R–groups for metaplectic groups*

Marcela Hanzer[†1]

[1]Department of Mathematics, University of Zagreb, Croatia

Abstract

In this short note, we completely describe a parabolically induced representation $\mathrm{Ind}_P^{\widetilde{Sp(2n,F)}}(\sigma)$, in particular, its length and multiplicities. Here, $\widetilde{Sp(2n,F)}$ is a p–adic metaplectic group, and σ is a discrete series representation of a Levi subgroup of P. A multiplicity one result follows.

1 Introduction

A knowledge of R–groups (in all their variants–classical, Arthur, etc.) is very important in understanding the representation theory of reductive p–adic groups, especially their unitary duals. Let G be a reductive p–adic group with a parabolic subgroup $P = MN$ where M is a Levi subgroup. Assume σ is a discrete series (complex) representation of M. Then, it is important to understand how the representation $\mathrm{Ind}_P^G(\sigma)$ reduces: whether it is reducible, and if it is reducible, whether all the irreducible subquotients appear with multiplicity one. These questions are answered by knowing the structure of the commuting algebra $C(\sigma)$; i.e., the intertwining algebra of $\mathrm{Ind}_P^G(\sigma)$. It is already known from the work of Casselman that the dimension of $C(\sigma)$ is bounded by the cardinality of $W(\sigma)$, the subgroup of the Weyl group of G fixing the representation σ. The precise structure of $C(\sigma)$ is given by a certain subgroup of $W(\sigma)$, called the R–group of σ. This approach to understanding the structure of $\mathrm{Ind}_P^G(\sigma)$ goes back to the work of Knapp and Stein on real groups and principal series representations. In various situations in the p-adic case, the R–groups were computed by Keys ([16],[17]), Winarsky ([30]), Herb ([13]), and, especially important for us, for the classical split groups, by Goldberg ([9], Theorems 4.9, 4.18 and 6.5). This calculation is generalized to hermitian quaternionic groups by this author ([10]).

The metaplectic groups over global fields are extremely important in number theory. Over local fields, besides their obvious importance to the global picture, they give us further information on the representations of classical groups through theta correspondence. The p–adic metaplectic groups, in which we are interested, are not linear algebraic groups. However, over a p–adic field F, they are l–groups, just as the groups of points in F of the classical p–adic groups. The p–adic metaplectic groups share a number of common properties with the (F–points of) reductive group: we can define tori, parabolic subgroups,

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†hanmar@math.hr

etc., so that parabolic induction, Jacquet modules and so on, are defined. We can define square-integrable and tempered representations, and the algebraic criteria of Casselman for representations to be square-integrable or tempered ([5], Theorem 4.4.6 and [27], Section 6) are quite similar between metaplectic and classical groups (cf. [3], Theorem 3.4). We also have the Langlands classification of irreducible representations which is analogous to the one for the reductive algebraic groups (cf. [3], Theorem 1.1).

Now, we could develop the theory of R–groups for metaplectic groups from scratch; i.e., by directly studying the structure of commuting algebra of $\mathrm{Ind}_P^G(\sigma)$ through it relation with the Weyl group, or we can use theta correspondence with the representations of the odd orthogonal groups, for which we know the R–groups (i.e. the structure of $\mathrm{Ind}_P^G(\sigma)$) by the work of Goldberg ([9]). We use the second approach to completely describe the R–groups for metaplectic groups. This is enabled by recent results of Gan and Savin on theta correspondence ([6]), which are very precise in handling tempered representations. We use their results, coupled with a more thorough analysis of the isotypic components in Kudla's filtration ([18], Theorem 2.8). This analysis is similar to that of Muić ([23], Section 3), and given in notation of [2], Section 5. To do that, we also use knowledge about L–packets, now available for metaplectic groups, to extract some information about Jacquet modules of the representations in question.

Our result completely describes the reducibility of $\mathrm{Ind}_P^G(\sigma)$, where G is a metaplectic group and P any parabolic subgroup, in terms of the representations induced from the maximal parabolic subgroups. The representation $\mathrm{Ind}_P^G(\sigma)$ is multiplicity free (for all the unexplained notation we refer to Preliminaries section):

Theorem. *Let $\delta_1, \ldots, \delta_k$ be (unitarizable) discrete series representations of $GL(m_i, F)$, $i = 1, 2, \ldots, k$. Let σ be an irreducible discrete series representation of $\widetilde{Sp(2n, F)}$. Then,*

$$\chi_\psi \delta_1 \times \chi_\psi \delta_2 \times \cdots \times \chi_\psi \delta_k \rtimes \sigma$$

is a direct sum of 2^m mutually inequivalent, irreducible, tempered representations. Here, m is the number of mutually inequivalent δ_i's such that $\chi_\psi \delta_i \rtimes \sigma$ reduces.

In other words, the R–group for the representation above is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^m$.

In this way, we have completed some basic results about representation theory of p–adic metaplectic groups, which we started in ([12]). All the results from [12] and this result about R–groups, confirm the similarities between the classical groups and the metaplectic ones, but the similarities go only so far: e.g., the standard module conjecture and the generalized injectivity conjecture which hold for classical groups ([22],[11]) do not hold for metaplectic groups (cf. Remark after Corollary 9.3. of [6]).

In the Preliminaries section we collect notation and results we need for our computation in the third section: we introduce the orthogonal and symplectic groups over a p–adic field F, and then introduce the metaplectic groups. In the third section we briefly recall theta correspondence and isotypic components. We recall Kudla's filtration of a Jacquet module of the Weil representation, and we give a basic technical result about some isotypic components in this filtration (Lemma 3.3). Then, the basic result is Proposition 3.4, which completely describes the R–group i.e., the reducibility of $\mathrm{Ind}_P^G(\sigma)$ for G metaplectic and P maximal (and a little more than that). After that, the general result easily follows and is given in Theorem 3.5.

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2 Preliminaries

Let F be a non–archimedean field of characteristic zero. All the representations of all the groups in this paper are assumed to be smooth, i.e. each vector in the representation space is fixed by some open compact subgroup. We now introduce the Zelevinsky notation for parabolic induction for general linear and classical p–adic groups (cf. [31]). Let $\pi_1,\dots,\pi_k$ be representations of $GL(n_i,F),\ i=1,\dots,k$. We fix a Borel subgroup consisting of upper triangular matrices in a matrix realization of $GL(n,F)$. The group $GL(n_1+n_2+\cdots+n_k,F)$ has a standard parabolic subgroup, say P, whose Levi subgroup M is isomorphic to $GL(n_1,F)\times GL(n_2,F)\times\cdots\times GL(n_k,F)$. Then we denote $\mathrm{Ind}_P^{GL(n_1+n_2+\cdots+n_k,F)}(\pi_1\otimes\pi_2\otimes\cdots\otimes\pi_k)$ (the normalized induction) by $\pi_1\times\pi_2\times\cdots\times\pi_k$. Analogously, if G is a classical group we fix a Borel subgroup consisting of upper triangular matrices inside the usual matrix realization of G (e.g. [27], Section 3). If a Levi subgroup M (of a standard parabolic subgroup P) of G is isomorphic to $GL(n_1,F)\times GL(n_2,F)\times\cdots\times GL(n_k,F)\times G'$, where G' is a classical group of the same type and smaller rank, and if $\pi_1,\dots,\pi_k$ are representations of $GL_{n_i},\ i=1,\dots,k$ and σ a representation of G', we denote $\mathrm{Ind}_P^G(\pi_1\otimes\pi_2\otimes\cdots\otimes\pi_k\otimes\sigma)$ by $\pi_1\times\pi_2\times\cdots\times\pi_k\rtimes\sigma$. We denote by ν a character of $GL(n,F)$ obtained by composing the determinant character of $GL(n,F)$ with the absolute value on F^*.

