TM Neka je $I \subseteq \mathbb{R}$ otvoreni interval, $f: I \to \mathbb{R}$ diferencijabilna.
Ako je $f'(x) > 0$ $(<)$, $\forall x \in I$, tada je f strogo rastuća (padajuća) na I.

TM Neka je $I \subseteq \mathbb{R}$ otvoren interval, $f: I \to \mathbb{R}$ diferencijabilna.
Tada f ima strogi lokalni maksimum (minimum) u $c \in I$ ako i samo ako je $f'(c) = 0$, $f'(x) > 0$ $(<)$ za $x \to c^-$, $f'(x) < 0$ $(>)$ za $x \to c^+$.

— || —

— || —

— || —

TM Neka je $I \subseteq \mathbb{R}$ otvoren interval, $f: I \to \mathbb{R}$ dva puta diferencijabilna.
Ako je $f''(x) > 0$ $(<)$, $\forall x \in I$, tada je f strogo konveksna (konkavna) na I.

TM Neka je $I \subseteq \mathbb{R}$ otvoren interval, $f: I \to \mathbb{R}$ dva puta diferencijabilna.
Tada f ima točku infleksije u $c \in I$ ako i samo ako je $f''(c) = 0$, $f''(x) > 0$ za $x \to c^-$, $f''(x) < 0$ za $x \to c^+$ ili
— || — $<$ — || — $>$

DEF Neka je $f: \langle a, b \rangle \to \mathbb{R}$. Kažemo da f ima vertikalnu asimptotu u a (b), ako je $\lim_{x \to a^+} f(x) = \infty$ ili $-\infty$.
— || — $(x \to b^-)$

DEF Neka je $f: \langle a, \infty \rangle \to \mathbb{R}$. Kažemo da f ima horizontalnu asimptotu u $+\infty$ ako je $\lim_{x \to \infty} f(x) = c \in \mathbb{R}$
[$(f: \langle -\infty, a \rangle \to \mathbb{R})$ $(-\infty)$ $(x \to -\infty)$]

DEF Neka je $f: \langle a, \infty \rangle \to \mathbb{R}$. Kažemo da f ima kosu asimptotu u $+\infty$ ako je $\lim_{x \to +\infty} \frac{f(x)}{x} = k \in \mathbb{R}$ i $\lim_{x \to +\infty} (f(x) - kx) = l \in \mathbb{R}$.
[$(f: \langle -\infty, a \rangle \to \mathbb{R})$ — || — $(-\infty)$ — || — $(x \to -\infty)$]. Tada je to pravac $y = kx + l$.

Odredite prirodnu domenu, lokalne ekstreme, točke infleksije, intervale monotonosti, konveksnosti i konkavnosti, asimptote, te skicirajte graf funkcije.

- $f(x) = x^4 - 2x^3 + 1$

$D(f) = \mathbb{R}$

$f'(x) = 4x^3 - 6x^2 = x^2(4x-6)$

$f''(x) = 12x^2 - 12x = 12x(x-1)$

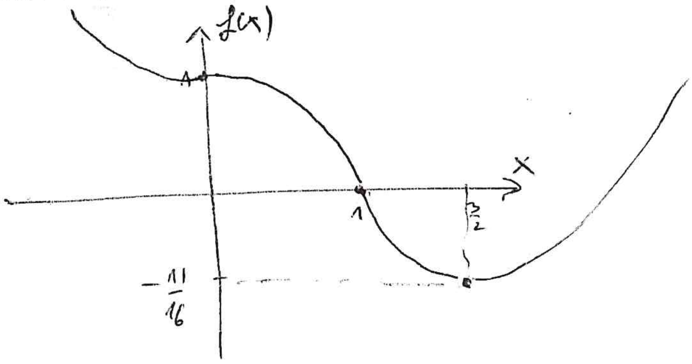
	$-\infty$	0		1		$3/2$	∞
f'	$-$	0	$-$		$-$	0	$+$
f''	$+$	0	$-$	0	$+$		$+$
f	$\searrow$ $\cup$		$\searrow$ $\cap$		$\searrow$ $\cup$		$\nearrow$ $\cup$

$f(0) = 1$ točka infleksije

$f(1) = 0$ točka infleksije

$f(\frac{3}{2}) = \frac{81}{16} - 2 \cdot \frac{27}{8} + 1 = \frac{81 - 108 + 16}{16} = -\frac{11}{16}$ lokalni minimum.

vertikalnih asimptota nema jer $D(f) = \mathbb{R}$.
horizontalnih i kosih nema jer je polinom stupnja > 1.



- $f(x) = xe^{-x}$

$D(f) = \mathbb{R}$

$f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x)$

$f''(x) = -e^{-x} - e^{-x} + xe^{-x} = e^{-x}(x-2)$

	$-\infty$	1		2	∞
f'	$+$	0	$-$		$-$
f''	$-$		$-$	0	$+$
f	$\nearrow$ $\cap$		$\searrow$ $\cap$		$\searrow$ $\cup$

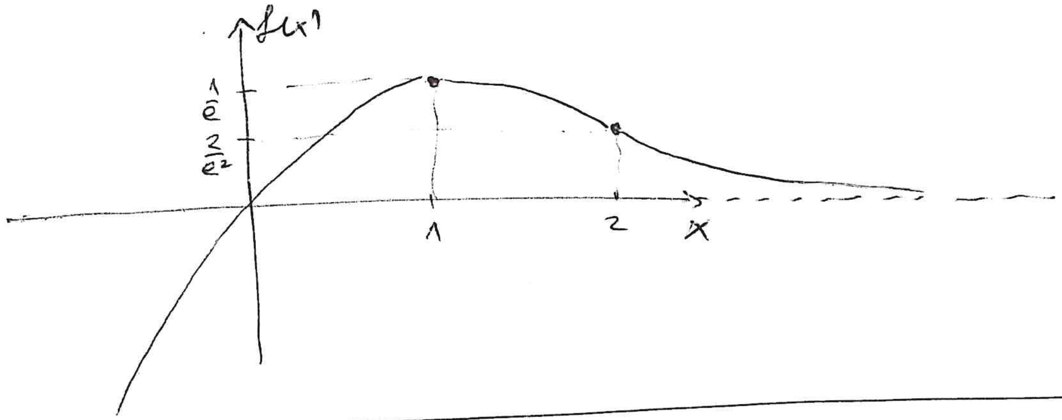
$f(1) = \frac{1}{e}$ ≈ 0.37 lokalni maksimum

