The conjectural relation between generalized Shalika models on $\mathrm{SO}_{4n}(F)$ and the symplectic linear model on $\mathrm{Sp}_{4n}(F)$. A Toy Example

Agnès David, Marcela Hanzer, Judith Ludwig

Abstract

We show that if an irreducible admissible representation of $\mathrm{SO}_4(F)$ has a generalized Shalika model, its theta lift to $\mathrm{Sp}_4(F)$ is non-zero and has a symplectic linear model.

1 Introduction

The recent progress towards proving the Local Langlands Conjectures for classical groups (cf. [1],[8],[7],[5],[16],[10] and many more) increased the interest in understanding characterizations of images of Langlands functorial transfers and the finer structures of L-and A-packets. One way of distinguishing representations is by the models they have. As an example of how models can be used to characterize images of transfers consider the following situation: Let $F/\mathbb{Q}_p$ be a finite extension and let τ be an irreducible unitary supercuspidal representation of $\mathrm{GL}_{2n}(F)$. Then (cf. [13] Theorem 1.1) τ is a local Langlands functorial transfer from $\mathrm{SO}_{2n+1}(F)$ if and only if τ has a Shalika model. Furthermore it turns out that the existence of certain models of representations of different groups is very much related through Langlands type correspondences. In this article we investigate how generalized Shalika models on the split group $\mathrm{SO}_4(F)$ are related to symplectic linear models on $\mathrm{Sp}_4(F)$ via the local theta correspondence.

More precisely in [12] Jiang, Nien and Qin conjecture the following:

Conjecture 1.1 ([12], p. 542)**.** Let π be an irreducible admissible representation of $\mathrm{SO}_{4n}(F)$ which has a generalized Shalika model. Then the representation $\theta(\pi)$ of $\mathrm{Sp}_{4n}(F)$ associated to π via the local theta correspondence is nonzero and has a symplectic linear model.

Here, and in the remainder of the paper, $F/\mathbb{Q}_p$ is a finite extension and $p \neq 2$. Furthermore $\theta(\pi, m)$ denotes the "small" theta lift of a representation π and $\Theta(\pi, m)$ denotes the "big" theta lift of π to the symplectic group $\mathrm{Sp}_{2m}(F)$

(cf. [17], p.33). If m is understood, we denote $\Theta(\pi, m)$ by $\Theta(\pi)$ and $\theta(\pi, m)$ by $\theta(\pi)$. The dual pair used in this theta correspondence consists of a symplectic and a full orthogonal group and the restriction to the special orthogonal group is explained below.

The goal of this article is to prove

Theorem (Theorem 5.2)**.** *Conjecture 1.1 is true for $n = 1$.*

The result that led to the conjecture in the first place and provides evidence for it is

Theorem 1.1 ([13] Theorem 1.1 and 1.2)**.** *Let τ be an irreducible unitary supercuspidal representation of $\mathrm{GL}_{2n}(F)$ which has a Shalika model. Then the induced representation $\tau\nu^{1/2} \rtimes_{\mathrm{SO}_{4n}(F)} 1$ has a unique Langlands quotient π which has a generalized Shalika model. The induced representation $\tau\nu^{1/2} \rtimes_{\mathrm{Sp}_{4n}(F)} 1$ has a unique Langlands quotient σ which has a symplectic linear model. Furthermore $\theta(\pi) = \sigma$.*

Here ν denotes the character of $\mathrm{GL}_{2n}(F)$ obtained by composing the determinant with the norm on the non-archimedean field F and we may regard any character of F^* as a character of $\mathrm{GL}_{2n}(F)$ analogously. Throughout the text we use Zelevinsky's notation for the parabolic induction for the general linear groups and for classical groups as introduced e.g. in [23] Sections 1 and 2.

We will eventually prove Theorem 5.2 by reducing it to a calculation of Jacquet-modules. The following two results make this reduction possible.

Theorem 1.2 ([14], Theorem 1.2)**.** *Let σ be an irreducible admissible representation of $\mathrm{SO}_{4n}(F)$ and assume it has a generalized Shalika model. Then there exists an irreducible admissible representation τ of $\mathrm{GL}_{2n}(F)$ such that σ is a quotient of the induced representation*

$$\tau\nu^{1/2} \rtimes_{\mathrm{SO}_{4n}(F)} 1 \twoheadrightarrow \sigma.$$

The following theorem is the specialization of Theorem 1.3 in [14] to our situation ($n = 1$).[1]

Theorem 1.3 ([14], Theorem 1.3)**.** *Let τ be an irreducible admissible representation of $\mathrm{GL}_2(F)$. If the induced representation $\tau\nu^{1/2} \rtimes_{\mathrm{SO}_4(F)} 1$ has a generalized Shalika model, then τ either has a symplectic model or a Shalika model.*

The strategy to prove Theorem 5.2 is as follows: The theorems quoted above allow to first study the representations τ. We determine the set of representations τ that admit a symplectic model or a Shalika model. We then analyse them case by case and study the representations obtained when we parabolically induce τ to representations of $\mathrm{SO}_4(F)$ and $\mathrm{Sp}_4(F)$ respectively. We check

[1] The $\theta_{\mathcal{N}_1}$ model is the Shalika model for GL_2, the other one the symplectic model, see the paragraph before Theorem 1.3 in [14].

when these induced representations have the respective models. In particular we need to make sure in any of the cases that the set of τ's for which the induction to $\mathrm{SO}_4(F)$ has a generalized Shalika model agrees with the set of τ's for which the induction to $\mathrm{Sp}_4(F)$ has a symplectic linear model. Finally we verify that the models factor through the relevant quotients of the inductions and that these quotients are related via the theta correspondence.

The key to get our hands on these representations is the following: We can show the existence of some of the models by proving that a certain (twisted) Jacquet-module has a trivial quotient. We prove the existence of these quotients by calculating the Jacquet-modules explicitly using the Geometric Lemma of Bernstein and Zelevinsky (see Theorem 5.1 in [3]).

As the groups we study have such small rank we can explicitly describe the possible representations τ and can then do explicit Jacquet-module calculations to find all the information on the models we need. In higher rank we cannot pin down the representations τ as explicitly. From the general version of Theorem 1.3 we know us that they all have a $\theta_{\mathcal{N}_r}$-model, but there is no explicit description of such representations. Therefore our method of analyzing everything explicitly case by case will not be successful and proving conjecture 1.1 in general will require a different approach.

The plan of the paper is as follows: In section 2 we recall the various models and determine the set of representations τ of $\mathrm{GL}_2(F)$ that admit a symplectic model or a Shalika model. We also give some background on the theta correspondence. We then prove Theorem 5.2 case by case in sections 3–5 depending on the properties of τ. Section 3 deals with the square-integrable case. In section 4 we study the case where τ is a character. We finish the proof by treating the case where τ is an irreducible principal series representation in section 5.

Acknowledgements. This project started at the WIN-Europe conference in October 2013. We would like to thank the organizers of the conference and the CIRM in Luminy for providing such excellent working conditions. MH has been supported in part by the Croatian Science Foundation under the project 9364. JL would like to thank Imperial College London for providing financial support in form of a Doris Chen Mobility Award.

2 Notation and Preliminaries

We recall the various models occurring in these notes specialized to the case at hand. For the general definitions we refer to [12] Section 2. Let $p \neq 2$, let $F/\mathbb{Q}_p$ be a finite extension and fix a non-trivial additive character $\psi : F \to \mathbb{C}^*$. Let

$$J_n := \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix} \in \mathrm{GL}_n(F)$$

and set $J := J_4$. In the special orthogonal group SO_4, whose F-points are given by

$$\mathrm{SO}_4(F) = \{A \in \mathrm{GL}_4(F) |\ {}^TAJA = J, \det A = 1\},$$

we fix the maximal diagonal torus T and the Borel subgroup B of upper triangular matrices. We let $P = MN$ be the standard maximal parabolic subgroup, whose Levi subgroup M is isomorphic to GL_2.

It is embedded via

$$\iota : \mathrm{GL}_2(F) \hookrightarrow \mathrm{SO}_4(F),\ g \mapsto \begin{pmatrix} g & 0 \\ 0 & J_2{}^Tg^{-1}J_2 \end{pmatrix}$$

and the F-points of its unipotent radical N are given by all matrices

$$y(X) = \begin{pmatrix} I_2 & X \\ 0 & I_2 \end{pmatrix},$$

such that ${}^TX = -J_2XJ_2$. We refer to P as the Siegel subgroup. The subgroup $\mathcal{H} \subset P(F)$ generated by all $\iota(g)$ for $g \in \mathrm{Sp}_2(F)$ and all $y \in N(F)$ is called the *generalized Shalika subgroup* of $\mathrm{SO}_4(F)$. We extend ψ to a character $\psi_{\mathcal{H}} : \mathcal{H} \to \mathbb{C}^*$ by $\psi_{\mathcal{H}}(y(X)) = \psi\left(tr\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} X\right)\right)$ and by demanding it is trivial on $\iota(\mathrm{Sp}_2(F))$.

Definition 2.1. An irreducible admissible representation π of $\mathrm{SO}_4(F)$ is said to have a generalized Shalika model if

$$\mathrm{Hom}_{\mathcal{H}}(\pi, \psi_{\mathcal{H}}) \neq 0.$$

Definition 2.2. Let τ be an irreducible admissible representation of $\mathrm{GL}_2(F)$.