Let ρ be an irreducible unitary cuspidal representation of $GL(m,F)$ and $k\in\mathbb{N}$. Then, the induced representation $\rho\nu^{\frac{k-1}{2}}\times\rho\nu^{\frac{k-1}{2}-1}\times\cdots\times\rho\nu^{-\frac{k-1}{2}}$ has a unique irreducible subrepresentation which we denote by $\delta(\rho\nu^{-\frac{k-1}{2}},\rho\nu^{\frac{k-1}{2}})$. This representation of $GL(mk,F)$ is square-integrable (mod center) and any square integrable representation of a general linear group is obtained in this way (cf. [31], Theorem 9.3).

2.1 Symplectic and orthogonal groups

For $n\in\mathbb{Z}_{\geq 0}$, let W_{2n} be a symplectic vector space over F of dimension $2n$. We fix a complete polarization as follows

$$W_{2n}=W_n'\oplus W_n'',\ W_n'=\mathrm{span}_F\{e_1,\dots e_n\},\ W_n''=\mathrm{span}_F\{e_1',\dots e_n'\},$$

where $e_i,e_i',\ i=1,\dots,n$ are basis vectors of W_{2n} and the skew–symmetric form on W_{2n} is described by the relations

$$\langle e_i,e_j\rangle=0,\ i,j=1,2,\dots,n,\ \langle e_i,e_j'\rangle=\delta_{ij}.$$

The group $Sp(W_{2n})$ fixes this form. Let P_j denote the maximal parabolic subgroup of $Sp(W_{2n})$ stabilizing the isotropic space $W_n'^j=\mathrm{span}_F\{e_1,\dots e_j\}$; then there is a Levi decomposition $P_j=M_jN_j$ where $M_j=GL(W_n'^j)$. By adding, in each step, a hyperbolic plane to the previous symplectic vector space, we obtain a tower of symplectic spaces and corresponding symplectic groups. We also use $Sp(2n,F)$ to denote $Sp(W_{2n})$.

Now we describe the orthogonal groups we consider. Let V_0 be an anisotropic quadratic space over F of odd dimension; then $\mathrm{dim}V_0\in\{1,3\}$. We fix such V_0. For the description of the invariants of this quadratic space, including the quadratic character χ_{V_0} describing the quadratic form on V_0, we refer to Chapter V of [19]. In each step, as for the symplectic situation, we add a hyperbolic plane and obtain a symmetric bilinear space, i.e., (since we are in the characteristic zero) a quadratic space. We choose a basis for this space analogously as for the symplectic spaces above. Consequently, we get a tower of quadratic spaces and a tower of corresponding orthogonal groups. Each of these groups has the same character χ_{V_0} attached to it, so this character is, in fact, attached to the whole tower. In the case in which r hyperbolic planes are added to the anisotropic space, the corresponding orthogonal group will be denoted $O(V_m)$, where $V_m = V_r' + V_0 + V_r''$ and V_r' and V_r'' are maximal isotropic subspaces defined analogously as for the symplectic space. Here $m = \dim V_m = 2r + \dim V_0$. Again, P_j will be the maximal parabolic subgroup stabilizing $\text{span}_F\{e_1, \dots e_j\}$. We will also use $O(m, F)$ to denote $O(V_m)$. We need to consider simultaneously two towers of quadratic spaces–one with the dimension of the anisotropic bottom space V_0 equal to 1, and the other with this dimension equal to 3– both anisotropic bottom spaces should have the same quadratic character attached to them. These two towers are referred to as "a pair of the orthogonal Witt towers" in Chapter V of [19].

Now we recall Goldberg's results on R–groups for odd orthogonal groups. These results ([9], Theorems 4.9, 4.18 and 6.5) are done in the setting of special orthogonal group $SO(2n+1, F)$, the connected component of $O(2n+1, F)$. We note that an analogous version holds for $O(2n+1, F)$. This follows easily from the following fact: since $O(2n+1, F) \cong SO(2n+1, F) \times \{\pm I\}$, for each irreducible representation π of $O(2n+1, F)$, the representation $\pi_{|SO(2n+1,F)}$ is irreducible (also cf. [21], Chapter 3, II. 5). Here, $\{\pm I\}$ is the center of $O(2n+1, F)$.

Theorem 2.1. *Let $\delta_1, \dots, \delta_k$ be (unitarizable) discrete series representations of $GL(m_i, F)$, $i = 1, 2, \dots, k$. Let σ be an irreducible discrete series representation of $SO(2n+1, F)$ (resp., $O(2n+1, F)$). Then,*

$$\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \sigma$$

is a direct sum of 2^m mutually inequivalent, irreducible, tempered representations of $SO(2n+1, F)$ (resp., $O(2n+1, F)$). Here m is the number of mutually inequivalent δ_i's such that $\delta_i \rtimes \sigma$ reduces.
In other words, the R–group for the representation above is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^m$.

Remark. The results in [9] are formulated for the split orthogonal groups, but it is easily checked that they hold in the same form in the non-split case.

The following is an immediate corollary of Theorem 2.1.

Corollary 2.2. *1. Let π be an irreducible tempered representation of $SO(2n+1, F)$, (resp., $O(2n+1, F)$) satisfying*

$$\pi \hookrightarrow \delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \sigma,$$

where $\delta_1, \dots, \delta_k$ are the discrete series representations of $GL(m_i, F)$, $i = 1, 2, \dots, k$ and σ an irreducible discrete series representation of $SO(2n+1, F)$ (resp., $O(2n+1, F)$). Then, the representation

$$\delta_i \rtimes \pi,$$

is irreducible, for each $i = 1, 2, \dots, k$.

2. Let π be an irreducible tempered representation of $SO(2n+1, F)$, (resp., $O(2n+1, F)$) and δ a discrete series representation of $GL(m, F)$ such that $\delta \rtimes \pi$ reduces. Then, it is a sum of two non-equivalent, irreducible tempered representations.

Proof. Let π be as in the first part of the proof. We have

$$\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \sigma = \pi \oplus \pi_2 \oplus \pi_3 \oplus \cdots \oplus \pi_{2^m},$$

for some other irreducible, tempered representations of $SO(2n+1, F)$, (resp., $O(2n+1, F)$), as in Theorem 2.1. Then, for each $i = 1, 2 \ldots, k$

$$\delta_i \times \delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \sigma = \delta_i \rtimes \pi \oplus \delta_i \rtimes \pi_2 \oplus \delta_i \rtimes \pi_3 \oplus \cdots \oplus \delta_i \rtimes \pi_{2^m}.$$

Again, by Theorem 2.1, the length of $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \sigma$ is the same as the length of $\delta_i \times \delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \sigma$. Thus, the right–hand sides of the two displayed equations above are of the same length, so $\delta_i \rtimes \pi$ is irreducible (as are all other representations on the right-hand side of the last displayed equation) and the first part of Corollary is proved.