$f(2) = \frac{2}{e^2} \approx 0.27$ točka infleksije

vertikalnih asimptota nema jer $D(f) = \mathbb{R}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{e^x} \overset{LH}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0 \rightarrow \text{horizontalna asimptota.}$$

$$\lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} e^{-x} = \infty \rightarrow \text{nema horizontalnih ni kosih asimptota za } -\infty.$$



- $f(x) = \frac{1}{x+1}$

$D(f) = \mathbb{R} \setminus \{-1\}$

$f'(x) = -\frac{1}{(x+1)^2}$

$f''(x) = \frac{2}{(x+1)^3}$

	$-\infty$	-1	∞
f'	$-$		$-$
f''	$-$		$+$
f	$\searrow$ $\cap$	ND	$\searrow$ $\cup$

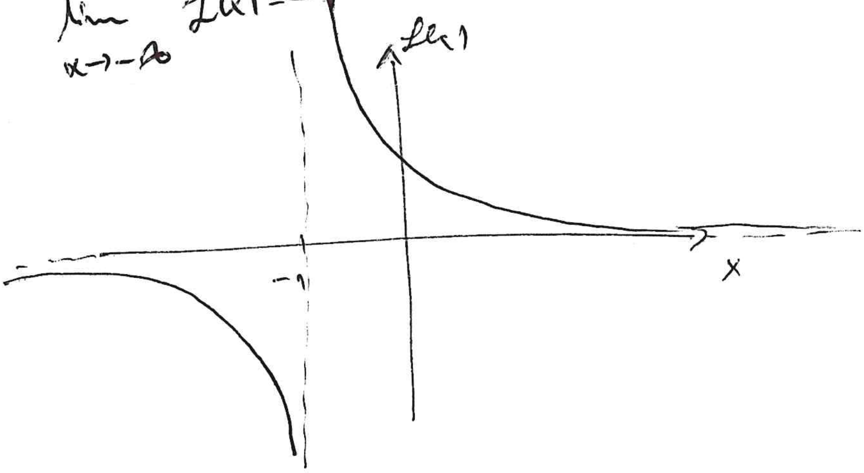
nema točaka infleksije niti lokalnih ekstrema

$\lim_{x\to-1^+} f(x) = \infty \rightarrow$ vertikalna asimptota

$\lim_{x\to-1^-} f(x) = -\infty \rightarrow$ vertikalna asimptota

$\lim_{x\to\infty} f(x) = 0 \rightarrow$ horizontalna asimptota

$\lim_{x\to-\infty} f(x) = 0 \rightarrow$ horizontalna asimptota



- $f(x) = x^3 e^{-\frac{x^2}{6}}$

$D(f) = \mathbb{R}$

$f'(x) = 3x^2 e^{-\frac{x^2}{6}} - \frac{x}{3} x^3 e^{-\frac{x^2}{6}} = 3x^2 e^{-\frac{x^2}{6}} - \frac{x^4}{3} e^{-\frac{x^2}{6}} = \frac{x^2}{3} e^{-\frac{x^2}{6}}(9 - x^2)$

$f''(x) = 6x e^{\wedge} - x^3 e^{\wedge} - \frac{4}{3}x^3 e^{\wedge} + \frac{x^5}{9} e^{\wedge} = \frac{1}{9} x e^{\wedge}(54 - 21x^2 + x^4)$

$(x^2-18)(x^2-3)$

	$-\infty$		$-\sqrt{18}$		-3		$-\sqrt{3}$		0		$\sqrt{3}$		3		$\sqrt{18}$		∞
f'		$-$		$-$	0	$+$		$+$	0	$+$		$+$	0	$-$		$-$	
f''		$-$	0	$+$		$+$	0	$-$	0	$+$	0	$-$		$-$	0	$+$	
f		$\searrow \frown$		$\searrow \smile$		$\nearrow \smile$		$\nearrow \frown$		$\nearrow \smile$		$\nearrow \frown$		$\searrow \frown$		$\searrow \smile$	

$f(-\sqrt{18}) = -\sqrt{18} \cdot 18 e^{-3} \approx -3.8$ točka infleksije ($\sqrt{18} \approx 4.24$)

$f(-3) = -27 e^{-\frac{3}{2}} \approx -6.02$ lokalni minimum

$f(-\sqrt{3}) = -\sqrt{3} \cdot 3 e^{-\frac{1}{2}} \approx -3.15$ točka infleksije ($\sqrt{3} \approx 1.73$)

$f(0) = 0$ točka infleksije

$f(\sqrt{3}) = \sqrt{3} \cdot 3 e^{-\frac{1}{2}} \approx 3.15$ točka infleksije

$f(3) = 27 e^{-\frac{3}{2}} \approx 6.02$ lokalni maksimum

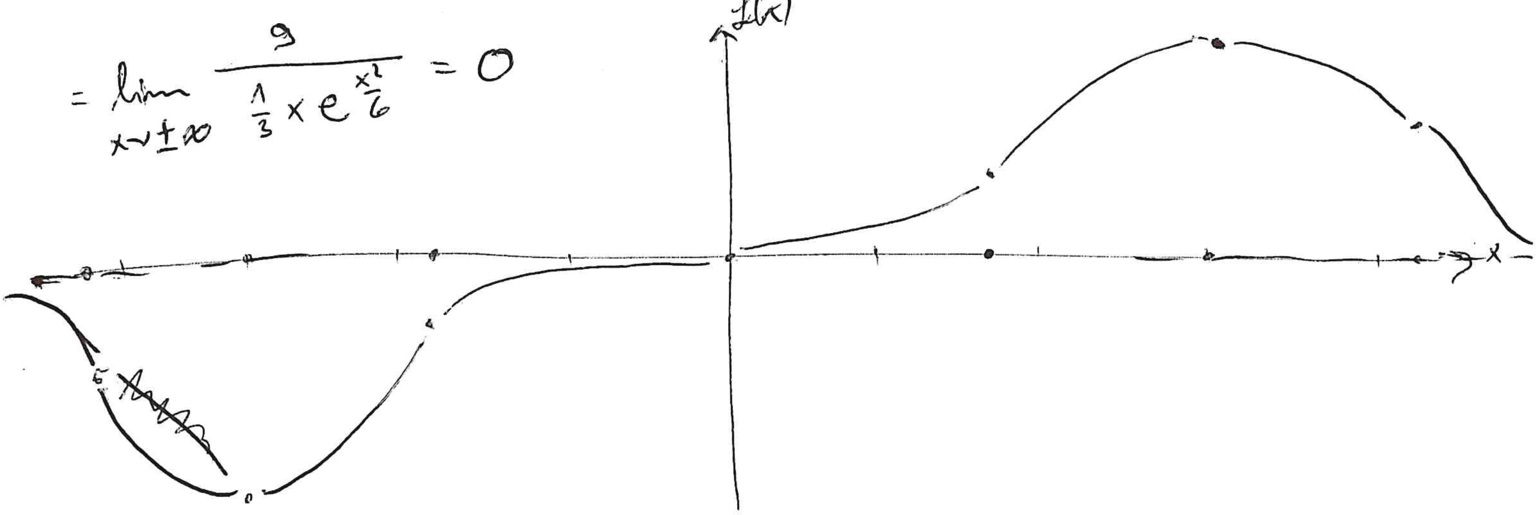
$f(\sqrt{18}) = \sqrt{18} \cdot 18 e^{-3} \approx 3.8$ točka infleksije

