\rho, \nu^{s_k}\rho]) \times \sigma'. \]

Again, let \(T_2\) be a generic tempered representation such that \(\delta([\nu^{a+1}\rho, \nu^{\beta_i}\rho]) \times T_2\) is an irreducible subrepresentation of \(\delta([\nu^{a+1}\rho, \nu^{s_k}\rho]) \times \sigma'. \) If we denote by \(T_3\) an irreducible tempered generic subrepresentation of \(\delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \times T_2\),
we get that $\pi_0 = \delta([\nu\rho, 1]) \times T_3$ (we get irreducibility of this representation by examining the long intertwining operator obtained when we appropriately embed $T_3$). At the end, we get

$$\delta_1 \times \cdots \times \delta_{k-1} \times \delta([\nu^{a+1}\rho, \nu^b\rho]) \times T_3 \leftrightarrow \delta_1 \times \cdots \times \delta_{k-1} \times \delta([\nu^{a+1}\rho, \nu^2\rho]) \times \delta_k \times \sigma.$$  

As in the previous case, we get that the left–hand side of this relation is isomorphic to

$$\delta([\nu^{a+1}\rho, \nu^b\rho]) \times \delta_1 \times \cdots \times \delta_{k-1} \times T_3.$$  

If $T_4$ is a generic tempered subrepresentation of $\delta_1 \times \cdots \times \delta_{k-1} \times T_3$, we get that $\delta([\nu^{a+1}\rho, \nu^2\rho]) \times T_4$ is an irreducible subrepresentation of $\delta_1 \times \cdots \times \delta_{k-1} \times \delta([\nu^{a+1}\rho, \nu^2\rho]) \times \delta_k \times \sigma$. By examining the kernels of the intertwining operators, we get that

$$\delta([\nu^{a+1}\rho, \nu^b\rho]) \times T_4 \leftrightarrow \delta([\nu^{a+1}\rho, \nu^2\rho]) \times \delta_1 \times \cdots \times \delta_{k-1} \times \delta_k \times \sigma,$$

and the claim follows. In the case $2l_2 + 1 \notin \mathrm{Jord}_\rho(\sigma), \ 2s_k + 1 \in \mathrm{Jord}_\rho(\sigma)$ we reason quite analogously; here the irreducible generic subrepresentation of $\delta([\nu^{a+1}\rho, \nu^2\rho]) \times \tau$ is $\delta([\nu^{a+1}\rho, \nu^b\rho]) \times T_3$, where $T_3$ is a unique generic tempered subrepresentation of $\delta_1 \times \cdots \times \delta_{k-1} \times T_2$, and $T_2$ is a generic tempered subrepresentation of $\delta([\nu^{-s_k}\rho, \nu^{s_k}\rho]) \times \sigma''$, where $\sigma''$ is an irreducible generic discrete series with $\mathrm{Jord}_\rho(\sigma'') = \mathrm{Jord}_\rho(\sigma) \setminus \{2\beta_1 + 1\} \cup \{2l_2 + 1\}$. In the case when $2s_k + 1, \ 2l_2 + 1 \in \mathrm{Jord}_\rho$ the generic irreducible subrepresentation of $\delta([\nu^{a+1}\rho, \nu^2\rho]) \times \tau$ is again of the form $\delta([\nu^{a+1}\rho, \nu^b\rho]) \times T_3$, where $T_3$ is a generic tempered subrepresentation of $\delta_1 \times \cdots \times \delta_{k-1} \times T_2$, and $T_2$ is the unique generic tempered subquotient of $\delta([\nu^{-s_k}\rho, \nu^{s_k}\rho]) \times \delta([\nu^{-l_2}\rho, \nu^{l_2}\rho]) \times \sigma''$. Here $\sigma''$ is a generic discrete series representation with $\mathrm{Jord}_\rho(\sigma'') = \mathrm{Jord}_\rho(\sigma) \setminus \{2\beta_1 + 1, 2l_2 + 1\}$. 

If the representation $\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \sigma$ does not reduce, the proof is analogous, but simpler.

\[\square\]

6 The case of split $SO(2n, F)$

One of the key ingredients of the Moeglin–Tadić classification is the structure formula for the calculation of the Jacquet modules of the induced representations. This formula ([22]) is valid for non-split, but quasi–split (nonconnected) orthogonal groups, and is also valid for the connected, non-split, but
quasi–split special orthogonal groups. In both these cases the Casselman’s square–integrability criterion takes the same form (the discussion on both topics can be found in the 16th section of [9]). The situation is different for split $SO(2n, F)$, because there the structure formula of [22] holds in that form only for $O(2n, F)$ (since split $SO(2n, F)$ has a different root system). So, to prove the generalized injectivity conjecture for split $SO(2n, F)$ we use $O(2n, F)$ case (where Moeglin-Tadić classification holds in the same form as for the symplectic and odd-orthogonal groups).