Now we prove the second part. For each irreducible tempered π, there exist irreducible discrete series representations $\delta_1, \ldots, \delta_k$ of general linear groups, and a discrete series representation σ of a smaller (special) orthogonal group, such that

$$\pi \hookrightarrow \delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \sigma.$$

Moreover, let

$$\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \sigma = \pi \oplus \pi_2 \oplus \pi_3 \oplus \cdots \oplus \pi_{2^m},$$

as in the first part of the proof. Then,

$$\delta \times \delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \sigma = \delta \rtimes \pi \oplus \delta \rtimes \pi_2 \oplus \delta \rtimes \pi_3 \oplus \cdots \oplus \delta \rtimes \pi_{2^m}.$$

Since $\delta \rtimes \pi$ is reducible, $\delta \times \delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \sigma$ and $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \sigma$ do not have the same length, i.e., by Theorem 2.1, the length of the former is 2^{m+1}. Now, if we want to employ the whole power of Arthur results ([1]), we can argue that $\pi, \pi_2, \ldots, \pi_{2^m}$ belong to the same L-packet, so we have the equality of Langlands-Shahidi L-functions $L(s, \delta \times \pi) = L(s, \delta \times \pi_i)$, $i = 2, 3, \ldots, 2^m$. The meromorphic properties of this function (together with $L(s, \delta, Sym^2)$) govern the reducibility of $\delta \rtimes \pi_i, i = 2, 3, \ldots, 2^m$, so, we get that all the representations $\delta \rtimes \pi_i, i = 2, 3, \ldots, 2^m$, are simultaneously reducible, thus, each has to be of length two. We can skip this argument which is short, but relies on some very deep results and proceed as follows. As we saw in Introduction, the structure of the induced representation we are interested in is governed by the intertwining algebra. Goldberg actually proved in [9] that the intertwining algebra

$$\mathrm{Hom}(\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \sigma, \delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \sigma)$$

is generated by the operators $\{R_i : i \in S\}$. Here R_i is induced from the operator $\delta_i \rtimes \sigma \to \delta_i \rtimes \sigma$, where S is the set of all mutually non-isomorphic δ_i such that $\delta_i \rtimes \sigma$ reduces. But this means that the intertwining algebra

$$\mathrm{Hom}(\delta \times \delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \sigma, \delta \times \delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \sigma)$$

is generated by the images of $\{R_i : i \in S\}$ under induction and by the image of the (long) intertwining operator $A : \delta \rtimes \sigma \to \delta \rtimes \sigma$ under induction. For each irreducible constituent $\pi' \in \{\pi, \pi_2, \ldots, \pi_{2^m}\}$ the images of $\{R_i : i \in S\}$ act on $\delta \rtimes \pi'$ as scalars, so $\mathrm{Hom}(\delta \rtimes \pi', \delta \rtimes \pi')$ is generated by the image of A. Since A has just two eigenspaces, this means that the length of $\delta \rtimes \pi'$ is at most 2. Since the length of $\delta \times \delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \sigma$ is 2^{m+1}, the length of $\delta \rtimes \pi'$ is equal to two for each $\pi' \in \{\pi, \pi_2, \ldots, \pi_{2^m}\}$. □

2.2 The metaplectic group

Let W_{2n} be the symplectic space as above. The metaplectic group $\widetilde{Sp(2n,F)}$ is given as a central extension

$$1 \to \mu_2 \to \widetilde{Sp(2n,F)} \to Sp(2n,F) \to 1 \quad (1)$$

where $\mu_2 = \{1,-1\}$ and the cocyle involved is Rao's cocycle ([24]). For a more thorough description of the structure theory of the metaplectic group we refer to [19],[24],[8],[12]. Specifically, for every subgroup G of $Sp(2n,F)$ we denote by $\widetilde{G}$ its preimage in $\widetilde{Sp(2n,F)}$. In this way, the standard parabolic subgroups of $\widetilde{Sp(2n,F)}$ are defined. Then, we have $\widetilde{P_j} = \widetilde{M_j}N'_j$, where N'_j is the image in $\widetilde{Sp(2n,F)}$ of the unique monomorphism from N_j (the unipotent radical of P_j) to $\widetilde{Sp(2n,F)}$. We emphasize that $\widetilde{M_j}$ is not a product of GL factors and a metaplectic group of smaller rank, but there is an epimorphism (this is the case of maximal parabolic subgroup, cf. [26])

$$\phi : \widetilde{GL(j,F)} \times \widetilde{Sp(2n-2j,F)} \to \widetilde{M_j}. \quad (2)$$

Here, we can view $\widetilde{GL(j,F)}$ as a two–fold cover of $GL(j,F)$ in its own right. In this way, an irreducible representation π of $\widetilde{M_j}$ can be considered as a representation $\rho \otimes \sigma$ of $\widetilde{GL(j,F)} \times \widetilde{Sp(2n,F)}$, where ρ and σ are irreducible representations, provided they are both trivial or both non–trivial when restricted to μ_2. We are concerned with the case where both of these representations are *genuine*, i.e., nontrivial on μ_2. Moreover, all the representations of $\widetilde{Sp(2n,F)}$ we are concerned with will be genuine. The epimorphism (2) justifies the use of Zelevinsky notation in the metaplectic case.

The pair $(Sp(2n,F), O(V_m))$ constitutes a dual pair in $Sp(2n, F \cdot m)$ ([19],[18]). Since m is odd, the group $Sp(2n,F)$ does not split in $\widetilde{Sp}(2nm)$, so the theta correspondence relates the representations of $\widetilde{Sp(2n,F)}$ and of $O(V_m)$. More on the theta correspondence will be recalled in Section 3 below.

From now on, we fix an additive, non–trivial character ψ of F related to the theta correspondence ([18] and [19], Chapter II), and the character χ_ψ on $\widetilde{GL(n,F)}$ given by

$$\chi_\psi(g,\epsilon) = \epsilon\gamma(\det g, \frac{1}{2}\psi)^{-1}. \quad (3)$$

Here, for $a \in F^*$ and a non-trivial additive character η of F, $\gamma(a,\eta)$ is defined as the normalized Weil index of the character of the second degree given by $x \mapsto \eta_a(x^2)$, where $\eta_a(x) = \eta(ax)$ (cf. [19], p. 17). Note that each representation δ of $GL(k,F)$ can be considered to be a non-genuine representation of $\widetilde{GL(k,F)}$, and, when tensored with χ_ψ (cf. (3)), becomes a genuine representation of $\widetilde{GL(k,F)}$.

3 Isotypic components and the main result

Let $\omega_{V_m,W_{2n},\psi}$ be the Weil representation of a reductive dual pair $(\widetilde{Sp(2n,F)}, O(m,F))$ with respect to a character ψ. Here $\widetilde{Sp(2n,F)}$ and $O(m,F)$ are groups described in the second section. We actually consider two orthogonal groups $O(m, F)$ corresponding to a pair of orthogonal towers as explained in Section 2.1. We are primarily interested in the representation theory of metaplectic groups, and theta correspondence is a tool we use for this analysis. So we choose to study the theta correspondence between metaplectic groups and orthogonal groups in a pair of towers attached to the trivial quadratic character, i.e., using notation from Section 2.1., χ_{V_0} is trivial. We make this choice because the classification of the representations of metaplectic groups in [6], which we use, is given through the theta correspondence with the pair of orthogonal towers attached to the trivial quadratic character. To distinguish between the orthogonal groups in the two towers, we denote by $O(m, F)^+$ the orthogonal group of the quadratic space of dimension m in a tower where the anisotropic space at the bottom is one-dimensional, and by $O(m, F)^-$ the orthogonal group of the quadratic space of dimension m in a tower where the anisotropic space at the bottom is three-dimensional; $SO(m, F)^+$ and $SO(m, F)^-$ denote their connected components, respectively.