vertikalnih asimptota nema jer $D(f) = \mathbb{R}$

horizontalna:

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x^3}{e^{\frac{x^2}{6}}} \overset{L'H}{=} \lim_{x \to \pm\infty} \frac{3x^2}{\frac{1}{3} x e^{\frac{x^2}{6}}} = \lim_{x \to \pm\infty} \frac{9x}{e^{\frac{x^2}{6}}} \overset{L'H}{=}$$

$$= \lim_{x \to \pm\infty} \frac{9}{\frac{1}{3} x e^{\frac{x^2}{6}}} = 0$$



- $f(x)=\dfrac{x^5}{5}-\dfrac{x^3}{3}$

$D(f)=\mathbb{R}$

$f'(x)=x^4-x^2=x^2(x^2-1)$

$f''(x)=4x^3-2x=2x(2x^2-1)$

	$-\infty$ – -1	-1 – $-\sqrt{\frac12}$	$-\sqrt{\frac12}$ – 0	0 – $\sqrt{\frac12}$	$\sqrt{\frac12}$ – 1	1 – ∞
f'	$+$ 0	$-$	$-$ 0	$-$	$-$ 0	$+$
f''	$-$	$-$ 0	$+$ 0	$-$ 0	$+$	$+$
f	$\nearrow$ $\cap$	$\searrow$ $\cap$	$\searrow$ $\cup$	$\searrow$ $\cap$	$\searrow$ $\cup$	$\nearrow$ $\cup$

$f(-1)=\frac{1}{3}-\frac{1}{5}=\frac{2}{15}$ lokalni maksimum

$f(-\sqrt{\frac12})=\frac13(\sqrt{\frac12})^3-\frac15(\sqrt{\frac12})^5\approx 0.08$ točka infleksije

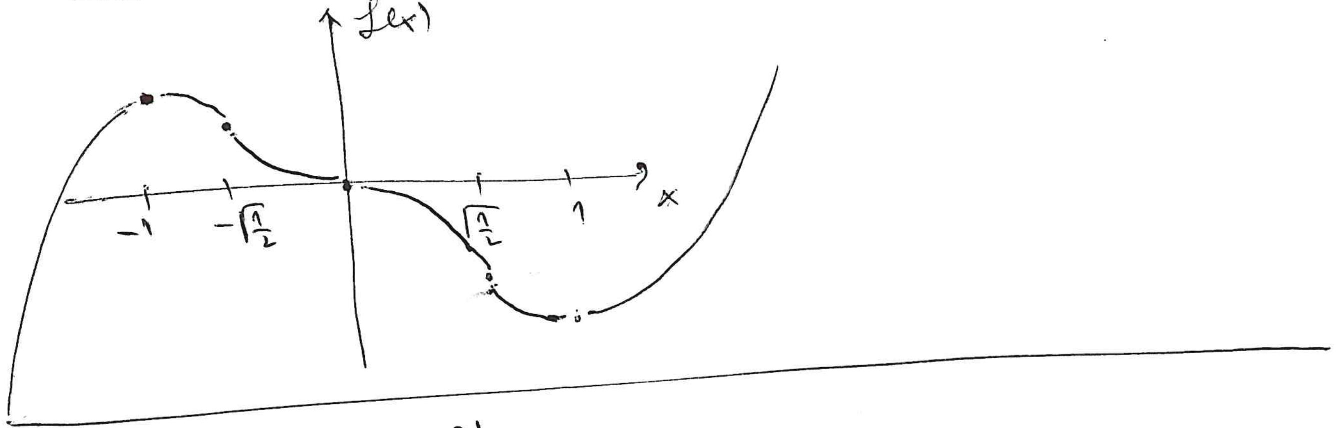
$f(0)=0$ točka infleksije

$f(\sqrt{\frac12})\approx-0.08$ točka infleksije

$f(1)=-\frac{2}{15}$ lokalni minimum

vertikalnih asimptota nema jer $D(f)=\mathbb{R}$

kosih i horizontalnih nema jer je polinom stupnja >1.



- $\operatorname{arctg} x-\frac12\ln(1+x^2)$

$D(f)=\mathbb{R}$

$$f'(x)=\frac{1}{1+x^2}-\frac12\cdot 2x\cdot\frac{1}{1+x^2}=\frac{1-x}{1+x^2}$$

$$f''(x)=\frac{-1-x^2-2x(1-x)}{(1+x^2)^2}=\frac{x^2-2x-1}{(1+x^2)^2}\qquad (x-1)^2-2$$

	$-\infty$ – $1-\sqrt2$	$1-\sqrt2$ – 1	1 – $1+\sqrt2$	$1+\sqrt2$ – ∞
f'	$+$	$+$ 0	$-$	$-$
f''	$+$ 0	$-$	$-$ 0	$+$
f	$\nearrow$ $\cup$	$\nearrow$ $\cap$	$\searrow$ $\cap$	$\searrow$ $\cup$

$f(1-\sqrt2)\approx-0.47$ točka infleksije

$f(1)=\frac{\pi}{4}-\frac12\ln 2\approx 0.44$ lok. maks.