- The representation τ has a Shalika model if

$$\mathrm{Hom}_{\mathcal{S}}(\tau, \psi_{\mathcal{S}}) \neq 0,$$

  where $\mathcal{S} = \left\{ \begin{pmatrix} a & x \\ 0 & a \end{pmatrix} \middle| a \in F^*, x \in F \right\} \subset \mathrm{GL}_2(F)$ is the Shalika subgroup and we have extended ψ to a character $\psi_{\mathcal{S}} : \mathcal{S} \to \mathbb{C}^*, \psi_{\mathcal{S}}\left(\begin{pmatrix} a & x \\ 0 & a \end{pmatrix}\right) = \psi(x/a)$.

- The representation τ has a symplectic model if

$$\mathrm{Hom}_{\mathrm{Sp}_2(F)}(\tau, 1_{\mathrm{Sp}_2(F)}) \neq 0.$$

- τ has a linear model if

$$\mathrm{Hom}_{\mathrm{GL}_1(F)\times\mathrm{GL}_1(F)}(\tau, 1_{\mathrm{GL}_1(F)\times\mathrm{GL}_1(F)}) \neq 0.$$

Theorem 2.1 ([11] Section 6)**.** *If an irreducible admissible representation τ of $\mathrm{GL}_n(F)$ has a Shalika model then τ has a linear model.*

In the symplectic group Sp_4, whose F-points are given by

$$\mathrm{Sp}_4(F) = \left\{ A \in \mathrm{GL}_4(F) \middle| \, {}^T A \begin{pmatrix} 0 & J_2 \\ -J_2 & 0 \end{pmatrix} A = \begin{pmatrix} 0 & J_2 \\ -J_2 & 0 \end{pmatrix} \right\},$$

we fix the maximal diagonal torus T and the Borel subgroup B of upper triangular matrices. We have a standard maximal parabolic subgroup $P = MN$ with Levi $M \cong \mathrm{GL}_2$ embedded via

$$\mathrm{GL}_2(F) \hookrightarrow \mathrm{Sp}_4(F), \; g \mapsto \begin{pmatrix} g & 0 \\ 0 & J_2{}^T g^{-1} J_2 \end{pmatrix}.$$

The group $\mathrm{Sp}_2(F) \times \mathrm{Sp}_2(F)$ injects into $\mathrm{Sp}_4(F)$ via

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} w & x \\ y & z \end{pmatrix} \right) \mapsto \begin{pmatrix} a & & & b \\ & w & x & \\ & y & z & \\ c & & & d \end{pmatrix}. \tag{1}$$

Definition 2.3. An irreducible admissible representation σ on $\mathrm{Sp}_4(F)$ has a symplectic linear model if

$$\mathrm{Hom}_{\mathrm{Sp}_2(F)\times\mathrm{Sp}_2(F)}(\sigma, 1_{\mathrm{Sp}_2(F)\times\mathrm{Sp}_2(F)}) \neq 0.$$

Remark. Note that in the definitions of models all representations are assumed to be irreducible. If the corresponding Hom-space for an admissible not necessarily irreducible representation in non-zero, we speak of *functionals* instead of *models*, so e.g. if π is a possibly reducible admissible representation of $\mathrm{SO}_4(F)$ such that $\mathrm{Hom}_{\mathcal{H}}(\pi, \psi_{\mathcal{H}}) \neq 0$, we say that π has a non-zero generalized Shalika functional.

Lemma 2.2. *Let τ be an irreducible admissible representation of* $\mathrm{GL}_2(F)$*.*

1. *If τ has a symplectic model, then τ is a character.*
2. *The representation τ has a Shalika model if and only if τ is generic with trivial central character.*

Proof. For the proof of the first part note that $\mathrm{Sp}_2 = \mathrm{SL}_2$. So τ has a symplectic model if and only if there exists a non-zero functional $\lambda \in \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, 1_{\mathrm{SL}_2})$. Then $V_\tau/ker(\lambda) \cong 1_{\mathrm{SL}_2(F)}$ as a representation of $\mathrm{SL}_2(F)$. The restriction of any smooth irreducible representation of $\mathrm{GL}_2(F)$ to $\mathrm{SL}_2(F)$ decomposes into a finite direct sum of irreducible representations, each occurring with multiplicity one (cf. [19], Lemma 2.4 & Lemma 2.6). Furthermore the representations occurring in the restriction are permuted by any set of representatives of

$\mathrm{GL}_2(F)/\mathrm{SL}_2(F)F^*$. Therefore if $\tau|_{\mathrm{SL}_2(F)}$ contains the trivial representation as a subrepresentation, $\tau = \chi$ is a character.

For the second part we unravel the definitions to see that any non-zero $\lambda \in \mathrm{Hom}_{\mathcal{S}}(\tau, \psi_{\mathcal{S}})$ is in fact a Whittaker functional on V_τ. Furthermore we have

$$\lambda(\omega_\tau(t)v) = \lambda\left(\tau\left(\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}\right)v\right) = \lambda(v)$$

for all $v \in V_\tau$ and $t \in F^*$ if and only if the central character ω_τ is trivial.

$\square$

So here are the options for τ:

1. The representation τ is supercuspidal. Note that having trivial central character implies that τ is unitary and so we are in special case of Theorem 1.1 above, where the implication of Conjecture 1.1 is known to hold.

2. τ is a generic subquotient of a reducible principal series. Then τ is an essentially square-integrable representation with trivial central character and in particular τ is unitary. It follows that

$$\tau \hookrightarrow \chi\nu^{1/2} \times \chi\nu^{-1/2} \cong \chi(\nu^{1/2} \times \nu^{-1/2})$$

and therefore $\tau \cong \chi\,\mathrm{St}_{\mathrm{GL}_2(F)}$, where $\mathrm{St}_{\mathrm{GL}_2(F)}$ denotes the Steinberg representation of $\mathrm{GL}_2(F)$. The condition that the central character is trivial furthermore implies that $\chi^2 = 1$.

3. The representation τ is a character. Note we can write $\chi = \chi_0\nu^s$, where χ_0 is unitary and $\nu^s = |det|^s$, where $s \in \mathbb{R}$.

4. τ is an irreducible principal series representation, $\tau \cong \chi_1\nu^{s_1} \times \chi_2\nu^{s_2}$, where χ_1, χ_2 are unitary characters and $s_1, s_2 \in \mathbb{R}$. The condition that the central character is trivial gives

$$\chi_1\chi_2 = 1 \quad \text{and} \quad s_1 + s_2 = 0.$$

Therefore $\tau \cong \chi\nu^s \times \chi^{-1}\nu^{-s}$ for a unitary character χ.

We briefly explain how to restrict the theta correspondence between symplectic and full orthogonal groups to the correspondence between representations of symplectic and special orthogonal groups. Let $\epsilon \in \mathrm{O}_{2n}(F)$ be the element

$$\epsilon = \begin{pmatrix} I_{n-1} & & & \\ & & 1 & \\ & 1 & & \\ & & & I_{n-1} \end{pmatrix}.$$

For an irreducible admissible representation τ of $\mathrm{SO}_{2n}(F)$, we denote by τ^ϵ representation of $\mathrm{SO}_{2n}(F)$ on the same space, defined by $\tau^\epsilon(g) = \tau(\epsilon g \epsilon^{-1})$. Recall that we can pass between irreducible admissible representations of $\mathrm{O}_{2n}(F)$ and $\mathrm{SO}_{2n}(F)$ as follows:

Lemma 2.3 (cf.[20] 3.II.5, Lemme)**.**

1. *Let π be an irreducible admissible representation of* $\mathrm{O}_{2n}(F)$*. Then* $\pi|_{\mathrm{SO}_{2n}(F)}$ *is irreducible if and only if* $\pi \ncong \pi \otimes det$*.*

2. *Let τ be an irreducible admissible representation of* $\mathrm{SO}_{2n}(F)$*. Then either (A) $\tau \ncong \tau^{\epsilon}$, in which case* $Ind_{\mathrm{SO}_{2n}(F)}^{\mathrm{O}_{2n}(F)}(\tau) =: \pi$ *is irreducible and satisfies* $\pi = \pi \otimes det$*, or (B) $\tau \cong \tau^{\epsilon}$ in which case* $Ind_{\mathrm{SO}_{2n}(F)}^{\mathrm{O}_{2n}(F)}(\tau)$ *is reducible and the direct sum of two non-equivalent irreducible representations π and $\pi \otimes det$.*

We fix a non-trivial additive character ϕ of F. All Weil representations occurring in this article will be with respect to this character. Furthermore for i=1,2 we fix the splittings $\mathrm{O}_{2i}(F) \times \mathrm{Sp}_{2i}(F) \to Mp_{4i^2}(F)$ and for later purposes $O_4(F) \times \mathrm{Sp}_2(F) \to Mp_8(F)$ as described explicitly in [18].