Let $\widetilde{P_k}$ denote a maximal standard parabolic subgroup of $\widetilde{Sp(2n, F)}$ defined above. Then, by $R_{\widetilde{P_k}}(\omega_{V_m,W_{2n},\psi})$ we denote the normalized Jacquet module of $\omega_{V_m,W_{2n},\psi}$ with respect to $\widetilde{P_k}$; it is an $\widetilde{GL(k, F)} \times \widetilde{Sp(2n-2k, F)} \times O(V_m)$–module (cf. (2)). We use Kudla's filtration (cf. [19], Theorem 8.1.) of $R_{\widetilde{P_k}}(\omega_{V_m,W_{2n},\psi})$ in the form given in [2], Lemma 5.1.

We need some notation: Assume that Π is a smooth representation of a product of l–groups $G_1 \times G_2$. Let ξ be an irreducible smooth representation of G_1; by $\Theta(\xi, \Pi)$ we denote the isotypic component of ξ in Π. More explicitly, with

$$W := \bigcap_{\substack{f:\Pi \to \xi \\ G_1 intertwining}} \mathrm{Ker} f$$

we have $\Theta(\xi, \Pi) = \Pi/W$. The representation $\Theta(\xi, \Pi)$ has a natural structure of G_2–module and

$$\mathrm{Hom}_{G_1}(\Pi, \xi)_\infty \cong \Theta(\xi, \Pi)^\vee; \tag{4}$$

here $\mathrm{Hom}_{G_1}(\Pi, \xi)_\infty$ denotes the smooth part of $\mathrm{Hom}_{G_1}(\Pi, \xi)$ and $\vee$ denotes the contragredient.

Now, we return to the theta correspondence. Let π be an irreducible smooth representation of $\widetilde{Sp(2n, F)}$. We say that the theta lift on the dimension level m (in one of the orthogonal towers) is non-zero if $\Theta(\pi, \omega_{V_m,W_{2n},\psi}) \neq 0$. We then call $\Theta(\pi, \omega_{V_m,W_{2n},\psi})$ *the full theta lift* of π on V_m and, to simplify the notation, we denote it by $\Theta(\pi, m)$. This is, as observed above, a representation of $O(m, F)$. Note that in this notation, the dependence on ψ is suppressed; also it is assumed that we know to which tower this lift refers (i.e., whether it is a representation of $O(m, F)^+$ or $O(m, F)^-$).

By the *Howe duality conjecture* (cf. [14], [15]), proved by Waldspurger when the residual characteristic is different from 2 ([29]), and in the general case by Gan and Takeda ([7]), the representation $\Theta(\pi, m)$ has a unique irreducible quotient which we call the *small theta lift* and denote by $\theta(\pi, m)$. Moreover, the correspondence

$$\pi \leftrightarrow \theta(\pi, m)$$

is a bijection between representations of $\widetilde{Sp(2n, F)}$ and $O(V_m)$ participating in the theta correspondence (i.e., having non-zero lifts).

It is known that there is exactly one odd orthogonal tower (in a pair, as above) such that the theta lift of π to that tower on the dimension level $2n+1$ is non-zero. This follows from *the conservation conjecture*, originally conjectured by Kudla and Rallis ([20]), and finally proved (in the general case) by Sun and Zhu ([25]). In [6] (cf. Introduction there) the following parameterization of the irreducible representations of $\widetilde{Sp(2n,F)}$ is given:

$$\mathrm{Irr}\widetilde{Sp(2n,F)} \longleftrightarrow \mathrm{Irr}SO(2n+1,F)^{+} \cup \mathrm{Irr}SO(2n+1,F)^{-}. \tag{5}$$

This bijection is given by the theta correspondence: for a given representation π of $\widetilde{Sp(2n,F)}$ we obtain a representation $\theta(\pi,2n+1) \neq 0$ in one of the towers, say $\varepsilon \in \{+,-\}$ (the lift to the other tower is zero) and then restrict it to a representation of $SO(2n+1,F)^{\varepsilon}$. This restriction remains irreducible, as we noted before. On the other hand, for a given irreducible representation σ of $SO(2n+1,F)^{\varepsilon}$, exactly one of the two possible extensions of this representation to $O(2n+1,F)^{\varepsilon}$ participates in the theta correspondence with the metaplectic group $\widetilde{Sp(2n,F)}$. We denote this (cf. (5)) slightly modified theta correspondence by $GS_{\psi}(\cdot)$. That is, if π is an irreducible representation of $\widetilde{Sp(2n,F)}$, then

$$GS_{\psi}(\pi,2n+1) = \theta(\pi,2n+1)|_{SO(2n+1,F)^{\varepsilon}}; \tag{6}$$

if σ is a representation of $SO(2n+1,F)^{\varepsilon}$, then

$$GS_{\psi}(\sigma,2n) = \theta(\sigma^{\delta},2n), \tag{7}$$

where σ^{δ} is the unique extension of σ to $O(2n+1,F)^{\varepsilon}$ whose lift to $\widetilde{Sp(2n,F)}$ is non-zero. Here $\delta \in \{1,-1\}$ denotes the value of the extended representation on $-I \in O(2n+1,F)^{\varepsilon} \setminus SO(2n+1,F)^{\varepsilon}$.

Now we give two results which we use later; although parts of this were known earlier in some form (e.g. [23], Theorem 6.2), we are taking them in the form given in [6]. In the formulations we take into account the fact that the Howe duality conjecture was meanwhile proved (cf. [7]).

Proposition 3.1. *(cf. Theorem 8.1.(i) and (ii) of [6]) For an irreducible tempered representation π of $\widetilde{Sp(2n,F)}$, $\Theta(\pi,2n+1)$ (the non-zero full lift on the appropriate tower) is irreducible and tempered. Moreover, if π is a discrete series representation, $\Theta(\pi,2n+1)$ is a discrete series representation (and of course, irreducible). An analogous claim holds for irreducible tempered representations of orthogonal groups $O(2n+1,F)^{\varepsilon}$.*

Proposition 3.2. *(cf. Theorem 8.1 (ii) and Theorem 1.3. (ii) of [6]) Let π be an irreducible tempered representation of $\widetilde{Sp(2n,F)}$ such that*

$$\pi \hookrightarrow \chi_{\psi}\tau_1 \times \chi_{\psi}\tau_2 \times \cdots \times \chi_{\psi}\tau_r \rtimes \pi_0,$$

where $\chi_{\psi}\tau_1,\ldots,\chi_{\psi}\tau_r$ are irreducible discrete series representations of $\widetilde{GL(n_1,F)},\ldots,\widetilde{GL(n_r,F)}$ and π_0 is an irreducible discrete series of $\widetilde{Sp(2n,F_0)}$. Then,

$$\Theta(\pi,2n+1) \hookrightarrow \tau_1 \times \tau_2 \times \cdots \times \tau_r \rtimes \Theta(\pi_0,2n_0+1).$$