$f(1+\sqrt2)\approx 0.22$ točka infleksije

Vertikalnih asimptota nema jer $D(f)=\mathbb{R}$

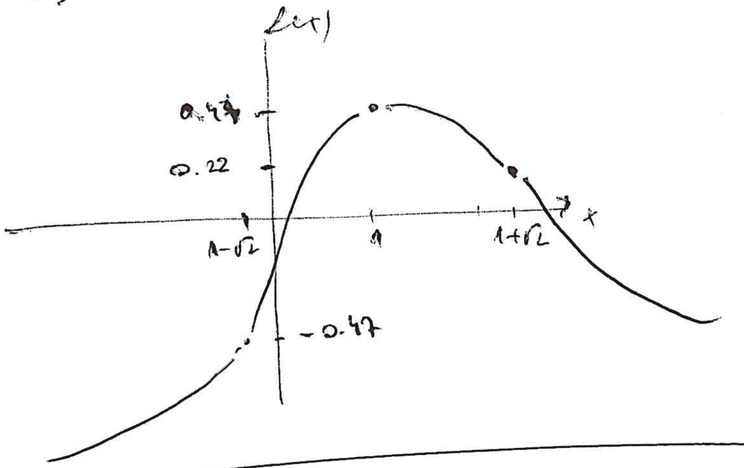
Kose asimptote:

$$\lim_{x\to\pm\infty} \frac{\operatorname{arctg} x - \frac{1}{2}\ln(1+x^2)}{x} = \underbrace{\lim_{x\to\pm\infty} \frac{\operatorname{arctg} x}{x}}_{=0\text{, po tm.o sendviču}} + \frac{1}{2}\lim_{x\to\pm\infty} \frac{\ln(1+x^2)}{x} =$$

$$\stackrel{L'H}{=} -\frac{1}{2}\lim_{x\to\pm\infty} \frac{2x}{1+x^2} = 0$$

$$\lim_{x\to\pm\infty}\left(f(x)-0\cdot x\right) = \lim_{x\to\pm\infty} \operatorname{arctg} x - \frac{1}{2}\ln(1+x^2) = -\infty$$

$\Rightarrow$ nema kosih ni horizontalnih asimptota.



- $\frac{(\ln x)^2}{x}$

$D(f)=\langle 0,\infty\rangle$

$$f'(x)=\frac{2\ln x\cdot\frac{1}{x}\cdot x-(\ln x)^2}{x^2}=\frac{2\ln x-(\ln x)^2}{x^2}=\frac{\ln x\,(2-\ln x)}{x^2}$$

$$f''(x)=\frac{\left(\frac{2}{x}-2\ln x\cdot\frac{1}{x}\right)x^2-2x\left(2\ln x-(\ln x)^2\right)}{x^4}=\frac{2x-2x\ln x-4x\ln x+2x(\ln x)^2}{x^4}$$

$$=\frac{2\left((\ln x)^2-3\ln x+1\right)}{x^3} \qquad \frac{3\pm\sqrt{9-4}}{2}=\frac{3}{2}\pm\frac{\sqrt{5}}{2}$$

	0		1		$e^{\frac{3}{2}-\frac{\sqrt{5}}{2}}$		e^2		$e^{\frac{3}{2}+\frac{\sqrt{5}}{2}}$		∞
f'		−	0	+		+	0	−		−	
f''		+		+	0	−		−	0	+	
f		↘ ∪		↗ ∪		↗ ∩		↘ ∩		↘ ∪	

$f(1)=0$ lok. min.

$f\left(e^{\frac{3}{2}-\frac{\sqrt{5}}{2}}\right)$ točka infleksije

$f(e^2)=\frac{4}{e^2}$ lok. maks.

$f\left(e^{\frac{3}{2}+\frac{\sqrt{5}}{2}}\right)$ točka infleksije

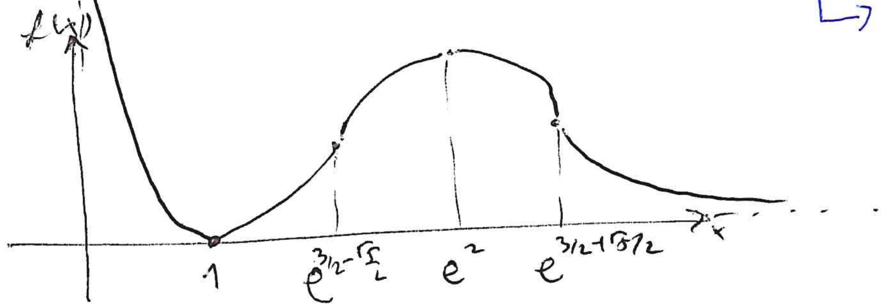
$$\lim_{x\to 0^+} \frac{(\ln x)^2}{x} \overset{L'H}{=} \lim_{x\to 0^+} \frac{2\ln x}{x} \overset{L'H}{=} \lim_{x\to 0^+} \frac{2}{x} = +\infty \longrightarrow \text{vertikalna asimptota}$$

$$\lim_{x\to \infty} \frac{(\ln x)^2}{x} = \quad -\|- \quad = 0 \longrightarrow \text{horizontalna asimptota}$$

ne, to je krivi argument!

$$\lim_{x\to 0^+} (\ln x)^2 \cdot \frac{1}{x} = +\infty \cdot +\infty = +\infty$$

(za L'H treba $\frac{\pm\infty}{\pm\infty}$ ili $\frac{0}{0}$).



- $f(x) = \dfrac{x^2-2x+2}{x-1}$

$D(f) = \mathbb{R} \setminus \{1\}$

$$f'(x) = \frac{(2x-2)(x-1)-(x^2-2x+2)}{(x-1)^2} = \frac{2x^2-4x+2-x^2+2x-2}{(x-1)^2} = \frac{x^2-2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$$

$$f''(x) = \frac{(2x-2)(x-1)^2-2(x-1)(x^2-2x)}{(x-1)^4} = \frac{2x^2-4x+2-2x^2+4x}{(x-1)^3} = \frac{2}{(x-1)^3}$$

	$-\infty$	0		1		2		∞
f'	+	0	−		−	0	+	
f''	−		−		+		+	
f	$\nearrow$ $\frown$		$\searrow$ $\frown$	ND	$\searrow$ $\smile$		$\nearrow$ $\smile$	

$f(0) = -2$ lok. maks.