Using the above lemma we can restrict the theta correspondence, i.e., we can relate the largest $\mathrm{Sp}_{2n}(F)$–invariant quotient which is an isotype of τ in the appropriate Weil representation with the similar quotient corresponding to π as follows: Let τ be an irreducible admissible representation of $\mathrm{SO}_{2n}(F)$, then

$$\theta(\tau, n) \cong \theta(\pi, n) \text{ if (A)} \tag{2}$$

$$\theta(\tau, n) := \theta(\pi, n) \oplus \theta(\pi \otimes det, n) \text{ if (B)}$$

Remark. We often use the following fact: assume that π is an irreducible representation of $\mathrm{O}_4(F)$ such that $\pi \cong \pi \otimes \det$. Then, the first occurrence index of π in theta correspondence, denoted by $n(\pi)$, is exactly 2, i.e. , π occurs for the first time in theta correspondence with $\mathrm{Sp}_4(F)$. This follows from the general fact ([22]):

Theorem 2.4. *Assume that σ is an irreducible admissible representation of the split* $\mathrm{O}_{2n}(F)$*. Then the following holds:*

$$n(\sigma) + n(\sigma \otimes \det) = 2n.$$

3 The case of square-integrable τ

From now on, if π is a standard representation of a classical group, we denote by $L(\pi)$ its Langlands quotient.

Lemma 3.1. *Let χ be a quadratic character of F^*. Then the representation* $\chi\,\mathrm{St}_{\mathrm{GL}_2(F)}\,\nu^{1/2} \rtimes_{\mathrm{Sp}_4(F)} 1$ *is of length three if $\chi \neq 1$ and of length two if $\chi = 1$ and in both cases has a unique irreducible quotient* $L(\chi\,\mathrm{St}_{\mathrm{GL}_2(F)}\,\nu^{1/2} \rtimes_{\mathrm{Sp}_4(F)} 1)$*. In the case $\chi = 1$ there is also an irreducible, tempered subrepresentation (necessarily generic), and in the case $\chi \neq 1$ two non-equivalent, irreducible, square-integrable subrepresentations. The Langlands quotient* $L(\chi\,\mathrm{St}_{\mathrm{GL}_2(F)}\,\nu^{1/2} \rtimes_{\mathrm{Sp}_4(F)} 1)$ *has a symplectic linear model.*

Proof. The reducibility issues are dealt with in [21], Proposition 5.4 and Theorem 5.2. On the other hand, we know that $\chi\,\mathrm{St}_{\mathrm{GL}_2(F)}$ has a Shalika model and so by Theorem 2.1 it also has a linear model. Then we reason as in [9], p. 878 to conclude that $\chi\,\mathrm{St}_{\mathrm{GL}_2(F)}\,\nu^{1/2} \rtimes_{\mathrm{Sp}_4(F)} 1$ has a non-zero symplectic linear functional. Since irreducible generic representations cannot have a symplectic linear model (cf. [9] Theorem 1), in the case of $\chi = 1$ we see that this functional factors through the Langlands quotient and gives a model. In the case $\chi \neq 1$, for a fixed non-degenerate character ψ, one of the two square–integrable representations is ψ–generic, and the other is not. It is not difficult to see that the other one is generic with respect to some other generic character, and the conclusion follows. □

Lemma 3.2. *Let χ be a quadratic character of F^*. Then the representation $\chi\,\mathrm{St}_{\mathrm{GL}_2(F)}\,\nu^{1/2} \rtimes_{\mathrm{SO}_4(F)} 1$ is of length two and has a unique irreducible quotient (the Langlands quotient) $L(\chi\,\mathrm{St}_{\mathrm{GL}_2(F)}\,\nu^{1/2} \rtimes_{\mathrm{SO}_4(F)} 1)$ which has a generalized Shalika model.*

Proof. Note that $\chi \rtimes_{\mathrm{O}_2(F)} 1$ is reducible. Thus, the same Jacquet module calculation as in (cf.[21], Proposition 5.4.) gives us that the representation $\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{O}_4(F)} 1$ is of length three and we have

$$\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{O}_4(F)} 1 = L(\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes 1) + \pi_1 + \pi_2,$$

where the $\pi_i, i = 1, 2$ are (mutually non-isomorphic) square integrable representations. Note that also $\pi_1 \hookrightarrow \chi\nu^1 \rtimes \chi$, and $\pi_2 \hookrightarrow \chi\nu^1 \rtimes \chi \otimes \det$, because $\chi \rtimes 1 = \chi 1_{\mathrm{O}_2(F)} \oplus \chi \det_{\mathrm{O}_2(F)}$. From this it follows that $\pi_1 \otimes \det \hookrightarrow \chi\nu^1 \rtimes \chi \otimes \det$, and analogously, $\pi_2 \otimes \det \hookrightarrow \chi\nu^1 \rtimes \chi$, so that $\pi_1 \cong \pi_2 \otimes \det$. We conclude that $L(\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{O}_4(F)} 1) \otimes \det \cong L(\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{O}_4(F)} 1)$. Therefore the restriction restriction of $\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{O}_4(F)} 1$ to $\mathrm{SO}_4(F)$ decomposes as

$$\begin{aligned}
&\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{O}_4(F)} 1|_{\mathrm{SO}_4(F)} \\
&= (\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{SO}_4(F)} 1) \oplus (\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{SO}_4(F)} 1)^{\epsilon} \\
&= L(\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{O}_4(F)} 1)|_{\mathrm{SO}_4(F)} + \pi_1|_{\mathrm{SO}_4(F)} + \pi_2|_{\mathrm{SO}_4(F)} \\
&= L(\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{SO}_4(F)} 1) + (L(\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{SO}_4(F)} 1))^{\epsilon} + \pi_1|_{\mathrm{SO}_4(F)} \\
&\quad + \pi_2|_{\mathrm{SO}_4(F)}
\end{aligned}$$

where $\pi_1|_{\mathrm{SO}_4(F)} = \pi_2|_{\mathrm{SO}_4(F)}$ and these two representations are irreducible (as $\pi_1 \ncong \pi_2$). We conclude

$$\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{SO}_4(F)} 1 = L(\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{SO}_4(F)} 1) + \pi_1|_{\mathrm{SO}_4(F)}.$$

The existence of the generalized Shalika model on $L(\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{SO}_4(F)} 1)$ now follows from the general result [15], Theorem 3.1. Alternatively we will see that we can argue directly: the subquotient $L(\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{SO}_4(F)} 1)$ appears again in Proposition 4.2 and in the proof there we can see the existence of the model directly. □

Next we determine the theta lift of $L(\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)}\rtimes_{\mathrm{SO}_4(F)}1)$. As far as we know, unlike in the case of an odd orthogonal-metaplectic pair (cf. [6]), there still is no explicit description of the theta correspondence for a symplectic-even orthogonal dual pair at the same level. We, therefore, directly calculate the lift by calculating enough bits of one of its Jacquet modules. The main ingredient in these calculations is Kudla's filtration of the Jacquet modules of the Weil representations involved (cf. [17] III.8).

For a parabolic subgroup P of a group G and an admissible representation π of $G(F)$ we denote by $r_P(\pi)$ the Jacquet module of π with respect to P. We define $P_1 = M_1N_1$ to be the standard parabolic subgroup of Sp_4 with Levi M_1 isomorphic to $\mathrm{GL}_1\times\mathrm{SL}_2$ and we define $Q_1 = M_1'N_1'$ to be the standard parabolic of O_4 with Levi M_1' isomorphic to $\mathrm{GL}_1\times\mathrm{O}_2$. The Jacquet-module functor r_{P_1} induces a functor from $Rep(\mathrm{O}_4(F)\times\mathrm{Sp}_4(F))$ to $Rep(\mathrm{O}_4(F)\times M_1)$, which we will again denote by r_{P_1}.

For $i = 1,2$ we denote by $\omega_{i,i}$ the Weil representation of the double cover $Mp_{4i^2}(F)$ of $\mathrm{Sp}_{4i^2}(F)$ viewed as a representation of a dual pair $(\mathrm{Sp}_{2i}(F),\mathrm{O}_{2i}(F))$ (inside $\mathrm{Sp}_{4i^2}(F)$). The Weil representation of $Mp_8(F)$ viewed as a representation of $(\mathrm{SL}_2(F),\mathrm{O}_4(F))$ will be denoted by $\omega_{1,2}$.

Proposition 3.3. *Assume $\chi^2 = 1$. Then the theta lift of the representation $L(\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)}\rtimes_{\mathrm{SO}_4(F)}1)$ to $\mathrm{Sp}_4(F)$ is*

$$\theta(L(\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)}\rtimes_{\mathrm{SO}_4(F)}1)) = L(\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)}\rtimes_{\mathrm{Sp}_4(F)}1).$$

Proof. Theorem 2.4 guarantees that

$$n(L(\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)}\rtimes_{\mathrm{SO}_4(F)}1)) = 2$$

with $\theta(L(\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)}\rtimes_{\mathrm{SO}_4(F)}1),2) = \theta(L(\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)}\rtimes_{\mathrm{O}_4(F)}1),2)$ by the calculation in the proof of Lemma 3.2 and (2). To determine the theta lift $\theta(L(\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)}\rtimes_{\mathrm{O}_4(F)}1),2)$ we compute a part of its r_{P_1}-Jacquet module. We have

$$\omega_{2,2}\twoheadrightarrow L(\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)}\rtimes_{\mathrm{O}_4(F)}1)\otimes\theta(L(\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)}\rtimes_{\mathrm{O}_4(F)}1),2),$$

so that there is a non-zero intertwining operator, say T, of $\mathrm{O}_4(F)\times\mathrm{GL}_1(F)\times\mathrm{SL}_2(F)$–modules such that

$$T: r_{P_1}(\omega_{2,2})\twoheadrightarrow L(\chi\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)}\rtimes_{\mathrm{O}_4(F)}1)\otimes r_{P_1}(\theta(L(\chi\nu^{\frac{1}{2}}\,\mathrm{St}_{\mathrm{GL}_2(F)}\rtimes_{\mathrm{O}_4(F)}1),2)).$$

Now, $r_{P_1}(\omega_{2,2})$ has a filtration

$$\{0\}\subset J_1^{(1)}\subset J_1^{(0)} = r_{P_1}(\omega_{2,2}),$$

such that $J_1^{(1)}\cong\mathrm{Ind}_{\mathrm{GL}_1(F)\times\mathrm{SL}_2(F)\times\mathrm{GL}_1(F)\times\mathrm{O}_2(F)}^{\mathrm{GL}_1(F)\times\mathrm{SL}_2(F)\times\mathrm{O}_4(F)}(\sigma_1\otimes\omega_{1,1})$, where σ_1 is the representation of $\mathrm{GL}_1(F)\times\mathrm{GL}_1(F)$ by left and right translations on the space of smooth, compactly supported functions on $\mathrm{GL}_1(F)$ (denoted by $S(\mathrm{GL}_1(F))$).