We realize split $O(2n, F)$ and $SO(2n, F)$ as a matrix groups in a usual way ([1]). Let

$$
epsilon = \begin{pmatrix} 1 & 1 \\ & \vdots \\ & 0 & 1 \\ & 1 & 0 \\ & & \ddots \\ & & & 1 & 1 \end{pmatrix} \in O(2n, F) \setminus SO(2n, F).$$

The element $\epsilon$ normalizes $SO(2n, F)$ and, for a representation $\sigma$ of $SO(2n, F)$, we denote by $\sigma^\epsilon$ a representation of the same group defined by $\sigma^\epsilon(g) = \sigma(\epsilon g \epsilon^{-1})$. Some of the standard parabolic subgroups of $SO(2n, F)$ are parametrized by the $k$–tuples $(n_1, n_2, \ldots, n_k)$ of positive integers such that $n_1 + \cdots + n_k = m \leq n$ (in the same way as for the odd-orthogonal and symplectic groups; here the corresponding Levi subgroup is isomorphic to $GL(n_1, F) \times \cdots \times GL(n_k, F) \times SO(n - m, F)$; to get a uniqueness of parametrization, we demand $m \neq n - 1$). The rest of standard parabolic subgroups are $\epsilon$–conjugates of standard parabolic subgroups attached to partitions of $n$ of type $(n_1, n_2, \ldots, n_k)$, $n_k > 1$ ([1],[13]). The parabolic induction in $O(2n, F)$ (as this groups has analogous standard Levi subgroups to the symplectic and odd-orthogonal groups) will be denoted, as before, with $\pi_1 \times \cdots \times \pi_k \rtimes SO \pi$. The parabolic induction in $SO(2n, F)$ will be denoted by $\pi_1 \times \cdots \times \pi_k \rtimes SO \pi$, if the induction is with respect to the first set of the standard parabolic subgroups, and by $\pi_1 \times \cdots \times \pi_k \rtimes SO \pi$, if the induction is with respect to the second set of standard parabolic subgroups.

The relation between an irreducible representation of $O(2n, F)$ and it’s restriction to $SO(2n, F)$ is given by the following proposition ([10], Lemma 5, p.60). Here $\text{sign}$ denotes the determinant character on $O(2n, F)$.  

35
Proposition 6.1. (i) If $\pi$ is an irreducible representation of $O(2n, F)$, then $\pi|_{SO}$ is irreducible if and only if $\pi \not\cong \pi \otimes \text{sign}$. In that case, if $\rho = \pi|_{SO}$, we have $\text{Ind}^{O(2n, F)}_{SO} \rho = \pi \oplus \pi \otimes \text{sign}$, and $\rho \cong \rho^\vee$.

(ii) If $\rho$ is an irreducible representation of $SO(2n, F)$, the representation $\text{Ind}^{O(2n, F)}_{SO(2n, F)} \rho$ is irreducible if and only if $\rho \not\cong \rho^\vee$. In that case if $\pi = \text{Ind}^{O(2n, F)}_{SO(2n, F)} \rho$, we have $\pi|_{SO(2n, F)} = \rho \oplus \rho^\vee$, and $\pi \cong \pi \otimes \text{sign}$.

We now give an easy lemma, which is very useful.

Lemma 6.2. Let $\pi$ be an irreducible representation of $O(2n, F)$, $n \geq 1$. Then, the following holds

$$(\pi_1 \times \cdots \pi_k \times \pi)|_{SO} \cong \pi_1 \times \cdots \pi_k \times (\pi|_{SO}).$$

Proof. By the straightforward calculation. \qed

To avoid misunderstanding when generic representation of $O(2n, F)$ is restricted to $SO(2n, F)$, we will again emphasize the non–degenerate character with respect to whom this representation is generic, i.e. “$\chi$–generic”, not only “generic.”