$f(2) = 2$ lok. min.

nema točaka infleksije

$$\lim_{x\to 1^+} \frac{x^2-2x+2}{x-1} = \lim_{x\to 1^+} \frac{1}{x-1} = \infty$$

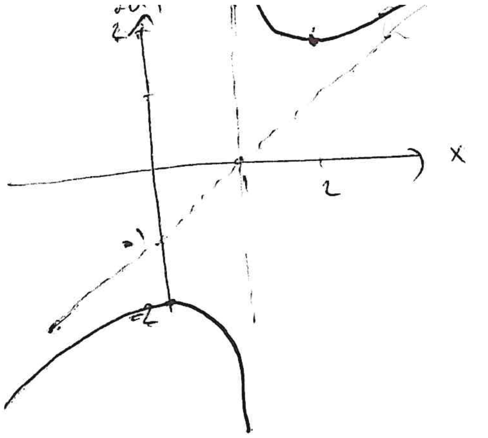
$$\lim_{x\to 1^-} \frac{x^2-2x+2}{x-1} = \lim_{x\to 1^-} \frac{1}{x-1} = -\infty$$

vertikalne asimptote

$$\lim_{x\to \pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{x^2-2x+2}{x^2-x} = 1$$

$$\lim_{x\to \pm\infty} f(x) - 1\cdot x = \lim_{x\to\pm\infty} \frac{x^2-2x+2}{x-1} - \frac{x^2-x}{x-1} = \lim_{x\to\pm\infty} \frac{-x+2}{x-1} = -1$$

$\Rightarrow$ kose asimptote u $+\infty$ i $-\infty$ su pravci $x-1$



	$-\infty$ … 0	0 … 1	1 … ∞
f'	$+$	$-$ 0	$+$
f''	$-$	$+$	$+$
f	$\nearrow$ $\cap$	$\searrow$ $\cup$	$\nearrow$ $\cup$

- $f(x) = xe^{\frac{1}{x}}$

$D(f) = \mathbb{R} \setminus \{0\}$

$f'(x) = e^{\frac{1}{x}} - \frac{1}{x} e^{1/x} = e^{1/x} \frac{1}{x}(x-1)$

$f''(x) = -\frac{1}{x^2} e^{1/x} + \frac{1}{x^3} e^{1/x} + \frac{1}{x^2} e^{1/x} = \frac{1}{x^3} e^{1/x}$

$f(1) = e$ je lokalni maksimum

nema točaka infleksije

vertikalne asimptote:

$\lim\limits_{x \to 0^-} xe^{\frac{1}{x}} = 0 \cdot e^{-\infty} = 0 \to$ nije asimptota

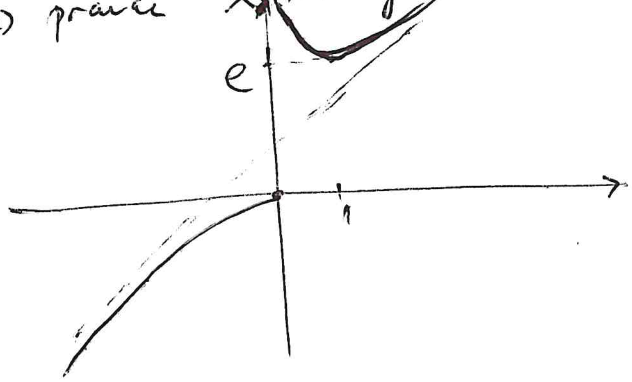
$\lim\limits_{x \to 0^+} xe^{\frac{1}{x}} = \lim\limits_{x \to 0^+} \frac{e^{\frac{1}{x}}}{\frac{1}{x}} \overset{L'H}{=} \lim\limits_{x \to 0^+} \frac{-\frac{1}{x^2} e^{\frac{1}{x}}}{-\frac{1}{x^2}} = \lim\limits_{x \to 0^+} e^{\frac{1}{x}} = \infty \to$ asimptota

kose asimptote:

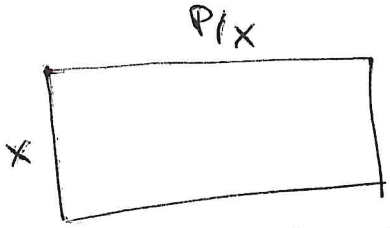
$\lim\limits_{x \to \pm\infty} \frac{f(x)}{x} = 1$

$\lim\limits_{x \to \pm\infty} (f(x) - 1 \cdot x) = \lim\limits_{x \to \pm\infty} x(e^{1/x} - 1) = \lim\limits_{x \to \pm\infty} \frac{e^{1/x} - 1}{\frac{1}{x}} \overset{L'H}{=} \lim\limits_{x \to \pm\infty} \frac{-\frac{1}{x^2} e^{\frac{1}{x}}}{-\frac{1}{x^2}} = 1$

$\Rightarrow$ pravac $x+1$ je kosa asimptota u $+\infty$ i $-\infty$



Dokažite da među svim pravokutnicima površine P, kvadrat ima najmanji opseg.



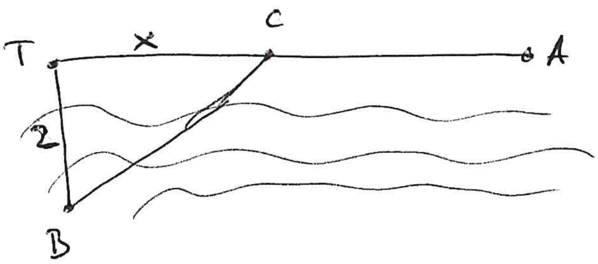
$$f(x) = 2x + \frac{2P}{x} \rightarrow \text{tražimo minimum za } x \in \langle 0, \infty\rangle$$

$$f'(x) = 2 - \frac{2P}{x^2} = \frac{2(x^2 - P)}{x^2}$$

	0		$\sqrt{P}$		∞
f'		$-$	0	$+$	
f		$\searrow$		$\nearrow$	

$\Rightarrow f$ ima minimum u $x = \sqrt{P}$. Druga stranica je tad $\frac{P}{\sqrt{P}} = \sqrt{P}$, dakle radi se o kvadratu.

Ribar se nalazi u točki B, 2 km od ravne obale od točke T. Želi stići u A na obali koji je 100 km udaljen od T. Brzina broda je 4 km/h, a brzina pješačenja 5 km/h. Prema kojoj točki C ribar mora ploviti kako bi što prije došao u A?



Označimo $|\overline{TC}| = x$.