Furthermore we know that $J_1^{(0)}/J_1^{(1)} \cong \nu^0 \otimes \omega_{1,2}$. We have $T|_{J_1^{(1)}} \neq 0$, as otherwise we would have

$$\nu^0 \otimes \omega_{1,2} \twoheadrightarrow L(\chi\nu^{1/2}\operatorname{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{O}_4(F)} 1) \otimes r_{P_1}(\theta(L(\chi\nu^{1/2}\operatorname{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{O}_4(F)} 1), 2))$$

which would mean $n(L(\chi\nu^{1/2}\operatorname{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{O}_4(F)} 1)) \leq 1$, and that is impossible. The second Frobenius reciprocity gives us that the space of $\mathrm{GL}_1(F) \times \mathrm{SL}_2(F) \times \mathrm{GL}_1(F) \times \mathrm{O}_2(F)$– intertwining operators from $\sigma_1 \otimes \omega_{1,1}$ to

$$r_{Q_1}(L(\chi\nu^{1/2}\widetilde{\operatorname{St}_{\mathrm{GL}_2(F)}} \rtimes_{\mathrm{O}_4(F)} 1)) \otimes r_{P_1}(\theta(L(\chi\nu^{1/2}\operatorname{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{O}_4(F)} 1), 2))$$

is non-zero. Now, we use the fact that

$$r_{Q_1}(L(\chi\nu^{\frac{1}{2}}\operatorname{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{O}_4(F)} 1)) = \chi\nu^0 \otimes \chi(\nu^1 \rtimes 1)$$

(we shall prove this in Lemma 3.4). This means that there is an $\mathrm{GL}_1(F)$–epimorphism from σ_1 to $\chi\nu^0$. But the maximal isotypic component of $\chi\nu^0$ in $S(\mathrm{GL}_1(F))$ (when we view it as a $\mathrm{GL}_1(F) \times \mathrm{GL}_1(F)$–module) is again $\chi\nu^0$. This means that $r_{P_1}(\theta(L(\chi\nu^{1/2}\operatorname{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{O}_4(F)} 1), 2))$ has an irreducible subquotient of the form $\chi\nu^0 \otimes *$, where $*$ is some representation of $\mathrm{SL}_2(F)$.

Now we settle the case of $\chi \neq 1$. Then, we can read off the cuspidal support of

$$\theta(L(\chi\nu^{1/2}\operatorname{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{SO}_4(F)} 1), 2)$$

(e.g [17]); it is a subquotient of $\chi\nu^1 \times \chi\nu^0 \rtimes_{\mathrm{Sp}_4(F)} 1$. The representation $\chi\nu^1 \times \chi\nu^0 \rtimes_{\mathrm{Sp}_4(F)} 1$ is of length six (cf. Proposition 5.4 of [21]), and has analogous subquotients as a representation $\chi\nu^1 \times \chi\nu^0 \rtimes_{\mathrm{O}_4(F)} 1$. This means that

$$r_{P_1}(L(\chi\nu^{1/2}\operatorname{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{Sp}_4(F)} 1)) = \chi\nu^0 \otimes \chi\nu^1 \rtimes_{\mathrm{SL}_2(F)} 1,$$

and the representation $L(\chi\nu^{1/2}\operatorname{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{Sp}_4(F)} 1)$ comes with the multiplicity two in $\chi\nu^1 \times \chi\nu^0 \rtimes_{\mathrm{Sp}_4(F)} 1$. Now, from the expression for $r_{P_1}(\chi\nu^1 \times \chi\nu^0 \rtimes_{\mathrm{Sp}_4(F)} 1)$ in the proof of Lemma 3.4, we see that $L(\chi\nu^{\frac{1}{2}}\operatorname{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{Sp}_4(F)} 1)$ is the only subquotient of $\chi\nu^1 \times \chi\nu^0 \rtimes_{\mathrm{Sp}_4(F)} 1$ having $\chi\nu^0 \otimes *$ in its $r_{P_1}-$ Jacquet module and the conclusion follows.

Now let $\chi = 1$. We continue with the analysis of the Jacquet module $r_{P_1}(\omega_{2,2})$. Since $\Theta(1_{\mathrm{SL}_2}, 2) = \nu^1 \rtimes_{\mathrm{O}_2(F)} 1$, we have a non-zero intertwining from $\nu^0 \otimes \nu^0 \otimes \omega_{1,1} \to \nu^0 \otimes \nu^0 \otimes 1_{\mathrm{SL}_2(F)} \otimes \nu^1 \rtimes_{\mathrm{O}_2(F)} 1$. Therefore,

$$r_{P_1}(\theta(L(\nu^{1/2}\operatorname{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{O}_4(F)} 1), 2)) \geq \nu^0 \otimes 1_{\mathrm{SL}_2(F)}$$

in the appropriate Grothendieck group. We conclude that

$$\theta(L(\nu^{1/2}\operatorname{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{O}_4(F)} 1), 2) \in \{L(\nu^{1/2}\operatorname{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{Sp}_4(F)} 1), L(\nu^1 \times \nu^0 \rtimes_{\mathrm{SL}_2(F)} 1)\}.$$

This follows from the fact that

$$\nu^0 \rtimes 1_{\mathrm{SL}_2(F)} = L(\nu^{1/2}\operatorname{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{Sp}_4(F)} 1) \oplus L(\nu^1 \times \nu^0 \rtimes 1),$$

and these two representations are the only irreducible subquotients of $\nu^1 \times \nu^0 \rtimes_{\mathrm{Sp}_4(F)} 1$ with $\nu^0 \otimes 1_{\mathrm{SL}_2(F)}$ in their Jacquet module.

Assume now that $\theta(L(\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{O}_4(F)} 1), 2) = L(\nu^1 \times \nu^0 \rtimes 1)$. Then, we have an epimorphism

$$r_{P_1}(\omega_{2,2}) \twoheadrightarrow L(\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{O}_4(F)} 1) \otimes r_{P_1}(L(\nu^1 \times \nu^0 \rtimes 1)),$$

and since we have an epimorphism $r_{P_1}(L(\nu^1 \times \nu^0 \rtimes 1)) \to \nu^{-1} \otimes \nu^0 \rtimes_{\mathrm{SL}_2(F)} 1$, there is a non-zero epimorphism, say T

$$r_{P_1}(\omega_{2,2}) \twoheadrightarrow L(\nu^{1/2}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{O}_4(F)} 1) \otimes \nu^{-1} \otimes \nu^0 \rtimes_{\mathrm{SL}_2(F)} 1.$$

Now we analyze the restrictions of T to the terms of the filtration of $r_{P_1}(\omega_{2,2})$. We get that there is a non-zero $\mathrm{GL}_1(F) \times \mathrm{GL}_1(F) \times \mathrm{SL}_2(F) \times \mathrm{O}_2(F)$–intertwining

$$\sigma_1 \otimes \omega_{1,1} \to \nu^0 \otimes \nu^1 \rtimes_{\mathrm{O}_2(F)} 1 \otimes \nu^{-1} \otimes \nu^0 \rtimes_{\mathrm{SL}_2(F)} 1,$$

which is impossible. This proves the proposition. $\square$

Lemma 3.4. *Assume $\chi^2 = 1$. Then*

$$r_{Q_1}(L(\chi\nu^{1/2}\,\mathrm{St}_{GL_2(F)} \rtimes_{O_4(F)} 1)) = \chi\nu^0 \otimes \chi\nu^1 \rtimes 1.$$

Proof. We use the structure formula (*) on page 2 of [2] to compute the Jacquet module of the induced representation $\pi := \chi\nu^1 \times \chi\nu^0 \rtimes_{O_4(F)} 1$ with respect to Q_1. We get that

$$\begin{aligned} r_{Q_1}(\pi) = {} & \chi\nu^1 \otimes \chi\mathrm{det} + \chi\nu^1 \otimes \chi 1_{O_2(F)} + \chi\nu^{-1} \otimes \chi\mathrm{det} \\ & + \chi\nu^{-1} \otimes \chi 1_{O_2(F)} + 2\chi\nu^0 \otimes \chi\nu^1 \rtimes 1. \end{aligned}$$

Since the multiplicity of $L(\chi\nu^{1/2}\,\mathrm{St}_{GL_2(F)} \rtimes_{O_4(F)} 1)$ in π is two the lemma follows. $\square$

4 The case of the representation $\chi\nu^s \rtimes 1$

In this section, we consider the case when the representation τ is a character of $\mathrm{GL}_2(F)$. We write τ as $\chi\nu^s$, with χ a unitary character and s in $\mathbb{R}$.