The analogous claim holds if we exchange $\Theta(\pi, 2n+1)$ *and* $\Theta(\pi_0, 2n_0+1)$ *with* $GS_\psi(\pi, 2n+1)$ *and* $GS_\psi(\pi_0, 2n_0+1)$ *(note that by Proposition 3.1,* $\Theta(\pi_0, 2n_0+1)$ *is an irreducible discrete series representation).*

To proceed, we need to calculate certain isotypic components for the subquotients of Kudla's filtration of $R_{\widetilde{P_k}}(\omega_{V_{2n+1+2k},W_{2n+2k},\psi})$ (cf. [18], Theorem 2.8). This is a representation of $\widetilde{GL_k(F)} \times \widetilde{Sp(2n,F)} \times O(V_{2n+1+2k})$. Let π be an irreducible representation of $\widetilde{Sp(2n,F)}$, and δ be an irreducible representation of $GL(k,F)$. Recall from Section 2.2. that $\chi_\psi\delta$ is a representation of $\widetilde{GL_k(F)}$. We concentrate on the odd dimensional tower for which $\Theta(\pi, 2n+1)$ is non-zero. We have

$$R_{\widetilde{P_k}}(\omega_{V_{2n+1+2k},W_{2n+2k},\psi}) = R_0 \supset R^1 \supset \cdots \supset R^k \supset R^{k+1} = 0,$$

where the successive quotients $J^a = R^a/R^{a+1}$ are $\widetilde{GL_k(F)} \times \widetilde{Sp(2n,F)} \times O(V_{2n+1+2k})$–invariant and described in Lemma 5.1 of [2]. In a manner very similar to what Muić has done in [23] (cf. Lemmas 3.1, 3.2 and 3.3 there) for the symplectic-even orthogonal pairs, we obtain an analogous result for the metaplectic-odd orthogonal pairs by using Kudla's filtration as given in Lemma 5.1 of [2]. These results of Muić were obtained by a very careful analysis of the successive subquotients J^a of Kudla filtration, primarily using the second Frobenius isomorphism, i.e., the second adjointness of Bernstein (cf. Chapter 3, Section 3 of [4]). The first part of the following lemma (dealing with the bottom part of the Kudla's filtration) and, especially, the third part (dealing with the top part of the filtration), are much easier to prove; the second part (dealing with the intermediate parts of the filtration) is more technical. The proof of all of them boils down to the use of the second adjointness which addresses the space of certain intertwinings and precisely gives (by (4)) the isotypic components in question.

Lemma 3.3.

- $\Theta(\chi_\psi\delta \otimes \pi, J^k) \neq 0$ *only if* $\Theta(\pi, 2n+1) \neq 0$ *and then*

$$\Theta(\chi_\psi\delta \otimes \pi, J^k) \cong \delta^\vee \rtimes \Theta(\pi, 2n+1).$$

- *Assume* $0 < a < k$. *Let* P_{ka} *be the standard parabolic subgroup of* $GL(k,F)$ *isomorphic to* $GL(k-a,F) \times GL(a,F)$, *and let* $R_{P_{ka}}(\delta^\vee)_{\Psi_{ka}^{-1}}$ *denote the maximal quotient of the Jacquet module* $R_{P_{ka}}(\delta^\vee)$ *on which* $GL(k-a,F)$ *acts as the character* $(\Psi_{ka})^{-1} = (|det|^{\frac{k-a}{2}})^{-1}$ *(so that* $R_{P_{ka}}(\delta^\vee)_{\Psi_{ka}^{-1}}$ *is* $GL(a,F)$*–module). Assume that* $R_{P_{ka}}(\delta^\vee)_{\Psi_{ka}^{-1}}$ *is irreducible if non-zero. Then,* $\Theta(\chi_\psi\delta \otimes \pi, J^a)$ *is non-zero only if* $R_{P_{ka}}(\delta^\vee)_{\Psi_{ka}^{-1}} \neq 0$ *and* $\Theta(\pi, 2n+1-2a+2k) \neq 0$. *In that case*

$$\Theta(\chi_\psi\delta \otimes \pi, J^a) \cong R_{P_{ka}}(\delta^\vee)_{\Psi_{ka}^{-1}} \rtimes \Theta(\pi, 2n+1-2a+2k).$$

- $\Theta(\chi_\psi\delta \otimes \pi, J^0) \neq 0$ *if and only if* $\delta \cong |\det|_k^{\frac{k}{2}}$ *and* $\Theta(\pi, 2n+1+2k) \neq 0$ *and then*

$$\Theta(\chi_\psi\delta \otimes \pi, J^0) \cong \Theta(\pi, 2n+1+2k).$$

Remark. We will also use the analogous claim involving the filtration of the Jacquet module $R_{P_k}(\omega_{V_{2n+1+2k},W_{2n+2k},\psi})$, where P_k is a maximal parabolic subgroup of $O(2n+1+2k)^{\varepsilon}$ with Levi subgroup isomorphic to $GL(k,F)\times O(2n+1,F)^{\varepsilon}$. Thus,

$$R_{P_k}(\omega_{V_{2n+1+2k},W_{2n+2k},\psi}) = L_0 \supset L^1 \supset \cdots \supset L^k \supset L^{k+1} = 0,$$

with $I^a = L^a/L^{a+1}$. So, if δ is an irreducible representation of $GL(k,F)$ and σ an irreducible representation of $O(2n+1,F)^{\varepsilon}$, there is a description of the isotypic components $\Theta(\delta\otimes\sigma, I^a)$ analogous to the one in Lemma 3.3.

The following proposition is the key proposition in this article, from which our main result readily follows. Recall that we have defined GS_{ψ} in (6) and (7).

Proposition 3.4. *Let δ be an irreducible, square integrable (unitarizable) representation of $GL_k(F)$ and π be an irreducible tempered representation of $\widetilde{Sp(2n,F)}$. Then, the representation $\chi_{\psi}\delta \rtimes \pi$ reduces if and only if $\delta \rtimes GS_{\psi}(\pi, 2n+1)$ does. If $\chi_{\psi}\delta \rtimes \pi$ reduces, then it is a sum of two non-equivalent tempered representations. Moreover, assume that*

$$\delta \rtimes \Theta(\pi, 2n+1) = T_1 \oplus T_2,$$

where T_1 and T_2 are irreducible and non-equivalent (cf. Corollary 2.2, 2.). Then

$$\chi_{\psi}\delta \rtimes \pi = \Theta(T_1, 2n+2k) \oplus \Theta(T_2, 2n+2k).$$

Here $\Theta(\pi, 2n+1)$, $\Theta(T_1, 2n+2k)$ and $\Theta(T_2, 2n+2k)$ denote the full non-zero lifts on the (same) appropriate tower.

Proof. First, assume that $\delta \rtimes GS_{\psi}(\pi, 2n+1)$ is irreducible. Recall that an irreducible representation of $O(2n+1,F)^{\varepsilon}$ is irreducible when restricted to $SO(2n+1,F)^{\varepsilon}$ ([21], Chapter 3, II. 5); by Proposition 3.1 $\delta \rtimes \Theta(\pi, 2n+1)$ is also irreducible. We prove that then $\chi_{\psi}\delta \rtimes \pi$ is irreducible.