$\Rightarrow |\overline{BC}| = \sqrt{x^2+4}$

$|AC| = 100 - x$

$$f(x) = \frac{\sqrt{x^2+4}}{4} + \frac{100-x}{5} \rightarrow \text{to je vrijeme potrebno da dođe u } A,$$

tražimo minimum na $x \in \langle 0, 100\rangle$

$$f'(x) = \frac{1}{4} \cdot \frac{1}{2} \frac{1}{\sqrt{x^2+4}} \cdot 2x - \frac{1}{5} = \frac{5x - 4\sqrt{x^2+4}}{20\sqrt{x^2+4}}$$

$$5x = 4\sqrt{x^2+4} \quad /^2 \text{ (jer } x > 0)$$

$$\Updownarrow$$

$$25x^2 = 16(x^2+4)$$

$$\Updownarrow$$

$$9x^2 = 64$$

$$\Updownarrow \quad (\text{jer } x > 0)$$

$$x = \frac{8}{3}$$

Isto tako, $5x > 4\sqrt{x^2+4} \iff x > \frac{8}{3}$, sve smijemo kvadrirati jer $x > 0$, pa dobivamo tablica

	0		8/3		100
f'		$-$	0	$+$	
f		$\searrow$		$\nearrow$	

$\Rightarrow f$ ima minimum u $\frac{8}{3}$, pa je to tražena pitanja.

Žica duljine L je presječena na dva dijela. Jedan dio se savije u krug, a drugi u kvadrat. Kako treba presjeći žicu da zbroj površina kruga i kvadrata bude minimalan?

---

x - opseg kruga $\Rightarrow \frac{x}{2\pi}$ - radijus kruga

$L-x$ - opseg kvadrata $\Rightarrow \frac{L-x}{4}$ - stranica kvadrata

$$f(x) = \left(\frac{x}{2\pi}\right)^2 \pi + \left(\frac{L-x}{4}\right)^2$$ $\longrightarrow$ želimo naći minimum za $x \in \langle 0, L\rangle$

$$f(x) = \frac{x^2}{4\pi} + \frac{x^2 - 2Lx + L^2}{4}$$

$$f'(x) = \frac{x}{2\pi} + \frac{x}{2} - \frac{L}{2} \Rightarrow f'(x) = 0 \iff x = \frac{L\pi}{1+\pi} \approx 0.76L$$

$$= \frac{x(1+\pi)}{2\pi} - \frac{L}{2}$$

	0		$L\pi/(1+\pi)$		L
f'		$-$	0	$+$	
f		$\searrow$		$\nearrow$	

$\Rightarrow f$ ima minimum u $\frac{L\pi}{1+\pi} \approx 0.76L$, pa tamo treba presjeći žicu.

TM Neka je $I \subseteq \mathbb{R}$ otvoren interval, $f: I \to \mathbb{R}$ dva puta diferencijabilna. Neka je $c \in I$. Ako je $f'(c) = 0$ i $f''(c) > 0$ $(<)$, tada f ima strogi lokalni minimum (maksimum) u c.

Odredite lokalne ekstreme funkcije

$$f(x) = x^2 e^x$$

$$f'(x) = 2xe^x + x^2e^x = e^x(x^2+2x)$$

$$f''(x) = 2e^x + 2xe^x + 2xe^x + x^2e^x = e^x(x^2+4x+2)$$

$f'(x) = 0 \Rightarrow x = 0$ ili $x = -2$

$f''(0) > 0$, $f''(-2) < 0$

$\Rightarrow$ 0 je lokalni minimum

-2 je lokalni maksimum.

DEF Neka je $I \subseteq \mathbb{R}$ otvoren interval, $f: I \to \mathbb{R}$ n puta derivabilna na I. Za $c \in I$, $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$ zovemo Taylorov polinom stupnja n za f u c.

Odredite Taylorov polinom za

- $f(x) = e^x$, $c = 0$, stupnja n

$f'(x) = e^x$ $\qquad f(0) = 1$

$\vdots \qquad \vdots$

$f^{(n)}(x) = e^x$ $\qquad f^{(n)}(0) = 1$

$$T_n(x) = 1 + x + \frac{x^2}{2} + \dots \frac{x^n}{n!}$$

- $f(x) = \sin(x)$, $c = 0$, stupnja 10

$f'(x) = \cos x$ $\qquad f(0) = 0$

$f''(x) = -\sin x$ $\qquad f'(0) = 1$

$f'''(x) = -\cos x$ $\qquad f''(0) = 0$

$f^{(4)}(x) = \sin x$ $\qquad f'''(0) = -1$

$\dots \qquad \dots$

$$T_{10}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9}$$

• $f(x) = \frac{1}{1-x}$, $c=0$, stopnja n

$f(x) = (1-x)^{-1}$ $\qquad f(0)=1$

$f'(x) = (1-x)^{-2}$ $\qquad f'(0)=1$

$f''(x) = 2(1-x)^{-3}$ $\qquad f''(0)=2$

$f'''(x) = 3!\,(1-x)^{-4}$ $\qquad f'''(0)=3!$

$\dots$ $\qquad \vdots$

$f^{(n)}(x) = n!\,(1-x)^{-n-1}$ $\qquad f^{(n)}(0)=n!$

$T_n(x) = 1 + x + x^2 + \dots + x^n$

• $f(x) = x^n$, $c=0$, stopnja n $\qquad f(0)=0$

$f'(x) = n x^{n-1}$ $\qquad f'(0)=0$

$f''(x) = n(n-1)\cancel{x}^{n-2}$ $\qquad \vdots$

$\vdots$

$f^{(n-1)}(x) = n(n-1)(n-2)\dots(n-(n-2))\,x = n!\,x$ $\qquad f^{(n-1)}(0)=0$

$f^{(n)}(x) = n!$ $\qquad f^{(n)}(0)=0$

$T_n(x) = 0 + \dots + \frac{n!}{n!}x^n = x^n$

• —॥— , stopnja $\underbrace{n-1}_{T_{n-1}(x)=0}$, te stopnja $\underbrace{n+1}_{T_{n+1}(x)=x^n}$

• $f(x) = x^4$, $c=1$, stopnja 4

$f'(x) = 4x^3$ $\qquad f(1)=1$

$f''(x) = 12x^2$ $\qquad f'(1)=4$

$f'''(x) = 24x$ $\qquad f''(1)=12$

$f^{(4)}(x) = 24$ $\qquad f'''(1)=24$

$f^{(4)}(1)=24$

$T_4(x) = 1 + 4(x-1) + \frac{12}{2}(x-1)^2 + \frac{24}{6}(x-1)^3 + \frac{24}{24}(x-1)^4$

$= 1 + 4(x-1) + 6(x-1)^2 + 4(x-1)^3 + (x-1)^4$

$\left(= (1+(x-1))^4 = x^4\right)$

Odredite neodređeni integral za

- $\int \frac{x^3}{2} dx = \frac{1}{8} x^4 + C$

- $\int \frac{1}{x} dx = \ln|x| + C$

- $\int 5^x dx = \frac{5^x}{\ln 5} + C$

- $\int \frac{1}{\sqrt[4]{x}} dx = \int x^{-1/4} dx = \frac{4}{3} x^{\frac{3}{4}} + C$