Lemma 6.3. Assume that $\pi$ is an irreducible $\chi$–generic representation of $O(2n, F)$ with $n \geq 1$, and $\pi_1, \ldots, \pi_k$ irreducible generic representations of $GL(n_i, F)$, $i = 1, \ldots, k$. Denote $\sigma = \pi_1 \times \cdots \times \pi_k \times \pi$. Then, we have the following

(i) If the representation $\pi|_{SO}$ is irreducible, the representation $\sigma$ has a unique irreducible $\chi$–generic subquotient (which appears with multiplicity one),

(ii) if $\pi|_{SO}$ reduces, and if a subquotient $\tau$ of $\sigma$ appears there as a unique subrepresentation (quotient), then $\tau|_{SO}$ also reduces.

Proof. We prove (i). By Lemma 6.2 the representation $\sigma|_{SO}$ has a unique irreducible generic subquotient $\tau$ which comes with multiplicity one (with respect to $\chi$, since $\pi|_{SO}$ is irreducible and $\chi$–generic). Since the hereditary property holds for $O(2n, F)$ (Theorem 6.4 (iii)), this $\tau$ came, by restriction, from some $\chi$–generic subquotient $\tau'$ of $\sigma$. So, this $\tau'$ is unique.
To prove (ii), we observe that if $\pi|_{SO}$ reduces, then $\pi \cong \pi \otimes \text{sign}$. The straightforward calculation then gives

$$\pi_1 \times \cdots \times \pi_k \times \pi \cong \pi_1 \times \cdots \times \pi_k \times (\pi \otimes \text{sign}) \cong \text{sign} \otimes \pi_1 \times \cdots \times \pi_k \times \pi.$$

We conclude that there exists an $O(2n,F)$–isomorphism $T$ intertwining the first and the third representation above. Let $(\tau, V)$ be a unique subrepresentation of $\sigma$ (on which $O(2n,F)$ acts by right translations $R$). Then $T(V)$ is a subrepresentation of $\sigma \otimes \text{sign}$, invariant for $R \otimes \text{sign}$, so by $R$ also. This means that $T(V) = V$, so $\tau \cong \tau \otimes \text{sign}$, and $\tau|_{SO}$ reduces. The analogous claim for unique quotient follows when we apply the contragredient. Note that this implies the corresponding relation for the Langlands quotient of the standard representation of $O(2n,F)$.

Now we collect main facts which are proved in the connected case, but easy Mackey–stile argument shows that they are valid in the non–connected case (of $O(2n,F)$). We still write down the details because we weren’t able to find the correct reference. The Langlands classification and the square–integrability criterion take the same form as for the symplectic and odd–orthogonal groups.

**Theorem 6.4.**

(i) The standard module conjecture ([13]) is also valid for $O(2n,F)$,

(ii) If a standard generic representation of $O(2n,F)$ has an irreducible, square–integrable (tempered) subquotient, then all it’s generic subquotients are square–integrable (tempered) (for the connected case, we refer to [13]),

(iii) The Whittaker model is hereditary (Theorem 2 of [18]); in the equivalent form: if $\pi_1, \ldots, \pi_k$ are irreducible, generic representations of $GL(n_i, F)$, $i = 1, \ldots, k$ and $\pi$ an irreducible representation of $O(2n,F)$, then the representation $\pi_1 \times \cdots \times \pi_k \times \pi$ is $\chi$–generic if and only if $\pi$ is $\chi$–generic.

**Proof.** Firstly, we prove (iii). This can be shown by following the Rodier’s original argument from [18]. Or, proceed like this: if $\pi$ is $\chi$–generic, then the non–trivial functional $\lambda$ on the representation space of $\pi$ (satisfying $\lambda(\pi(u)v) = \chi(u)\lambda(v)$ for $u$ from a maximal unipotent subgroup) has to be non–trivial on, at least, one subspace of $\rho$ or $\rho^\vee$. By abusing the notation, let
say that $\lambda|_{\rho} \neq 0$. Then $\lambda|_{\rho}$ also satisfies the compatibility condition with respect to $\chi$ (since $O(2n, F)$ and $SO(2n, F)$ have the same maximal unipotent subgroup), i.e. $\rho$ is $\chi$–generic. Then, by Lemma 6.2

$$\pi_1 \times \cdots \times \pi_k \times \pi|_{SO} \cong \pi_1 \times \cdots \times \pi_k \times \pi|_{SO} \rho \oplus \pi_1 \times \cdots \times \pi_k \times \pi|_{SO} \rho^\oplus.$$ 

Now, by using the hereditary property for $SO(2n, F)$, one implication follows. On the other hand, if $\pi_1 \times \cdots \times \pi_k \times \pi$ is $\chi$–generic, there exist a nontrivial ($\chi$–compatible) functional on that representation space. It’s restriction to the at least one of the $SO$–invariant subspaces from the above relation (say $\pi_1 \times \cdots \times \pi_k \times \rho$) has to be non–trivial. This forces $\rho$ to be $\chi$–generic (with respect to some functional $\lambda$). Then, by Proposition 6.1, $\pi \cong \text{Ind}^\rho_{SO} \rho$ and we can define a non–trivial, $\chi$–compatible functional $\lambda'$ on $\text{Ind}^\rho_{SO} \rho$ by putting $\lambda'(f) = \lambda(f(e))$. If $\pi|_{SO}$ is irreducible, the argument is similar, but simpler.