We apply Lemma 3.3 with δ and π as in this Proposition. Since δ is a unitary representation, the third possibility in the above Lemma cannot happen. Now we discuss the second possibility. Having in mind what the Jacquet module with respect to a maximal parabolic subgroup P_{ka} for a discrete series representation of a general linear group looks like (cf. [31], Section 3), if $GL(k-a,F)$ acts as a character, we must have $k-a=1$. But then $\delta = \delta(\chi\nu^{-\frac{k-1}{2}}, \chi\nu^{\frac{k-1}{2}})$ for some unitary character χ. Moreover, $R_{P_{k,k-1}}(\delta^{\vee}) = \chi^{\vee}\nu^{\frac{k-1}{2}} \otimes \delta(\chi^{\vee}\nu^{-\frac{k-1}{2}}, \chi^{\vee}\nu^{\frac{k-1}{2}-1})$. But, $\frac{k-1}{2} \neq -\frac{1}{2}$, so the second possibility cannot happen.

We conclude that $\Theta(\chi_{\psi}\delta\otimes\pi, J^a) \neq 0$ only if $a=k$. This guarantees that the restriction map

$$\mathrm{Hom}_{\widetilde{GL(k,F)}\times\widetilde{Sp(W_{2n})}}(R_{\widetilde{P_k}}(\omega_{V_{2n+1+2k},W_{2n+2k},\psi}), \chi_{\psi}\delta\otimes\pi)_{\infty} \to \mathrm{Hom}_{\widetilde{GL(k,F)}\times\widetilde{Sp(W_{2n})}}(R^k, \chi_{\psi}\delta\otimes\pi)_{\infty}$$

is injective. Thus, using (4) and Lemma 3.3, we get

$$\mathrm{Hom}_{GL(\tilde{k},F)\times\widetilde{Sp(W_{2n})}}(R_{\widetilde{P_k}}(\omega_{V_{2n+1+2k},W_{2n+2k},\psi}), \chi_{\psi}\delta\otimes\pi)_{\infty} \hookrightarrow \delta \rtimes \Theta(\pi, 2n+1)^{\vee}.$$

Note that $\Theta(\pi, 2n+1)$ is an irreducible (tempered) representation of an odd orthogonal group, thus $\Theta(\pi, 2n+1)^{\vee} \cong \Theta(\pi, 2n+1)$ (cf. [21], Chapter 4, II.1). The Frobenius reciprocity then gives

$$\mathrm{Hom}_{\widetilde{Sp(W_{2n+2k})}}(\omega_{V_{2n+1+2k},W_{2n+2k},\psi}, \chi_{\psi}\delta \rtimes \pi)_{\infty} \hookrightarrow \delta \rtimes \Theta(\pi, 2n+1).$$

On the other hand, let $\chi_\psi\delta \rtimes \pi = T_1 \oplus T_2 \oplus \cdots \oplus T_l$, where T_i is irreducible tempered. Note that $\Theta(T_i, 2n+2k+1) \neq 0$ (we examine the lift on the same orthogonal tower for which we have $\Theta(\pi, 2n+1) \neq 0$). Indeed, the T_i's and π share the same discrete series support on the smaller metaplectic group and the tower of the non-zero lift is determined by this discrete series (cf. Proposition 3.2). Thus

$$\begin{aligned}&0 \neq \oplus_{i=1}^{l} \mathrm{Hom}_{\widetilde{Sp(W_{2n+2k})}}(\omega_{V_{2n+1+2k}, W_{2n+2k}, \psi}, T_i)_\infty \hookrightarrow \\ &\mathrm{Hom}_{\widetilde{Sp(W_{2n+2k})}}(\omega_{V_{2n+1+2k}, W_{2n+2k}, \psi}, \chi_\psi\delta \rtimes \pi)_\infty \hookrightarrow \\ &\delta \rtimes \Theta(\pi, 2n+1);\end{aligned}$$

note that we assumed that $\delta \rtimes \Theta(\pi, 2n+1)$ is irreducible. Thus $l = 1$ and the representation $\chi_\psi\delta \rtimes \pi$ is irreducible.

Totally analogously, we prove that if $\chi_\psi\delta \rtimes \pi$ is irreducible, so is $\delta \rtimes GS_\psi(\pi, 2n+1)$. We just emphasize the following subtlety. Assume that $\delta \rtimes \Theta(\pi, 2n+1) = T_1 \oplus \cdots \oplus T_l$. As above, we get

$$\begin{aligned}&\oplus_{i=1}^{l} \mathrm{Hom}_{O(V_{2n+1+2k})}(\omega_{V_{2n+1+2k}, W_{2n+2k}, \psi}, T_i)_\infty \hookrightarrow \\ &\mathrm{Hom}_{O(V_{2n+1+2k})}(\omega_{V_{2n+1+2k}, W_{2n+2k}, \psi}, \delta \rtimes \Theta(\pi, 2n+1))_\infty \hookrightarrow \\ &\chi_\psi\delta \rtimes \pi^\vee.\end{aligned} \tag{8}$$

We have to see if $\mathrm{Hom}_{O(V_{2n+1+2k})}(\omega_{V_{2n+1+2k}, W_{2n+2k}, \psi}, T_i)_\infty \neq 0$, i.e. is T_i the extension of $T_i|_{SO(2n+1+2k)^\varepsilon}$ which participates in the theta correspondence with $\widetilde{Sp(2n, F}+2k)$?(We know that $\Theta(\pi, 2n+1)$ is!) This follows immediately from (8). Indeed, assume that $T_i \otimes det^{\varepsilon_0}$, where $\varepsilon_0 \in \{0, 1\}$, is such that $\Theta(T_i \otimes det^{\varepsilon_0}, 2n+2k) \neq 0$. Then, we can repeat the same reasoning as in (8), but for $T_i \otimes det^{\varepsilon_0}$ and $\delta \rtimes (\Theta(\pi, 2n+1) \otimes det^{\varepsilon_0})$, and we get

$$0 \neq \mathrm{Hom}_{O(V_{2n+1+2k})}(\omega_{V_{2n+1+2k}, W_{2n+2k}, \psi}, T_i \otimes det^{\varepsilon_0})_\infty \hookrightarrow \chi_\psi\delta \rtimes \Theta(\Theta(\pi, 2n+1) \otimes det^{\varepsilon_0}, 2n).$$

This means that $\Theta(\Theta(\pi, 2n+1) \otimes det^{\varepsilon_0}, 2n) \neq 0$, so $\varepsilon_0 = 0$. Thus, we indeed have

$$\mathrm{Hom}_{O(V_{2n+1+2k})}(\omega_{V_{2n+1+2k}, W_{2n+2k}, \psi}, T_i)_\infty \neq 0.$$