TM Neka je $I \subseteq \mathbb{R}$ otvoreni interval, $f: I \to \mathbb{R}$ neprekidna, $F: I \to \mathbb{R}$ antiderivacija od f. Tada za svaki segment $[a, b] \subseteq I$ vrijedi $\int_a^b f(x) dx = F(x)\Big|_a^b = F(b) - F(a)$. (Newton-Leibniz formula)

Izračunajte određeni integral za

- $\int_0^{\pi/2} (4\sin^2 x - 3\cos^2 x) dx$

$$= \int_0^{\pi/2} (7\sin^2 x - 3) dx = \int_0^{\pi/2} \left(7 \cdot \frac{1-\cos 2x}{2} - 3\right) dx =$$

$$\frac{1}{2} \int_0^{\pi/2} 1 \, dx - \frac{7}{2} \int_0^{\pi/2} \cos 2x \, dx = \frac{1}{2} x \Big|_0^{\pi/2} - \frac{7}{2} \frac{\sin 2x}{2} \Big|_0^{\pi/2} =$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - 0\right) - \frac{7}{2}\left(\frac{\sin(\pi)}{2} - \frac{\sin(0)}{2}\right) = \frac{\pi}{4}$$

- $\int_1^2 \frac{x\sqrt{x} + x^2}{\sqrt[3]{x}} dx$

$$= \int_1^2 (x^{3/2 - 1/3} + x^{2 - 1/3}) dx = \int_1^2 (x^{7/6} + x^{5/3}) dx =$$

$$\frac{6}{13} x^{13/6} \Big|_1^2 + \frac{3}{8} x^{8/3} \Big|_1^2 = \frac{6}{13}\left(2^{\frac{13}{6}} - 1\right) + \frac{3}{8}\left(2^{\frac{8}{3}} - 1\right)$$

TM (o supstituciji varijabli u integralu).

Ako su sve funkcije dobro definirane i neprekidne na odgovarajućim domenama, tada

$$\int_a^b f(\varphi(x))\varphi'(x)\,dx = \left\{\begin{matrix} t=\varphi(x) & a\to\varphi(a) \\ dt=\varphi'(x)\,d\cancel{x} & b\to\varphi(b)\end{matrix}\right\} = \int_{\varphi(a)}^{\varphi(b)} f(t)\,dt,$$

$$\int f(\varphi(x))\cancel{f}'(x)\,dx = \left\{\begin{matrix} t=\varphi(x) \\ dt=\varphi'(x)dx\end{matrix}\right\} = \int f(t)\,dt = F(t)+C = F(\varphi(x))+C$$

Odredite integrale

- $$\int_0^2 2xe^{x^2} = \left\{\begin{matrix} t=x^2 & 0\to 0 \\ dt=2x\,dx & 2\to 4\end{matrix}\right\} = \int_0^4 e^t\,dt = e^t\Big|_0^4 = e^4-1$$

- $$\int x^2(2x^3+4)^4\,dx = \left\{\begin{matrix} t=2x^3+4 \\ dt=6x^2dx\end{matrix}\right\} = \int \frac{1}{6}t^4\,dt =$$

$$= \frac{t^5}{30}+C = \frac{(2x^3+4)^5}{30}+C$$

- $$\int_0^{\ln 2} \frac{e^x}{\sqrt{e^x+1}}\,dx = \left\{\begin{matrix} t=e^x+1 & 0\to e^0+1=2 \\ dt=e^x dx & \ln 2\to e^{\ln 2}+1=3\end{matrix}\right\} =$$

$$= \int_2^3 \frac{1}{\sqrt{t}}\,dt = \int_2^3 t^{-1/2}\,dt = 2t^{1/2}\Big|_2^3 = 2(\sqrt{3}-\sqrt{2})$$

- $$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \left\{\begin{matrix} t=\sqrt{x} \\ dt=\frac{1}{2\sqrt{x}}dx\end{matrix}\right\} = 2\int \cos t\,dt =$$

$$= 2\sin t + C = 2\sin\sqrt{x}+C$$

- $$\int_2^3 x\sqrt{x^2-4}\,dx = \left\{\begin{matrix} t=x^2-4 & 2\to 0 \\ dt=2x\,dx & 3\to 5\end{matrix}\right\} = \frac{1}{2}\int_0^5 \sqrt{t}\,dt =$$

$$= \frac{1}{2}\cdot\frac{2}{3}t^{3/2}\Big|_0^5 = \frac{1}{3}(5^{3/2}-0) = \frac{1}{3}\sqrt{125}$$

- $\int \frac{x}{\sqrt{x+1}}\,dx = \left\{\begin{matrix} t = x+1 & x = t-1 \\ dt = dx & \end{matrix}\right\} = \int \frac{t-1}{\sqrt{t}}\,dt =$

$= \int t^{1/2}\,dt - \int t^{-1/2}\,dt = \frac{2}{3}t^{3/2} - 2t^{1/2} + C = \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C$

- $\int_0^{\pi/2} \cos^4 x \sin^3 x\,dx = \int_0^{\pi/2} \cos^4(x)(1-\cos^2 x)\sin x\,dx =$

$\left\{\begin{matrix} t = \cos x & 0 \to 1 \\ dt = -\sin x\,dx & \frac{\pi}{2} \to 0 \end{matrix}\right\} = \int_1^0 t^4(1-t^2)(-1)\,dt =$

$\int_0^1 (t^4 - t^6)\,dt = \frac{t^5}{5}\Big|_0^1 - \frac{t^7}{7}\Big|_0^1 = \frac{1}{5} - \frac{1}{7} = \frac{2}{35}$

- $\int_0^{\pi/2} \sin(2x)\sqrt{1+\sin^2 x}\,dx = \left\{\begin{matrix} t = 1+\sin^2 x & 0 \to 1 \\ dt = \underbrace{2\sin x\cos x\,dx}_{\sin(2x)} & \pi/2 \to 2 \end{matrix}\right\}$

$= \int_1^2 \sqrt{t}\,dt = \frac{2}{3}t^{3/2}\Big|_1^2 = \frac{2}{3}\left(2^{3/2} - 1\right)$

TM (parcijalna integracija)

Ako su sve funkcije dobro definirane i neprekidne na odgovarajućim domenama, tada vrijedi:

$\int u\,dv = uv - \int v\,du \quad , \quad \int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$