We now prove (i). Let $\tau$ be an irreducible, tempered, $\chi$–generic representation of $O(2n, F)$, $n \geq 1$ and let $\delta_1, \ldots, \delta_k$ be essentially square–integrable representations of $GL(n_i, F)$, $i = 1, \ldots, k$ such that $\delta_1 \times \cdots \times \delta_k \times \tau$ is standard (and $\chi$–generic, by (iii)). Assume that $L(\delta_1, \ldots, \delta_k; \tau)$ is generic. If $\tau|_{SO}$ is irreducible, it is straightforward that $L(\delta_1, \ldots, \delta_k; \tau|_{SO}) \leq L(\delta_1, \ldots, \delta_k; \tau)_{SO}$. Our assumption then forces $L(\delta_1, \ldots, \delta_k; \tau|_{SO})$ to be generic, and this is possible only if $\delta_1 \times \cdots \times \delta_k \times \tau$ is irreducible (by the standard modul conjecture for $SO$–groups). This means that $\delta_1 \times \cdots \times \delta_k \times \tau$ has to be irreducible.

If $\tau|_{SO} = \tau_1 \oplus \tau_2$, where $\tau_1$ and $\tau_2$ are tempered, using Lemma 6.3, we get that

$$L(\delta_1, \ldots, \delta_k; \tau)|_{SO} = L(\delta_1, \ldots, \delta_k; \tau_1) \oplus L(\delta_1, \ldots, \delta_k; \tau_2).$$

Both summands from the right–hand side are generic $SO$–representations, so $\delta_1 \times \cdots \times \delta_k \times \tau_i$, $i = 1, 2$ are irreducible. If we assume that, in an appropriate Grothendieck group, we have $\delta_1 \times \cdots \times \delta_k \times \tau = \tau_1 \oplus \tau_2$, for some genuine representations $\pi_1$ and $\pi_2$, we get $\pi_i|_{SO} = \delta_1 \times \cdots \times \delta_k \times \pi_i$, $i = 1, 2$. This means, by Proposition 6.1 (i), that

$$(\delta_1 \times \cdots \times \delta_k \times \pi_1)^e \cong \delta_1 \times \cdots \times \delta_k \times (\tau_1|_{SO})^e \cong \delta_1 \times \cdots \times \delta_k \times (\tau_1|_{SO}) \cong \delta_1 \times \cdots \times \delta_k \times \pi_2.$$ 

By the same proposition, we would then further have

$$\text{Ind}^\rho_{SO}(\delta_1 \times \cdots \delta_k \times \pi_1) = \pi_1 \oplus \pi_1 \otimes \text{sign}.$$ 

38
We can write the left–hand side of the above relation as

\[ \text{Ind}_{SO}^{O} \delta_1 \times \cdots \times \delta_k \rtimes \tau_1 \cong \delta_1 \times \cdots \times \delta_k \rtimes \text{Ind}_{SO}^{O} \tau_1 = \delta_1 \times \cdots \times \delta_k \rtimes \tau. \]

In this way the representation \( \delta_1 \times \cdots \times \delta_k \rtimes \tau \) wouldn't have a unique quotient, which can not be true. The case of \( \tau = 1 \) (the trivial representation of the trivial group) the discussion is similar.

To prove (ii), we use the fact that an irreducible representation of \( O(2n, F) \) is square–integrable (tempered) if and only if its restriction to \( SO(2n, F) \) is irreducible and square–integrable (tempered), or the the sum of such representations. We use the notation from (i). If \( \tau|_{SO} \) does not reduce, the discussion is straightforward.