4.1 $\mathrm{SO}_4(F)$–side

First of all we prove that for any unitary character χ and any $s \in \mathbb{R}$, the representation $\pi := \chi\nu^s \rtimes 1$ of $\mathrm{SO}_4(F)$ has a generalized Shalika model by showing that a certain twisted Jacquet-module of π has a trivial quotient.

The calculation is done by applying the Geometric Lemma and we adapt the notation as follows: As before let $\mathcal{H}$ be the Shalika subgroup of $\mathrm{SO}_4(F)$. We let $P := P(F) = MV$ denote the F-points of the Siegel subgroup, and we write $\mathcal{H} = NV$, where $N \cong \mathrm{SL}_2(F)$. We form the twisted Jacquet module $r_{V,\psi}(\pi)$ of

π with respect to the group V and the character $\psi := \psi_{\mathcal{H}}|_V$. Recall that it is defined as $\mathrm{SL}_2(F)$-module given by the quotient of π by the space

$$\pi_{V,\psi} := span_{\mathbb{C}}\{\pi(X)f - \psi(X)f : X \in V, f \in \pi\}.$$

Lemma 4.1. *For any unitary character χ of $\mathrm{GL}_2(F)$ and any s in $\mathbb{R}$, the representation $\chi\nu^s \rtimes_{\mathrm{SO}_4(F)} 1$ has a non-zero generalized Shalika functional.*

Proof. It follows from the definitions that if $r_{V,\psi}(\chi\nu^s \rtimes_{\mathrm{SO}_4(F)} 1)$ has a trivial quotient, $\chi\nu^s \rtimes_{\mathrm{SO}_4(F)} 1$ has a non-zero generalized Shalika functional. The geometric lemma [3] gives a description of the composition of functors $F := r_{V,\psi} \circ i_{P,\mathrm{SO}_4(F)}$ from $\mathrm{GL}_2(F)$-representations to $\mathrm{SL}_2(F)$-representations. Here $i_{P,\mathrm{SO}_4(F)}$ denotes the functor of normalized parabolic induction. In order to apply it note the following: Firstly, under the action of $\mathcal{H}$ by right translation, the space $P \setminus \mathrm{SO}_4(F)$ decomposes into two orbits $P \setminus \mathrm{SO}_4(F) = P \cup Pw_1\mathcal{H}$ (we easily get that $Pw_1P = Pw_1\mathcal{H}$), where $w_1 = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}$. Furthermore all the requirements like good decomposition, etc. of section 5.1. of [3] for the triples (P, M, V) and $(\mathcal{H}, N, V)$ are satisfied. Abbreviate $Y := Pw_1\mathcal{H}$. There is an $\mathrm{SL}_2(F)$-invariant subspace τ_Y of $\chi\nu^s \rtimes 1$ which consists of all functions in $\chi\nu^s \rtimes 1$ which vanish outside of $Pw_1\mathcal{H}$. We apply the Jacquet functor $r_{V,\psi}$ to the filtration of $\mathrm{SL}_2(F)$-representations

$$\{0\} \subset \tau_Y \subset \chi\nu^s \rtimes 1.$$

Then Theorem 5.2 of [3] implies that $r_{V,\psi}(\chi\nu^s \rtimes 1/\tau_Y)$ is the $\mathrm{SL}_2(F)$-module given by the restriction of $\chi\nu^s$ from $\mathrm{GL}_2(F)$ to $\mathrm{SL}_2(F)$. Similarly we get for $r_{V,\psi}(F_1)$. We conclude that the $\mathrm{SL}_2(F)$ representation $r_{V,\psi}(\chi\nu^s \rtimes 1)$ has length two and each subquotient is isomorphic to the trivial representation. We conclude that $r_{V,\psi}(\chi\nu^s \rtimes 1)$ has a trivial quotient. $\square$

We remind the reader that we need to determine the irreducible quotients of $\chi\nu^s \rtimes 1$ and decide when they have a generalized Shalika model.

Proposition 4.2. *Whenever the representation $\chi\nu^s \rtimes 1$ of $\mathrm{SO}_4(F)$ is irreducible it has a generalized Shalika model.*

1. *Assume that $\chi^2 \neq 1$. Then the representation $\chi\nu^s \rtimes 1$ of $\mathrm{SO}_4(F)$ is irreducible.*

2. *Assume that $\chi^2 = 1$. The representation $\chi\nu^s \rtimes_{\mathrm{SO}_4(F)} 1$ is reducible if and only if $s = \pm\frac{1}{2}$. Then, in the appropriate Grothendieck group, $\chi\nu^{\frac{1}{2}} \rtimes_{\mathrm{SO}_4(F)} 1 = L(\chi\,\mathrm{St}_{\mathrm{GL}_2(F)}\,\nu^{\frac{1}{2}} \rtimes_{\mathrm{SO}_4(F)} 1) + L(\chi\nu^1 \rtimes \chi)$. The representation $L(\chi\,\mathrm{St}_{\mathrm{GL}_2(F)}\,\nu^{\frac{1}{2}} \rtimes_{\mathrm{SO}_4(F)} 1)$ has a generalized Shalika model and $L(\chi\nu^1 \rtimes \chi)$ doesn't admit one.*

Proof. Note that if the representation $\chi\nu^s \rtimes 1$ of $\mathrm{O}_4(F)$ is irreducible, then the representation $\chi\nu^s \rtimes_{\mathrm{SO}_4(F)} 1$ of $\mathrm{SO}_4(F)$ is irreducible, since we saw that

$\chi\nu^s \rtimes 1|_{\mathrm{SO}_4(F)} = \chi\nu^s \rtimes_{\mathrm{SO}_4(F)} 1 + (\chi\nu^s \rtimes_{\mathrm{SO}_4(F)} 1)^\epsilon$. So, the reducibility points for $\chi\nu^s \rtimes_{\mathrm{SO}_4(F)} 1$ are among the reducibility points for $\chi\nu^s \rtimes 1$. Necessary condition for reducibility here is $\chi^2 = 1$. In that case, using the spinor norm, we have $\chi\nu^s \rtimes 1 \cong \chi(\nu^s \rtimes 1)$. We can extend, as we already saw, the Jacquet module calculations from the case of $\mathrm{Sp}_4(F)$ to $\mathrm{O}_4(F)$ case if the rank-one reducibilities are the same (i.e. in the case when $\chi \rtimes 1$ reduces in $\mathrm{SL}_2(F)$). We conclude (cf.[21], Proposition 5.4.) that the only cases of reducibility are $s = \pm\frac{1}{2}$, and the length of the representation $\chi\nu^s \rtimes 1$ is three. Now, analogously as in the case of the representation $\chi\nu^{\frac{1}{2}}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes 1$, when restricting to $\mathrm{SO}_4(F)$, we obtain that the length of $\chi\nu^{\frac{1}{2}} \rtimes_{\mathrm{SO}_4(F)} 1$ is two. Namely, in $\mathrm{O}_4(F)$

$$\chi\nu^{\frac{1}{2}} \rtimes 1 = L(\chi\nu^{\frac{1}{2}}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes 1) + L(\chi\nu^1 \rtimes \chi) + L(\chi\nu^1 \rtimes \chi \otimes \det).$$

We get that $L(\chi\nu^{\frac{1}{2}}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes 1) \cong L(\chi\nu^{\frac{1}{2}}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes 1) \otimes \det$ and $L(\chi\nu^1 \rtimes \chi)|_{\mathrm{SO}_4(F)} = L(\chi\nu^1 \rtimes \chi \otimes \det)|_{\mathrm{SO}_4(F)}$, so that

$$\chi\nu^{\frac{1}{2}} \rtimes_{\mathrm{SO}_4(F)} 1 = L(\chi\nu^{\frac{1}{2}}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{SO}_4(F)} 1) + L(\chi\nu^1 \rtimes \chi)|_{\mathrm{SO}_4(F)}.$$

We already know that $L(\chi\nu^{\frac{1}{2}}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{SO}_4(F)} 1)$ has a generalized Shalika model. Take $\chi = 1$ for a moment. Then, it is easy to see that $L(\nu^1 \rtimes 1)|_{\mathrm{SO}_4(F)}$ is actually the trivial representation of $\mathrm{SO}_4(F)$ and it does not admit a generalized Shalika model, since ψ is a non-trivial character of $\mathcal{H}$. Similarly, if $\chi \neq 1$, then χ composed with the spinor norm is equal to one, since the spinor norm on the Shalika subgroup is trivial (cf.[17], Lemma 2.2 on p.79 and [24] for the unipotent elements in $\mathcal{H}$). We can thus also directly see that $L(\chi\nu^{\frac{1}{2}}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{SO}_4(F)} 1)$ has a generalized Shalika model, since $\chi\nu^{-\frac{1}{2}} \rtimes_{\mathrm{SO}_4(F)} 1$ has it and $L(\chi\nu^1 \rtimes \chi)|_{\mathrm{SO}_4(F)}$ does not have it, and $L(\chi\nu^{\frac{1}{2}}\,\mathrm{St}_{\mathrm{GL}_2(F)} \rtimes_{\mathrm{SO}_4(F)} 1)$ is a quotient of $\chi\nu^{-\frac{1}{2}} \rtimes_{\mathrm{SO}_4(F)} 1$. $\square$

4.2 $\mathrm{Sp}_4(F)$–side

We now analyze the representation $\chi\nu^s \rtimes_{\mathrm{Sp}_4(F)} 1$, where as above χ is unitary and $s \in \mathbb{R}$.