Now assume that $\delta \rtimes GS_\psi(\pi, 2n+1)$ is reducible (so is $\delta \rtimes \Theta(\pi, 2n+1)$). This forces $\delta^\vee \cong \delta$. Then, $\delta \rtimes \Theta(\pi, 2n+1) = T_1 \oplus T_2$, where T_1 and T_2 are non-equivalent and tempered (cf. Corollary 2.2, 2.). By previous reasoning, $\chi_\psi\delta \rtimes \pi$ is reducible. By the remark just above, T_1 and T_2 do participate in the theta correspondence with $\widetilde{Sp(2n, F}+2k)$. We further have

$$\begin{aligned}\omega_{V_{2n+1+2k}, W_{2n+2k}, \psi} &\twoheadrightarrow T_1 \otimes \Theta(T_1, 2n+2k), \\ R_{P_k}(\omega_{V_{2n+1+2k}, W_{2n+2k}, \psi}) &\twoheadrightarrow \delta \otimes \Theta(\pi, 2n+1) \otimes \Theta(T_1, 2n+2k),\end{aligned}$$

where, in the second line, we have an $GL_k(F) \times O(2n+1, F) \times \widetilde{Sp(2n, F)}$–intertwining. We have calculated above that (recall that I^a are subquotients in the filtrations, cf. Remark after Lemma 3.3)

$$\begin{aligned}&\mathrm{Hom}_{GL(k,F) \times O(2n+1,F)}(R_{P_k}(\omega_{V_{2n+1+2k}, W_{2n+2k}, \psi}), \delta \otimes \Theta(\pi, 2n+1))_\infty \hookrightarrow \\ &\quad \mathrm{Hom}_{GL(k,F) \times O(2n+1,F)}(I^k, \delta \otimes \Theta(\pi, 2n+1))_\infty \cong \chi_\psi\delta \rtimes \pi^\vee.\end{aligned}$$

Thus, $\Theta(T_1, 2n+2k) \leq \chi_\psi \delta^\vee \rtimes \pi$. The same calculation holds for T_2, so that $\Theta(T_1, 2n+2k) + \Theta(T_2, 2n+2k) \leq \chi_\psi \delta \rtimes \pi$. We will prove that we actually have equality. Recall that $\Theta(T_i, 2n+2k),\ i=1,2$ is irreducible (cf. Proposition 3.1) and, by the Howe duality conjecture, $\Theta(T_1, 2n+2k) \not\cong \Theta(T_2, 2n+2k)$.

Now we calculate the multiplicity of $\chi_\psi \delta \otimes \pi$ in the Jacquet module $R_{\widetilde{P_k}}(\chi_\psi \delta \rtimes \pi)$. Let $\delta = \delta(\rho\nu^{-\frac{t-1}{2}}, \rho\nu^{\frac{t-1}{2}})$, where $\rho \cong \rho^\vee$ is a cuspidal representation of $GL(m_\rho, F)$ with $m_\rho t = k$. Using Tadić's formula (cf. [28]) checked for the metaplectic groups in [12] (cf. Section 3 and Proposition 4.5), we count the multiplicity with which $\chi_\psi \delta \otimes \pi$ appears in

$$\sum_{\delta', \sigma_1} \sum_{i=0}^{t} \sum_{j=0}^{i} \chi_\psi \left(\delta(\rho\nu^{i-\frac{t-1}{2}}, \rho\nu^{\frac{t-1}{2}}) \times \delta(\rho\nu^{\frac{t+1}{2}-j}, \rho\nu^{\frac{t-1}{2}}) \right) \times \delta' \otimes \delta(\rho\nu^{\frac{t+1}{2}-i}, \rho\nu^{\frac{t-1}{2}-j}) \rtimes \sigma_1.$$

Here the first sum goes over all $\delta' \otimes \sigma_1$ which are in the Jacquet module of π with respect to any maximal parabolic subgroup of $\widetilde{Sp(2n, F)}$. We easily get that the multiplicity is two (for $i = j = t$ and $\delta' \otimes \sigma_1 = 1 \otimes \pi$ and $i = j = 0$ and $\delta' \otimes \sigma_1 = 1 \otimes \pi$) plus the multiplicity of all the pieces of the form $\chi_\psi \delta(\rho\nu^{-\frac{t-1}{2}}, \rho\nu^{\frac{t-1}{2}}) \otimes \sigma_1$ appearing in the appropriate Jacquet module of π, with the additional property that $\pi \leq \chi_\psi \delta(\rho\nu^{-\frac{t-1}{2}}, \rho\nu^{\frac{t-1}{2}}) \rtimes \sigma_1$.

We now prove that the latter multiplicities are zero, i.e., that the multiplicity of $\chi_\psi \delta \otimes \pi$ in the Jacquet module $R_{\widetilde{P_k}}(\chi_\psi \delta \rtimes \pi)$ equals two.

Assume that $\chi_\psi \delta(\rho\nu^{-\frac{t-1}{2}}, \rho\nu^{\frac{t-1}{2}}) \otimes \sigma_1 \leq \mu^*(\pi)$. Using transitivity of Jacquet modules and projectivity of cuspidal representations in the category of smooth representations, we get that there exists an irreducible representation σ_2 of $\widetilde{Sp(2n, F}-2k)$ such that

$$\pi \hookrightarrow \chi_\psi \left(\rho\nu^{\frac{t-1}{2}} \times \rho\nu^{\frac{t-1}{2}-1} \times \cdots \times \rho\nu^{-\frac{t-1}{2}} \right) \rtimes \sigma_2.$$

Now, the Casselman temperedness criterion for π (Theorem 3.4. of [3]) forces σ_2 to be tempered, too. On the other hand,

$$\Pi(t_1, t_2, \ldots, t_{l-1}, t_l) := \delta(\rho\nu^{t_{l-1}+1}, \rho\nu^{\frac{t-1}{2}}) \times \delta(\rho\nu^{t_{l-2}+1}, \rho\nu^{t_{l-1}}) \times \cdots \times \delta(\rho\nu^{-\frac{t-1}{2}}, \rho\nu^{t_1})$$

is a subrepresentation of

$$\rho\nu^{\frac{t-1}{2}} \times \rho\nu^{\frac{t-1}{2}-1} \times \cdots \times \rho\nu^{-\frac{t-1}{2}} \tag{9}$$

for any choice of

$$-\frac{t-1}{2} \leq t_1 < t_2 < \cdots < t_{l-1} < t_l = \frac{t-1}{2},$$

where $t_i - \frac{t-1}{2} \in \mathbb{Z}$. Note that if $l = t$ we get that $\Pi(t_1, t_2, \ldots, t_{l-1}, t_l)$ is exactly the representation (9), so there are choices of $t_1, \ldots, t_l$ such that

$$\pi \hookrightarrow \chi_\psi \Pi(t_1, t_2, \ldots, t_l) \rtimes \sigma_2. \tag{10}$$

Let l be the smallest integer such that (10) holds. Assume that $l \geq 2$. Then, the minimality of l guarantees that π can be embedded in the representation induced by any essentially discrete series attached to any permutation of l intervals in $\Pi(t_1, t_2, \ldots, t_l)$ (indeed, one can examine the kernel of the attached "permutation" GL–induced intertwining operators). Thus,

$$\pi \hookrightarrow \chi_\psi \left(\delta(\rho\nu^{-\frac{t-1}{2}},\rho\nu^{t_1}) \times \delta(\rho\nu^{t_{l-1}+1},\rho\nu^{\frac{t-1}{2}}) \times \delta(\rho\nu^{t_{l-2}+1},\rho\nu^{t_{l-1}}) \times \cdots \times \delta(\rho\nu^{t_1+1},\rho\nu^{t_2})\right) \rtimes \sigma_2.$$

Since $t_1 < \frac{t-1}{2}$, this violates the temperedness criterion for π. This means $l = 1$, i.e.,

$$\pi \hookrightarrow \chi_\psi \delta(\rho\nu^{-\frac{t-1}{2}},\rho\nu^{\frac{t-1}{2}}) \rtimes \sigma_2.$$