Now, let \( \tau|_{SO} = \tau_1 \oplus \tau_2 \). Let \( \delta \) be a square–integrable subquotient of \( \pi \). Then, we have

\[ \delta|_{SO} \leq \delta_1 \times \cdots \times \delta_k \rtimes_{SO} \tau_1 \oplus \delta_1 \times \cdots \times \delta_k \rtimes_{SO} \tau_2. \]

This means that at least one of the representations \( \delta_1 \times \cdots \times \delta_k \rtimes_{SO} \tau_i, \ i = 1, 2 \) has a square–integrable subquotient, and then it has a square–integrable generic subquotient. But, the appearance of a generic discrete series subquotient can be expressed in terms \( L \)–functions (i.e. \( \gamma \)–factors) ([13], Theorem 3.1), and the multiplicativity of \( \gamma \)–factors ([19]) guarantees that both of the representations satisfy the same condition (since \( \tau_2 = \tau_1^* \), all the \( \gamma \)–factors involved are the same). The case when \( \tau = 1 \) is analogous to the previous one. The other part of the claim involving the tempered representations is proved analogously.

\[ \square \]

**Lemma 6.5.** Let \( \pi \) be a \( \chi \)–generic representation of \( O(2n, F) \), \( n \geq 1 \) such that \( \pi|_{SO} = \sigma \oplus \sigma^{*} \). Then, both \( \sigma \) and \( \sigma^{*} \) are \( \chi \)–generic.

**Proof.** Let 1 denote the trivial character of \( F^{*} \). We prove this lemma by considering three cases: \( \pi \) is cuspidal, tempered, and non–tempered. Assume firstly that \( \pi \) is cuspidal. We study the representation \( 1 \rtimes \pi \), which is \( \chi \)–generic (Theorem 6.4 (iii)). Observe that this representation must be reducible (namely, if it wasn’t, the representations \( 1 \rtimes_{SO} \sigma \) and \( 1 \rtimes_{SO} \sigma^{*} \) would be irreducible, and \( 1 \rtimes_{SO} \sigma^{*} \not\cong 1 \rtimes_{SO} \sigma \). But, the nontrivial element \( w_{0} \) of the appropriate relative Weyl group transforms \( w_{0}(1 \otimes \sigma) \) = \( \bar{1} \otimes \sigma^{*} \) ([1], p.192). Since \( \sigma \not\cong \sigma^{*} \), the cuspidal data is not self–associate, and the representation
$1 \times \sigma$ is irreducible ([3]), but then also $1 \times_{SO} \sigma \cong 1 \times_{SO} \sigma'$ ([1], p.192)). We have $1 \times \pi = T_1 \oplus T_2$, $T_1 \not\cong T_2$ and by restricting to $SO$, we have

$$1 \times_{SO} \sigma \oplus 1 \times_{SO} \sigma' = T_1|_{SO} \oplus T_2|_{SO},$$

and $T_1|_{SO} \cong T_2|_{SO} = 1 \times \sigma$ and, further, when we induce back to the full orthogonal group,

$$1 \times \pi = T_1 \oplus T_1 \otimes \text{sign}.$$  

We now see that at least one of the $T_i$, $i = 1, 2$ has to be generic, and since $T_2 = T_1 \otimes \text{sign}$, they both are. This means that $1 \times_{SO} \sigma \cong 1 \times_{SO} \sigma'$ also is, so by the hereditary property, so are $\sigma$ and $\sigma'$. In the case of the square–integrable $\pi$, we have the same discussion as in the cuspidal case, due to Bruhat results which also demand a self–associativity of data to have reducibility of $1 \times_{SO} \sigma$. For the tempered $\pi$ we proceed as follows: assume that $\delta_1, \ldots, \delta_k$ are square–integrable representations of $GL(n_i, F)$, $i = 1, \ldots, k$, and $\delta$ a square–integrable representation, such that

$$\pi \hookrightarrow \delta_1 \times \cdots \times \delta_k \rtimes \delta.$$  

Such square–integrable representations $\delta_i$, $i = 1, \ldots, k$ are unique, up to permutations and taking contragredients, and $\delta$ is unique ([16], Theorem 1.1). If $\delta$ is a representation of $O(2m, F)$, $m \geq 1$, this forces $\delta \cong \delta \otimes \text{sign}$, i.e. $\delta|_{SO}$ reduces. For the moment, we keep this assumption. Also observe that $\delta$ is $\chi$–generic. We note there is no $\delta_i$ (among $\delta_1, \ldots, \delta_k$) such that $n_i$ is odd and $\delta_i \cong \delta_i$. Indeed, assume that such $\delta_i$ exists, and, for simplicity, let it be $\delta_1$. Then $\pi \hookrightarrow \delta_2 \times \cdots \times \delta_k \times \delta_1 \rtimes \delta$. Just as in the previous discussion, when we analyzed $1_F^*$ instead of more general $\delta_1$, we obtain that $\delta_1 \rtimes \delta = T \otimes T \otimes \text{sign}$, where $T \not\cong T \otimes \text{sign}$. When we restrict to $SO$, we obtain (with $\rho = T|_{SO}$ irreducible tempered)

$$(\delta_1 \times \cdots \times \delta_k \rtimes \delta)|_{SO} = \delta_2 \times \cdots \times \delta_k \times_{SO} \rho \oplus \delta_2 \times \cdots \times \delta_k \times_{SO} \rho.$$  