Let $S(\mathrm{SL}_2(F))$ denote the space of smooth, compactly supported functions on $\mathrm{SL}_2(F)$. It comes equipped with an action R of the group $\mathrm{SL}_2(F) \times \mathrm{SL}_2(F)$ given by $R(g_1, g_2)\phi(x) = \phi(g_1^{-1} x g_2)$, for $\phi \in S(\mathrm{SL}_2(F))$, $(g_1, g_2) \in \mathrm{SL}_2(F) \times \mathrm{SL}_2(F)$. In the study of symplectic linear functionals on $\chi\nu^s \rtimes_{\mathrm{Sp}_4(F)} 1$, the following lemma will be very useful.

Lemma 4.3. *We have an exact sequence of* $\mathrm{SL}_2(F) \times \mathrm{SL}_2(F)$*-representations*

$$0 \to S(\mathrm{SL}_2(F)) \to \chi\nu^s \rtimes 1|_{\mathrm{SL}_2(F)\times\mathrm{SL}_2(F)} \to \chi\nu^{s+1/2} \rtimes 1 \otimes \chi\nu^{s+1/2} \rtimes 1 \to 0. \quad (3)$$

Proof. Again, we use Theorem 5.2 of [3] and adapt our notation. So, $P = MU$ denotes the Siegel parabolic subgroup of $\mathrm{Sp}_4(F)$ (with our previous choice of the Borel subgroup). We let $Q = NV$ with $Q = N = \mathrm{SL}_2(F) \times \mathrm{SL}_2(F)$ and

$V = \{e\}$. We decompose $r_{V,1} \circ i_{P,\mathrm{Sp}_4(F)}(\chi\nu^s)$. Here $r_{V,1}$ turns out to be just the restriction to $\mathrm{SL}_2(F) \times \mathrm{SL}_2(F)$. To meet the requirements of the geometric lemma, (decomposability with respect to Q) we have to take $M = P$ and $U = \{e\}$. We describe $P \setminus \mathrm{Sp}_4(F)/Q$ using ([17], Chapter 4, Proposition 2.1). We have just two orbits of Q–action on $P \setminus \mathrm{Sp}_4(F)$; there is an open orbit $Pw^{-1}Q$, where $w = \begin{bmatrix} I_2 & 0 \\ -I_2 & I_2 \end{bmatrix}$, and (closed) orbit PQ. We have the following filtration ($\tau = \chi\nu^s \rtimes 1$)

$$0 \subset \tau_1 \subset \tau,$$

where τ_1 is a subset of functions in τ vanishing outside of PwQ. In the notation of ([3]) we get that two subgroups which we use to decompose $r_{V,1} \circ i_{P,\mathrm{Sp}_4(F)}$ are

$$M' = \left\{ \begin{bmatrix} a & b & 0 & -b \\ c & d & -c & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & -c & d \end{bmatrix} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(F) \right\}$$

and

$$N' = wM'w^{-1} = \left\{ \begin{bmatrix} a & 0 & 0 & -b \\ 0 & d & -c & 0 \\ 0 & -b & a & 0 \\ -c & 0 & 0 & d \end{bmatrix} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(F) \right\}.$$

Then, restricted to τ_1, $r_{V,1}$ acts as $i_{N',Q} \circ w \circ r_{V',1}(\chi\nu^s)$. Note that the restriction $r_{V',1} : AlgP \to AlgM'$ applied to $\chi\nu^s$ gives the trivial character of M', and with conjugation by w, it again gives the trivial character of N', so we have $i_{N',Q}(1)$ (compact induction). Note that N' is isomorphic with $\mathrm{SL}_2(F)$, so that this isomorphism gives an embedding $\mathrm{SL}_2(F) \to \mathrm{SL}_2(F) \times \mathrm{SL}_2(F)$ with $g \mapsto (g, J_2 g J_2)$. By Proposition 2.3 of Chapter 4 in [17] $i_{N',Q}(1) \cong S(\mathrm{SL}_2(F))$. The intertwining operator from $i_{N',Q}(1)$ to $S(\mathrm{SL}_2(F))$ is given by $T(f)(x) = f(1, J_2 x J_2)$.

On the other hand $r_{V,1}$ on τ/τ_1 is composed of $i_{N',Q} \circ e \circ r_{V',1} : AlgP \to AlgQ$, where now $N' = M' = P \cap Q$, so that $r_{V',1}$ denotes the restriction of the representation of P to representation of $P \cap Q$ and $i_{N',Q}$ is compact induction from representations of $P \cap Q$ to representations of Q. It is easy to see that $P \cap Q$ consists of matrices of the form

$$\begin{bmatrix} a_1 & 0 & 0 & b_1 \\ 0 & a_2 & b_2 & 0 \\ 0 & 0 & a_2^{-1} & 0 \\ 0 & 0 & 0 & a_1^{-1} \end{bmatrix}.$$

Note that, in the case of the normalized induction, $\chi\nu^s \rtimes 1$ as a representation of $\mathrm{Sp}_4(F)$ is actually induced from the representation $\chi\nu^s\delta_P^{1/2}$, with our choice of $M = P$ (so that $U = \{e\}$). When we restrict to $P \cap Q$, we get $(\chi\nu^{s+3/2} \otimes \chi\nu^{s+3/2}) \left(\begin{bmatrix} a_1 & b_1 \\ 0 & a_1^{-1} \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ 0 & a_2^{-1} \end{bmatrix} \right)$. When we then induce to $\mathrm{SL}_2(F) \times \mathrm{SL}_2(F)$, we get $\chi\nu^{s+3/2} \rtimes' 1 \otimes \chi\nu^{s+3/2} \rtimes' 1$ as a representations of $\mathrm{SL}_2(F) \times \mathrm{SL}_2(F)$ (the prime denotes the unnormalized induction, so in our usual notation of the normalized induction, we get $\chi\nu^{s+1/2} \rtimes 1 \otimes \chi\nu^{s+1/2} \rtimes 1$). $\square$

We now analyze (3) to see if $\chi\nu^s \rtimes 1|_{\mathrm{SL}_2(F)\times \mathrm{SL}_2(F)}$ has a trivial quotient. We use the following fact: for any irreducible smooth representation σ of $\mathrm{SL}_2(F)$, the largest σ–isotypic component of $S(\mathrm{SL}_2(F))|_{R_{\mathrm{SL}_2(F)\times 1}}$ is $\sigma \otimes \widetilde{\sigma}$, as a $\mathrm{SL}_2(F) \times \mathrm{SL}_2(F)$-module.

Proposition 4.4. *Let χ be a unitary character and $s \in \mathbb{R}$. Assume $(\chi, s) \neq (1, -\frac{3}{2})$. Then the representation $\chi\nu^s \rtimes_{\mathrm{Sp}(4)} 1$ has a symplectic linear model. The representation $\nu^{-3/2} \rtimes_{\mathrm{Sp}_4(F)} 1$ has the trivial representation as a subquotient, and the trivial representation obviously has a symplectic linear model.*

Proof. Assume χ is a ramified character. Then, using the Bernstein center decomposition, we get that in that case the epimorphism $S(\mathrm{SL}_2(F)) \to 1 \otimes 1$ can be extended to $\chi\nu^s \rtimes 1|_{\mathrm{SL}_2(F)\times\mathrm{SL}_2(F)}$, since it is non-zero anyway only on the Bernstein component in which $1 \otimes 1$, as a representation of $\mathrm{SL}_2(F) \times \mathrm{SL}_2(F)$, lies. This component is different from the component in which $\chi\nu^{s+1/2} \rtimes 1 \otimes \chi\nu^{s+1/2} \rtimes 1$ lies. This means that $\chi\nu^s \rtimes 1$ has a non-zero linear symplectic model.

Assume that $\chi = 1$ and $s = \frac{1}{2}$, so that $\chi\nu^{s+1/2} \rtimes 1 \otimes \chi\nu^{s+1/2} \rtimes 1$ has $1 \otimes 1$ as a trivial quotient. Then obviously, $\chi\nu^s \rtimes 1|_{\mathrm{SL}_2(F)\times\mathrm{SL}_2(F)}$ has a trivial quotient.