But, by Proposition 3.2, then

$$\Theta(\pi, 2n+1) \hookrightarrow \delta(\rho\nu^{-\frac{t-1}{2}},\rho\nu^{\frac{t-1}{2}}) \rtimes T$$

for some tempered representation T. According to Corollary 2.2 1. this means that $\delta(\rho\nu^{-\frac{t-1}{2}},\rho\nu^{\frac{t-1}{2}}) \rtimes \Theta(\pi, 2n+1)$ is irreducible, and this contradicts our assumption. Thus, there is no piece of the form $\delta(\rho\nu^{-\frac{t-1}{2}},\rho\nu^{\frac{t-1}{2}}) \otimes \sigma_1$ appearing in the appropriate Jacquet module of π, with the additional property that $\pi \le \delta(\rho\nu^{-\frac{t-1}{2}},\rho\nu^{\frac{t-1}{2}}) \rtimes \sigma_1$. This means that the multiplicity of $\chi_\psi\delta \otimes \pi$ in the appropriate Jacquet module of $\chi_\psi\delta \rtimes \pi$ is two. Therefore, $\chi_\psi\delta \rtimes \pi$ cannot have other summands except $\Theta(T_1, 2n+2k)$ and $\Theta(T_2, 2n+2k)$.

□

From this proposition, the theorem about R–groups for metaplectic groups readily follows.

Theorem 3.5. *Let $\delta_1,\ldots,\delta_k$ be (unitarizable) discrete series representations of $GL(m_i, F)$, $i = 1, 2, \ldots, k$. Let σ be an irreducible discrete series representation of $\widetilde{Sp(2n, F)}$. Then,*

$$\chi_\psi\delta_1 \times \chi_\psi\delta_2 \times \cdots \times \chi_\psi\delta_k \rtimes \sigma$$

is a direct sum of 2^m mutually inequivalent, irreducible tempered representations. Here m is the number of mutually inequivalent δ_i's such that $\chi_\psi\delta_i \rtimes \sigma$ reduces.

Proof. We denote $2(m_1 + \cdots + m_k) = 2s$. We prove this Theorem by induction over k. Here we include in the induction the following claim

Claim

If $\chi_\psi\delta_1 \times \chi_\psi\delta_2 \times \cdots \times \chi_\psi\delta_k \rtimes \sigma = T_1 \oplus \cdots \oplus T_r$, where $T_i,\ i = 1, 2, \ldots, r$ is an irreducible tempered representation, then

$$\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \Theta(\sigma, 2n+1) = \Theta(T_1, 2n+1+2s) \oplus \cdots \oplus \Theta(T_r, 2n+1+2s). \quad (11)$$

Here, as before, $\Theta(\sigma, 2n+1)$ denotes the full non-zero lift on the appropriate tower; note that, by Proposition 3.1, $\Theta(\sigma, 2n+1)$ is in irreducible discrete series, so we can apply Theorem 2.1 to the left-hand side of (11). As discussed above, all the lifts $\Theta(T_i, 2n+1+2s)$ (appearing on the right-hand side of (11)) on the same tower are non-zero (and irreducible by Proposition 3.1).

For $k = 1$ (for both Theorem and the **Claim**) we get a special case of Proposition 3.4. Assume that the Theorem is valid for each $k \leq l - 1$. We introduce some notation: let

$$\Pi_{l-1} = \delta_2 \times \cdots \times \delta_l \rtimes \Theta(\sigma, 2n+1),$$
$$\Pi_l = \delta_1 \times \delta_2 \times \cdots \times \delta_l \rtimes \Theta(\sigma, 2n+1) = \delta_1 \rtimes \Pi_{l-1},$$
$$\Pi_{\psi,l-1} = \chi_\psi \delta_2 \times \cdots \times \chi_\psi \delta_l \rtimes \sigma,$$
$$\Pi_{\psi,l} = \chi_\psi \delta_1 \times \chi_\psi \delta_2 \times \cdots \times \chi_\psi \delta_l \rtimes \sigma = \chi_\psi \delta_1 \rtimes \Pi_{\psi,l-1}.$$

Then, let

$$\Pi_{\psi,l-1} = T_1 \oplus T_2 \oplus \cdots \oplus T_{2^m}, \tag{12}$$

where the2^m irreducible representations T_i are non-isomorphic tempered representations.

Now, we examine

$$\Pi_{\psi,l} = \chi_\psi \delta_1 \rtimes T_1 \oplus \cdots \oplus \chi_\psi \delta_1 \rtimes T_{2^m}. \tag{13}$$

By our induction assumption (of the **Claim**) applied to (12),

$$\Pi_{l-1} = \Theta(T_1, 2n+1+2m_2+\ldots+2m_l) \oplus \cdots \oplus \Theta(T_{2^m}, 2n+1+2m_2+\ldots+2m_l), \tag{14}$$

so that

$$\Pi_l = \delta_1 \rtimes \Theta(T_1, 2n+1+2m_2+\ldots+2m_l) \oplus \cdots \oplus \delta_1 \rtimes \Theta(T_{2^m}, 2n+1+2m_2+\ldots+2m_l). \tag{15}$$

• Assume that $\chi_\psi \delta_1 \rtimes \sigma$ is irreducible. By Proposition 3.4, this means that $\delta_1 \rtimes \Theta(\sigma, 2n+1)$ is irreducible. Then, by Theorem 2.1, the length of Π_{l-1} equals the length of Π_l. This also means that the lengths of the right-hand sides of (14) and (15) are the same, so $\delta_1 \rtimes \Theta(T_i, 2n+1+2m_2+\ldots+2m_l)$ is irreducible for each $T_i,\ i = 1, 2, \ldots, 2^m$. According to Proposition 3.4, $\chi_\psi \delta_1 \rtimes T_i$ is irreducible for each $i = 1, 2, \ldots, 2^m$. Thus, by (13), the length of $\Pi_{\psi,l}$ equals the length of $\Pi_{\psi,l-1}$, i.e. it is equal to 2^m.
• Assume that $\chi_\psi \delta_1 \rtimes \sigma$ is reducible. By Proposition 3.4, this means that $\delta_1 \rtimes \Theta(\sigma, 2n+1)$ is reducible. Assume that there exists $\delta_i,\ i \in \{2, 3, \ldots, l\}$ such that $\delta_1 \cong \delta_i$; by Theorem 2.1, the length of Π_l is equal to the length of Π_{l-1}. Now again using the arguments like those in the previous case, we have that the length of $\Pi_{\psi,l}$ equals the length of $\Pi_{\psi,l-1}$, i.e., 2^m.
• Assume that $\chi_\psi \delta_1 \rtimes \sigma$ is reducible and $\delta_1 \not\cong \delta_i,\ i \in \{2, 3, \ldots, l\}$. Then $\delta_1 \rtimes \Theta(\sigma, 2n+1)$ is reducible and, according to Theorem 2.1, the length of Π_l is 2^{m+1}. Thus, by (14), (15) and the proof of Corollary 2.2, 2., $\delta_1 \rtimes \Theta(T_i, 2n+1+2m_2+\ldots+2m_l)$ is reducible for each i. By Proposition 3.4, every $\chi_\psi \delta_1 \rtimes T_i$ is reducible of length two, so by (12) and (13) the length of $\Pi_{\psi,l}$ is 2^{m+1}. We have proved the theorem.

□

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