This forces any irreducible tempered subrepresentation $\pi'$ of $\delta_1 \times \cdots \times \delta_k \rtimes \delta$ to be irreducible when restricted to $SO$. Indeed, if $\pi'|_{SO}$ reduces, then each of the attached constituents can come only by restriction from $\pi'$, and no other tempered representation suffices. But this would force the representation $\delta_1 \times \cdots \times \delta_k \rtimes \delta$ not to be multiplicity free, which is false. The same conclusion on $\delta'_i$s can be obtained if $\delta = 1$ (trivial representation of trivial group), because then again $\delta_1 \rtimes 1 = T \oplus T \otimes \text{sign}$. This conclusion on $\delta'_i$s enables us
to apply the same strategy as for the discrete series case, with the aid of the reducibility results of Goldberg ([5], theorems 5.16, 6.5, 6.11). Namely, if \( \delta \) is a representation of \( O(2m, F) \), \( m \geq 1 \) with \( \delta|_{SO} = \delta^\prime \oplus \delta^\prime' \), we obtain that \( 1 \rtimes \pi \) reduces (since \( 1 \rtimes \delta \) reduces; it is the sum of two pieces distinguished by sign, so both are \( \chi \)-generic) and \( 1 \rtimes_{SO} \sigma \cong 1 \rtimes_{SO} \sigma^\prime \) is irreducible. We again get that \( \sigma \) and \( \sigma^\prime \) are \( \chi \)-generic. If \( \delta = 1 \) we use ([5], theorems 5.8 and 6.8) to obtain the same results.

If \( \pi \) is non-tempered, we use the standard module conjecture to write \( \pi = \delta_1 \times \cdots \times \delta_k \rtimes \tau \), where the induced representation is standard, irreducible and \( \tau \) is \( \chi \)-generic. Assume firstly that \( \tau \) is a representation of \( O(2^m, F) \), \( m \geq 1 \). Then

\[
\sigma \oplus \sigma^\prime = \delta_1 \times \cdots \times \delta_k \rtimes_{SO} (\tau|_{SO}).
\]

\( \tau|_{SO} \) has to reduce (the uniqueness of Langland’s quotient). Then, our previous cases force that both of the restrictions of \( \tau \) are \( \chi \)-generic, so are \( \sigma \) and \( \sigma^\prime \). If \( \tau = 1 \), then

\[
\sigma \oplus \sigma^\prime = \delta_1 \times \cdots \times \delta_k \rtimes_{SO} 1 \oplus \delta_1 \times \cdots \times \delta_k \rtimes_{SO} 1,
\]

and again the claim follows.

Theorem 6.4 gathered all the facts, besides the existence of the Moeglin–Tadić classification of discrete series, we have used in the proof of the generalized injectivity for the connected case. We owe few remarks.

**Lemma 6.6.** Assume that \( \sigma \) is a \( \chi \)-generic square-integrable representation of \( O(2n, F) \), \( \rho \cong \tilde{\rho} \) cuspidal, irreducible representation of \( GL(n_{\rho}, F) \). Assume that \( 2l_1 + 1, 2l_2 + 1 \in \mathbb{Z}_{\geq 0} \) with \( l_1 < l_2 \) satisfy the parity conditions and assume that \( [2l_1 + 1, 2l_2 + 1] \cap \text{Jord}_\rho(\sigma) = \emptyset \). Then, only one of the discrete series subrepresentations (say, \( \sigma_1 \) and \( \sigma_2 \)) of \( \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma \) is \( \chi \)-generic, except in the case when \( \sigma|_{SO} \) reduces, \( n_{\rho} \) is odd and \( 2l_1 + 1 \) is odd; this forces \( \text{Jord}_\rho(\sigma) = \emptyset \) (we call this an exceptional case).