Assume that χ is unramified, but $\chi\nu^{s+1/2} \neq \nu^{\pm 1}$. For a smooth representation π of $\mathrm{SL}_2(F) \times \mathrm{SL}_2(F)$ we denote by $r_{V_1,V_2}(\pi)$ the Jacquet module of π with respect to the upper-triangular unipotent subgroup of the first and, then, of the second copy of $\mathrm{SL}_2(F)$ (we can view it as $r_{V_1}(r_{V_2}(\pi))$). We get then a smooth $\mathrm{GL}_1(F) \times \mathrm{GL}_1(F)$ module. We apply this Jacquet functor, which is exact, on the exact sequence (3). Since $S(\mathrm{SL}_2(F)) \twoheadrightarrow 1_{\mathrm{SL}_2(F)} \times 1_{\mathrm{SL}_2(F)}$, we have $r_{V_1,V_2}(S(\mathrm{SL}_2(F)) \twoheadrightarrow \nu^{-1} \otimes \nu^{-1}$, so that $\nu^{-1} \otimes \nu^{-1}$ is a subquotient of $r_{V_1,V_2}(\chi\nu^s \rtimes 1|_{\mathrm{SL}_2(F)\times\mathrm{SL}_2(F)})$. Projectivity of cuspidal representations (Lemma 26. of [4]) gives us an epimorphism

$$r_{V_1,V_2}(\chi\nu^s \rtimes 1|_{\mathrm{SL}_2(F)\times\mathrm{SL}_2(F)}) \twoheadrightarrow \nu^{-1} \otimes \nu^{-1}.$$

Actually, Lemma 26. of [4] requires finite-length representations, but we can just apply it to the representation $\chi\nu^s \rtimes 1/W$, where W is a subspace of $S(\mathrm{SL}_2(F))$ such that $S(\mathrm{SL}_2(F))/W \cong 1_{\mathrm{SL}_2(F)} \times 1_{\mathrm{SL}_2(F)}$, i.e., we apply it on $r_{V_1,V_2}(\chi\nu^s \rtimes 1/W)$ which is then clearly of the finite length (as a $\mathrm{GL}_1(F) \times \mathrm{GL}_1(F)$–module). Frobenius reciprocity then gives rise to an $\mathrm{SL}_2(F) \times \mathrm{GL}_1(F)$– intertwining operator, say T,

$$T : r_{V_2}(\chi\nu^s \rtimes 1) \to \nu^{-1} \rtimes 1 \otimes \nu^{-1}.$$

If the image of this operator is $\nu^{-1} \rtimes 1 \otimes \nu^{-1}$, then, there would exist an epimorphism

$$T_1 : r_{V_2}(\chi\nu^s \rtimes 1) \twoheadrightarrow \mathrm{St}_{\mathrm{SL}_2(F)} \otimes \nu^{-1}.$$

If $T_1|_{r_{V_2}(S(\mathrm{SL}_2(F))}=0$, this would give us an epimorphism

$$r_{V_2}(\chi\nu^{s+1/2}\rtimes 1\otimes\chi\nu^{s+1/2}\rtimes 1)=\chi\nu^{s+1/2}\rtimes 1\otimes r_{V_2}(\chi\nu^{s+1/2}\rtimes 1)\twoheadrightarrow \mathrm{St}_{\mathrm{SL}_2(F)}\otimes\nu^{-1},$$

which is impossible, by the requirement $\chi\nu^{s+1/2}\neq\nu^{-1}$. So, we conclude that $T_1|_{r_{V_2}(S(\mathrm{SL}_2(F))}\neq 0$, so that we have an epimorphism

$$r_{V_2}(S(\mathrm{SL}_2(F))\twoheadrightarrow \mathrm{St}_{\mathrm{SL}_2(F)}\otimes\nu^{-1}.$$

By the Frobenius reciprocity, this would give us an non-zero intertwining

$$S(\mathrm{SL}_2(F))\to \mathrm{St}_{\mathrm{SL}_2(F)}\otimes\nu^{-1}\rtimes 1.$$

The image of this intertwining is $\mathrm{St}_{\mathrm{SL}_2(F)}\otimes\nu^{-1}\rtimes 1$ or $\mathrm{St}_{\mathrm{SL}_2(F)}\otimes 1_{\mathrm{SL}_2(F)}$. Note that the maximal isotypic quotient of $\mathrm{St}_{\mathrm{SL}_2(F)}$ in $S(\mathrm{SL}_2(F))$ is $\Theta(\mathrm{St}_{\mathrm{SL}_2(F)})=\mathrm{St}_{\mathrm{SL}_2(F)}$, so we would have an epimorphism $\mathrm{St}_{\mathrm{SL}_2(F)}\twoheadrightarrow\nu^{-1}\rtimes 1$ or $\mathrm{St}_{\mathrm{SL}_2(F)}\twoheadrightarrow 1_{\mathrm{SL}_2(F)}$, which is impossible in both of the cases.

We conclude that we have an epimorphism $r_{V_2}(\chi\nu^s\rtimes 1)\to 1_{\mathrm{SL}_2(F)}\otimes\nu^{-1}$.

Now we continue analogously: the Frobenius isomorphism gives us a non-zero $\mathrm{SL}_2(F)\times\mathrm{SL}_2(F)$–intertwining, say T_2, $\chi\nu^s\rtimes 1\to 1_{\mathrm{SL}_2(F)}\otimes\nu^{-1}\rtimes 1$. If the image of this intertwining were to be $1_{\mathrm{SL}_2(F)}\otimes\nu^{-1}\rtimes 1$, we would have an epimorphism, say T_3, from $\chi\nu^s\rtimes 1$ to $1_{\mathrm{SL}_2(F)}\otimes\mathrm{St}_{\mathrm{SL}_2(F)}$. If T_3 restricted to $S(\mathrm{SL}_2(F))$ is zero, this would give an epimorphism

$$\chi\nu^{s+1/2}\rtimes 1\otimes\chi\nu^{s+1/2}\rtimes 1\twoheadrightarrow 1_{\mathrm{SL}_2(F)}\otimes\mathrm{St}_{\mathrm{SL}_2(F)},$$

which is impossible by our choice of $\chi\nu^s$. So, T_3 restricted to $S(\mathrm{SL}_2(F))$ is non-zero, but this is impossible with the form of the isotypic component, recalled above. Thus, the image of T is $1_{\mathrm{SL}_2(F)}\otimes 1_{\mathrm{SL}_2(F)}$, what is what we wanted. □

Using Propositions 4.2 and 4.4, we conclude

Proposition 4.5. *For any irreducible subquotient π of $\chi\nu^s\rtimes_{\mathrm{SO}_4(F)}1$ having a generalized Shalika model, its ("small") theta-lift to $\mathrm{Sp}_4(F)$ is non-zero and has a linear symplectic model.*

Proof. Assume $\chi^2\neq 1$. Then, the representations $\chi\nu^s\rtimes_{\mathrm{SO}_4(F)}1$ and $\chi\nu^s\rtimes_{\mathrm{Sp}_4(F)}1$ are irreducible. From Theorem 2.4 we get that $n(\chi\nu^s\rtimes_{\mathrm{SO}_4(F)}1)=2$. Moreover, in the same way as in Proposition 3.3, we get that $\theta(\chi\nu^s\rtimes_{\mathrm{SO}_4(F)}1,2)=\chi\nu^s\rtimes_{\mathrm{Sp}_4(F)}1$ and then apply Propositions 4.2 and 4.4. Assume that $\chi^2=1$. Then, if $\chi\neq 1$ then for $s\neq\pm\frac{1}{2}$ the representations $\chi\nu^s\rtimes_{\mathrm{SO}_4(F)}1$ and $\chi\nu^s\rtimes_{\mathrm{Sp}_4(F)}1$ are both irreducible and the conclusion follows as previously. For $s=\frac{1}{2}$, we know that $L(\chi\nu^{\frac{1}{2}}\mathrm{St}_{\mathrm{GL}_2(F)}\rtimes_{\mathrm{SO}_4(F)}1)$ is a subquotient of $\chi\nu^s\rtimes_{\mathrm{SO}_4(F)}1$ and has a non-zero generalized Shalika model, and in Proposition 3.3 we have already proved that $\theta(L(\chi\nu^{\frac{1}{2}}\mathrm{St}_{\mathrm{GL}_2(F)}\rtimes_{\mathrm{SO}_4(F)}1))=L(\chi\nu^{\frac{1}{2}}\mathrm{St}_{\mathrm{GL}_2(F)}\rtimes_{\mathrm{Sp}_4(F)}1)$. The other subquotients of $\chi\nu^s\rtimes_{\mathrm{SO}_4(F)}1$ do not have the generalized Shalika models (Proposition 4.2). We have also proved that $\theta(L(\nu^{\frac{1}{2}}\mathrm{St}_{\mathrm{GL}_2(F)}\rtimes_{\mathrm{SO}_4(F)}1))=L(\nu^{\frac{1}{2}}\mathrm{St}_{\mathrm{GL}_2(F)}\rtimes_{\mathrm{Sp}_4(F)}1)$ (the case of $\chi=1$) in (Proposition 3.3), so the conclusion is the same. Note that in the case $s=\frac{3}{2}$ the small theta lift of $\nu^{3/2}\rtimes_{\mathrm{SO}_4(F)}1$ is the trivial representation of $\mathrm{Sp}_4(F)$ (cf. Theorem 5.1 (ii) of [17]), and all the cases are covered. □

5 Final case and proof of the main theorem

5.1 The case of irreducible principal series

In this section, we consider the case where τ is an irreducible principal series of $\mathrm{GL}_2(F)$ with trivial central character. The representation τ is of the form $\chi\nu^s \times \chi^{-1}\nu^{-s}$, with χ a unitary character and s in $\mathbb{R}$. The irreducibility condition for τ is : $(\chi^2, |s|) \neq (1, \frac{1}{2})$.