**Proof.** Firstly, assume that \( n \geq 1 \). If \( \sigma|_{SO} \) does not reduce, the claim follows from Lemma 6.3.

Now, suppose \( \sigma|_{SO} = \sigma' \oplus \sigma'' \). Observe that, by Lemma 6.5, \( \sigma' \) and \( \sigma'' \) are \( \chi \)-generic. As in the Remark after Proposition 3.1 \( \sigma_i \leftrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes T_i, i = 1, 2 \) where \( T_1 \oplus T_2 = \delta([\nu^{-l_i}\rho, \nu^{l_i}\rho]) \rtimes \sigma \). \( \sigma_i \) is \( \chi \)-generic if and only if \( T_i \) is \( \chi \)-generic, \( i = 1, 2 \). Now we examine the consequences of the reducibility...
of $\delta([\nu^{-1}\rho, \nu^l\rho]) \rtimes \sigma$; we use the discussion in Lemma 2.5 of [7]. We denote $\delta_u = \delta([\nu^{-1}\rho, \nu^l\rho])$. We have two possibilities: either $\delta_u \rtimes_{SO} \sigma'$ reduces, or $\delta_u \rtimes_{SO} \sigma'$ is irreducible, and $\delta_u \rtimes_{SO} \sigma' \cong \delta_u \rtimes_{SO} \sigma''$. We discuss the first possibility; we have

$$\delta_u \rtimes_{SO} \sigma' = T_3 \oplus T_4,$$

for some tempered representations $T_3$ and $T_4$, only one of them is $\chi$–generic; say $T_3$. When we induce to $O(2n, F)$ the above relation, we get

$$\delta_u \rtimes \sigma = \text{Ind}_{SO}^O T_3 \oplus \text{Ind}_{SO}^O T_4 = T_1 \oplus T_2.$$

If we put $T_1 = \text{Ind}_{SO}^O T_3$, this forces $T_1$ to be $\chi$–generic. Further, $T_1|_{SO} = T_3 \oplus T_3'$, and both $T_3$ and $T_3'$ are $\chi$–generic. Since $\delta_u \rtimes_{SO} \sigma'' = T_3' \oplus T_4'$, this means that $T_4'$ is not $\chi$–generic, and $T_4$ is also not generic, so $T_1$ is $\chi$–generic and $T_2$ is not, and the claim follows. Note that the reducibility of $\delta_u \rtimes_{SO} \sigma'$ forces $(2l_1 + 1)n_\rho$ to be is even (and a certain condition on an appropriate $L$–function has to be satisfied).

If we assume the other possibility (i.e. $\delta_u \rtimes_{SO} \sigma'$ is irreducible), this forces $(2l_1 + 1)n_\rho$ to be odd. As in the discussion of Lemma 6.5 this possibility forces $T_2 \cong T_i \otimes \text{sign}$, and $T_i$, $i = 1, 2$ are both $\chi$–generic. Let $(2a+1, \rho) \in \text{Jord}_\rho(\sigma)$. Then $a \in \mathbb{Z}_{\geq 0}$ and $\delta([\nu^{-a}\rho, \nu^a\rho]) \rtimes \sigma$ is irreducible. But, this cannot hold, because, when we restrict this representation to $SO$ we get a contradiction (since $(2a+1)n_\rho$ is odd, we would get $\delta([\nu^{-a}\rho, \nu^a\rho]) \rtimes \sigma' \cong \delta([\nu^{-a}\rho, \nu^a\rho]) \rtimes \sigma''$). This means $\text{Jord}_\rho(\sigma)$ is empty.

If $\sigma = 1$ (the trivial representation of the trivial group), the discussion is analogous to the one above, since $(\delta_u \rtimes 1)|_{SO} = \delta_u \rtimes_{SO} 1 \oplus \delta_u \rtimes_{SO}' 1$.

\[ \square \]

**Proposition 6.7.** Let $\tau$ be an irreducible, generic tempered representation of some $O(2m, F)$, $m \geq 1$. Then, all the generic subquotients of $\delta([\nu^{-1}\rho, \nu^2\rho]) \rtimes \tau$ are subrepresentations.

**Proof.** We review the differences in the proofs of the the sections 3, 4, 5. We emphasise that we can keep the assumptions (5) since the theorems cited before this assumption are valid for $O(2n, F)$ (they are formulated in that settings). The main point is that, in the case of $O(2n, F)$, in general, there is no uniqueness of the Whittaker model, so we have to bypass the uniqueness of the model–style arguments. The proof of Proposition 3.1 can be repeated without changes. Having in mind remark right after Proposition 3.1 and Lemma 6.6, we immediately see that the generic discrete series $\sigma$ of $O(2n, F)$
have the following property: assume that $\varepsilon$ function is defined on the elements of $\text{Jord}_\rho(\sigma)$ (not only on pairs). Then, on all the elements of $\text{Jord}_\rho(\sigma)$ it attains value one, except in the case originated in the exceptional case of Lemma 6.6, when it can attain value $-1$ on all the elements of $\text{Jord}_\rho(\sigma)$ (then, the “all 1’s” generic discrete series differs from “all $-1$’s discrete series” by sign—of course, we assume they have the same Jordan block and cuspidal support). From this we conclude if $\sigma_1$ appears as a $\chi$–generic discrete series subquotient of $\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$, with $\sigma$ $\chi$–generic and $\text{Jord}_\rho(\sigma) \neq \emptyset$, it is a unique $\chi$–generic subquotient of that representation (since the 1’s or $-1$’s values on $\text{Jord}_\rho(\sigma_1)$ are already determined by the corresponding data of $\sigma$).

If we have a situation when we handle a discrete series in the exceptional case, we bypass the obstacle very easy: we will discuss only the situation in Proposition 4.4, the case of $m$ even. If we are in the exceptional case and $\text{Jord}_\rho(\sigma') \neq \emptyset$, and if we assume $\sigma_1 \leq \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$, and

$$\sigma \hookrightarrow \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-a_{m-1}}\rho, \nu^{a_m}\rho]) \rtimes \sigma'$$

with

$$\sigma_1 \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \delta([\nu^{-a_1}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{-a_{m-1}}\rho, \nu^{a_m}\rho]) \rtimes (\sigma' \otimes \text{sign}),$$

then $\sigma_1 \otimes \text{sign} \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$. Then, there exists a generic discrete series $\pi$ such that $\text{Jord}_\rho(\pi) = \emptyset$ and

$$\sigma \oplus \sigma \otimes \text{sign} \hookrightarrow \Pi \rtimes \pi,$$

for some generic representation $\Pi$. We immediately get

$$\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma \oplus \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes (\sigma \otimes \text{sign}) \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \Pi \rtimes \pi.$$

The right–hand side has exactly two discrete series $\chi$–generic representations, namely $\sigma_1$ and $\sigma_1 \otimes \text{sign}$ (these two are necessarily different), so they cannot be both subquotients of $\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \sigma$.

If $\text{Jord}_\rho(\sigma') = \emptyset$, the argument is the same, but here instead of $\pi$, we immediately put $\sigma'$.

We can check all the claims in the sections 3, 4 and 5 and modify the arguments there in the similar simple way as illustrated above, to obtain the validity for $O(2n, F)$. \qed
Now, let $\tau$ be an irreducible, $\chi$–generic tempered representation of $SO(2m, F)$, $m \geq 1$. Then, the following holds:

**Proposition 6.8.** The unique irreducible $\chi$–generic subquotient of $\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes_{SO} \tau$ is a subrepresentation.

**Proof.** If $\tau' \not\cong \tau$ the representation $\tau' = \text{Ind}_{SO}^O \tau$ is an irreducible tempered representation of $O(2m, F)$, and we have

$$\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \tau' \cong \text{Ind}_{SO}^O \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes_{SO} \tau.$$ 

Let $\pi$ be an irreducible $\chi$–generic subrepresentation (Proposition 6.7) of the left–hand side. When we view both sides as representations of $SO(2m, F)$, we get

$$\pi|_{SO} \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes_{SO} \tau \oplus \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes_{SO} \tau'.$$

Since each irreducible $\chi$–generic representation of $SO(2m, F)$ comes from restriction of the generic representation of $O(2m, F)$ there must exist $\pi$ as above such that an irreducible subrepresentation of $\pi|_{SO}$ is a subrepresentation of $\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes_{SO} \tau$.

On the other hand, if $\tau' \cong \tau$, then $\text{Ind}_{SO}^O \tau = \tau' \oplus \tau' \otimes \text{sign}$, where $\tau'$ is a generic tempered representation of $O(2m, F)$ with $\tau'|_{SO} = \tau$. Then

$$(\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \tau')|_{SO} \cong \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes_{SO} (\tau'|_{SO}) \cong \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes_{SO} \tau.$$ 

If $\pi$ is a generic subrepresentation of $\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \tau'$, we have

$$\pi|_{SO} \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes_{SO} \tau,$$

and the claim is proved. \hfill $\square$

**References**