Proposition 5.1. *Let χ be a unitary character of $\mathrm{GL}_2(F)$ and s in $\mathbb{R}$, with $(\chi^2, |s|) \neq (1, \frac{1}{2})$, and let τ be the representation $\chi\nu^s \times \chi^{-1}\nu^{-s}$.*

Then, the representation $\tau\nu^{\frac{1}{2}} \rtimes_{\mathrm{SO}_4(F)} 1$ has $\left(\chi\nu^s \rtimes_{\mathrm{SO}_4(F)} 1\right)^\epsilon$ as unique irreducible quotient. Its theta lift to $\mathrm{Sp}_4(F)$ is:

(i) $\chi\nu^s \rtimes_{\mathrm{Sp}_4(F)} 1$ *if* $(\chi, s) \neq (1, \pm\frac{3}{2})$ *;*

(ii) $1_{\mathrm{Sp}_4(F)}$ *if* $(\chi, s) = (1, \pm\frac{3}{2})$.

This theta lift has a symplectic model.

Proof. We first note that the map sending f to $s \mapsto f(\epsilon s \epsilon)$ gives an isomorphism:

$$\begin{aligned}(\tau\nu^{\frac{1}{2}} \rtimes_{\mathrm{SO}_4(F)} 1)^\epsilon &= \left(\left(\chi\nu^{s+\frac{1}{2}} \times \chi^{-1}\nu^{-s+\frac{1}{2}}\right) \rtimes_{\mathrm{SO}_4(F)} 1\right)^\epsilon \\ &\cong \left(\chi\nu^{s+\frac{1}{2}} \times \chi\nu^{s-\frac{1}{2}}\right) \rtimes_{\mathrm{SO}_4(F)} 1 = \chi\nu^s(\nu^{\frac{1}{2}} \times \nu^{-\frac{1}{2}}) \rtimes_{\mathrm{SO}_4(F)} 1.\end{aligned}$$

The last representation has a unique irreducible quotient, namely $\chi\nu^s \rtimes_{\mathrm{SO}_4(F)} 1$. This gives that the representation $\tau\nu^{\frac{1}{2}} \rtimes_{\mathrm{SO}_4(F)} 1$ has a unique irreducible quotient, $\left(\chi\nu^s \rtimes_{\mathrm{SO}_4(F)} 1\right)^\epsilon$.

Since $\left(\chi\nu^s \rtimes_{\mathrm{SO}_4(F)} 1\right)^\epsilon$ and $\chi\nu^s \rtimes_{\mathrm{SO}_4(F)} 1$ are non-isomorphic, these two representations have the same theta lift to $\mathrm{Sp}_4(F)$ (Lemma 2.3 and Relation (2)). Proposition 4.5 and its proof then give the desired result. $\square$

5.2 Main theorem

Theorem 5.2. *Let π be an irreducible smooth admissible representation of $\mathrm{SO}_4(F)$ with a generalized Shalika model. Then $\theta(\pi)$ is non-zero and has a symplectic linear model.*

Proof. By Theorem 1.2 the representation π is a quotient of $\tau\nu^{1/2} \rtimes_{\mathrm{SO}_4(F)} 1$ for an irreducible admissible representation τ of $\mathrm{GL}_2(F)$. If τ is supercuspidal, everything is known by Theorem 1.1. So we may assume that τ is not supercuspidal.

Lemma 2.2 combined with Lemma 3.2, Propositions 4.2 and 5.1 implies that π must then be one of the representations in the first column of the following table.

π	$\theta(\pi)$
$L(\chi\,\mathrm{St}_{\mathrm{GL}_2(F)}\,\nu^{1/2} \rtimes_{\mathrm{SO}_4(F)} 1)$	$L(\chi\,\mathrm{St}_{\mathrm{GL}_2(F)}\,\nu^{1/2} \rtimes_{\mathrm{Sp}_4(F)} 1)$
$\chi\nu^s \rtimes_{\mathrm{SO}_4(F)} 1$, s.th $(\chi^2, s) \neq (1, \pm\frac{1}{2})$ and $(\chi, s) \neq (1, \pm\frac{3}{2})$	$\chi\nu^s \rtimes_{\mathrm{Sp}_4(F)} 1$
$\nu^{\pm 3/2} \rtimes_{\mathrm{SO}_4(F)} 1$	$1_{\mathrm{Sp}_4(F)}$
$\left(\chi\nu^s \rtimes_{\mathrm{SO}_4(F)} 1\right)^{\epsilon}$, s.th $(\chi^2, \lvert s\rvert) \neq (1, 1/2)$	$\chi\nu^s \rtimes_{\mathrm{Sp}_4(F)} 1$

Note that in the first column, the first three entries indeed all have a generalized Shalika model. The last entry of the first column might have a generalized Shalika model. Note furthermore that all entries in the second column are non-zero. We have shown in Lemma 3.1 and Proposition 4.4 that all representations in the second column have a symplectic linear model. Finally by Proposition 3.3 and 4.5 a representation in the second column is indeed the theta lift of the representation in the same line in the first column, which completes the proof. □

References

[1] J. Arthur, *The endoscopic classification of representations*, vol. 61 of American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, RI, 2013. Orthogonal and symplectic groups.

[2] D. Ban, *Parabolic induction and Jacquet modules of representations of* $\mathrm{O}(2n, F)$, Glas. Mat. Ser. III, 34(54) (1999), pp. 147–185.

[3] I. N. Bernstein and A. V. Zelevinsky, *Induced representations of reductive p-adic groups. I*, Ann. Sci. École Norm. Sup. (4), 10 (1977), pp. 441–472.

[4] J. Bernstein, *Draft of: Representations of p-adic groups*, (1992).

[5] J. W. Cogdell, I. I. Piatetski-Shapiro, and F. Shahidi, *Functoriality for the quasisplit classical groups*, in On certain L-functions, vol. 13 of Clay Math. Proc., Amer. Math. Soc., Providence, RI, 2011, pp. 117–140.

[6] W. T. Gan and G. Savin, *Representations of metaplectic groups I: epsilon dichotomy and local Langlands correspondence*, Compos. Math., 148 (2012), pp. 1655–1694.

[7] W. T. Gan and S. Takeda, *The local Langlands conjecture for Sp(4)*, Int. Math. Res. Not. IMRN, (2010), pp. 2987–3038.

[8] ——, *The local Langlands conjecture for* $\mathrm{GSp}(4)$, Ann. of Math. (2), 173 (2011), pp. 1841–1882.

[9] D. Ginzburg, S. Rallis, and D. Soudry, *On a correspondence between cuspidal representations of* GL_{2n} *and* $\widetilde{\mathrm{Sp}}_{2n}$, J. Amer. Math. Soc., 12 (1999), pp. 849–907.

[10] ——, *Generic automorphic forms on* $\mathrm{SO}(2n+1)$*: functorial lift to* $\mathrm{GL}(2n)$*, endoscopy, and base change*, Internat. Math. Res. Notices, (2001), pp. 729–764.

[11] H. Jacquet and S. Rallis, *Uniqueness of linear periods*, Compositio Math., 102 (1996), pp. 65–123.

[12] D. Jiang, C. Nien, and Y. Qin, *Symplectic supercuspidal representations and related problems*, Sci. China Math., 53 (2010), pp. 533–546.

[13] ——, *Symplectic supercuspidal representations of $GL(2n)$ over p-adic fields*, Pacific J. Math., 245 (2010), pp. 273–313.

[14] ——, *Generalized Shalika models of p-adic SO_{4n} and functoriality*, Israel J. Math., 195 (2013), pp. 135–169.

[15] D. Jiang and Y. Qin, *Residues of Eisenstein series and generalized Shalika models for* SO_{4n}, J. Ramanujan Math. Soc., 22 (2007), pp. 101–133.

[16] D. Jiang and D. Soudry, *Appendix: On the local descent from* $\mathrm{GL}(n)$ *to classical groups [appendix to mr2931222]*, Amer. J. Math., 134 (2012), pp. 767–772.

[17] S. Kudla, *Notes on the local theta correspondence (lectures at the European School in Group Theory)*, (1996).

[18] S. S. Kudla, *Splitting metaplectic covers of dual reductive pairs*, Israel J. Math., 87 (1994), pp. 361–401.

[19] J.-P. Labesse and R. P. Langlands, *L-indistinguishability for* $\mathrm{SL}(2)$, Canad. J. Math., 31 (1979), pp. 726–785.

[20] C. Mœglin, M.-F. Vignéras, and J.-L. Waldspurger, *Correspondances de Howe sur un corps p-adique*, vol. 1291 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1987.

[21] P. J. Sally, Jr. and M. Tadić, *Induced representations and classifications for* $\mathrm{GSp}(2,F)$ *and* $\mathrm{Sp}(2,F)$, Mém. Soc. Math. France (N.S.), (1993), pp. 75–133.

[22] B. Sun and C.-B. Zhu, *Conservation relations for local theta correspondence*, http://arxiv.org/pdf/1204.2969v2.pdf, (2012).

[23] M. Tadić, *On reducibility of parabolic induction*, Israel J. Math., 107 (1998), pp. 29–91.

[24] H. Zassenhaus, *On the spinor norm*, Arch. Math., 13 (1962), pp. 434–451.