Odredite integrale

- $\int x e^x\,dx = \left\{\begin{matrix} u = x & du = dx \\ dv = e^x dx & v = e^x \end{matrix}\right\} = xe^x - \int e^x\,dx$

$= xe^x - e^x + C$

$$\bullet \int_1^2 \frac{\ln x}{x^2}\,dx = \left\{\begin{matrix} u = \ln x & du = \frac{dx}{x} \\ dv = \frac{dx}{x^2} & v = -\frac{1}{x} \end{matrix}\right\} = -\frac{\ln x}{x}\Big|_1^2 + \int_1^2 \frac{dx}{x^2}$$

$$= -\frac{\ln 2}{2} - \frac{1}{x}\Big|_1^2 = -\frac{\ln 2}{2} - \frac{1}{2} + 1 = \frac{1-\ln 2}{2}$$

$$\bullet \int \ln x\,dx = \left\{\begin{matrix} u = \ln x & du = \frac{dx}{x} \\ dv = dx & v = x \end{matrix}\right\} = x\ln x - \int dx$$

$$= x\ln x - x + C$$

$$\bullet \int_0^{\pi/2} e^x \sin x\,dx = \left\{\begin{matrix} u = \sin x & du = \cos x\,dx \\ dv = e^x dx & v = e^x \end{matrix}\right\} =$$

$$= \underbrace{e^x \sin x\Big|_0^{\pi/2}}_{= e^{\pi/2}} - \int_0^{\pi/2} e^x \cos x\,dx = \left\{\begin{matrix} u = \cos x & du = -\sin x\,dx \\ dv = e^x dx & v = e^x \end{matrix}\right\} =$$

$$= e^{\pi/2} - e^x \cos x\Big|_0^{\pi/2} - \int_0^{\pi/2} e^x \sin x\,dx = e^{\pi/2} + 1 - \int_0^{\pi/2} e^x \sin x$$

$$\Rightarrow \int_0^{\pi/2} e^x \sin x = \frac{1}{2}\left(e^{\pi/2} + 1\right)$$

$$\bullet \int \operatorname{arctg} x\,dx = \left\{\begin{matrix} u = \operatorname{arctg} x & du = \frac{1}{x^2+1}\,dx \\ dv = dx & v = x \end{matrix}\right\} =$$

$$= x\operatorname{arctg} x - \int \frac{x}{x^2+1}\,dx = \left\{\begin{matrix} t = x^2 \\ dt = 2x\,dx \end{matrix}\right\} = x\operatorname{arctg} x - \frac{1}{2}\int \frac{1}{t+1}\,dt$$

$$= x\operatorname{arctg} x - \frac{1}{2}\ln|t+1| + C = x\operatorname{arctg} x - \frac{1}{2}\ln(x^2+1) + C$$

• $$\int_{-\frac{\pi}{4}}^{\pi/4} \frac{x}{\cos^2 x}\,dx = \left\{ \begin{matrix} u = x & du = dx \\ dv = \frac{1}{\cos^2 x}\,dx & v = tg\, x \end{matrix} \right\} = \underbrace{x\, tg\, x \Big|_{-\pi/4}^{\pi/4}}_{= \frac{\pi}{4} - \frac{\pi}{4} = 0} - \int_{-\frac{\pi}{4}}^{\pi/4} tg\, x\, dx$$

$$= - \int_{-\pi/4}^{\pi/4} \frac{\sin x}{\cos x}\,dx = \left\{ \begin{matrix} t = \cos x & -\pi/4 \to \frac{\sqrt{2}}{2} \\ dt = -\sin x\, dx & \pi/4 \to \frac{\sqrt{2}}{2} \end{matrix} \right\} =$$

$$= \int_{\sqrt{2}/2}^{\sqrt{2}/2} \frac{1}{t}\,dt = 0$$ (ovo smo mogli odmah zaključiti — neparna funkcija na simetričnoj domeni)

• $$\int x^3 e^x\,dx = \left\{ \begin{matrix} u = x^3 & du = 3x^2\,dx \\ dv = e^x\,dx & v = e^x \end{matrix} \right\} = x^3 e^x - \int 3x^2 e^x\,dx =$$

$$\left\{ \begin{matrix} u = 3x^2 & du = 6x\,dx \\ dv = e^x\,dx & v = e^x \end{matrix} \right\} = x^3 e^x - 3x^2 e^x + \int 6x e^x\,dx =$$

$$\left\{ \begin{matrix} u = 6x & du = 6\,dx \\ dv = e^x\,dx & v = e^x \end{matrix} \right\} = x^3 e^x - 3x^2 e^x + 6x e^x - \int 6 e^x\,dx$$

$$= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C$$

TM (rastav na parcijalne razlomke)

Neka su $P(x)$ i $Q(x)$ polinomi pri čemu je P manjeg stupnja od Q. Neka je $Q(x) = a(x+b_1)^{k_1} \cdot \ldots \cdot (x+b_m)^{k_m}(x^2+c_1x+d_1)^{l_1} \cdot \ldots \cdot (x^2+c_mx+d_m)^{l_m}$ rastav polinoma Q na njegove nultočke i kvadratne polinome bez realnih nultočaka.

Tada postoje konstante $*$ t.d. $\frac{P(x)}{Q(x)} =$

$$\frac{*}{x+b_1} + \frac{*}{(x+b_1)^2} + \ldots + \frac{*}{(x+b_1)^{k_1}} + \ldots + \frac{*}{x+b_m} + \frac{*}{(x+b_m)^2} + \ldots + \frac{*}{(x+b_m)^{k_m}} +$$

$$\frac{*x+*}{x^2+c_1x+d_1} + \ldots + \frac{*x+*}{(x^2+c_1x+d_1)^{l_1}} + \ldots + \frac{*x+*}{x^2+c_mx+d_m} + \ldots + \frac{*x+*}{(x^2+c_mx+d_m)^{l_m}}$$

Rastavite na parcijalne razlomke

- $\dfrac{2}{x^3-1}$

$x^3-1=(x-1)(x^2+x+1)$

$$\Rightarrow \frac{2}{x^3-1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}=\frac{Ax^2+Ax+A+Bx^2-Bx+Cx-C}{x^3-1}$$

$$=\frac{x^2(A+B)+x(A-B+C)+(A-C)}{x^3-1}$$

$$\left.\begin{array}{l} A+B=0 \\ A-B+C=0 \end{array}\right\} \left.\begin{array}{l} 2A+C=0 \\ \\ \end{array}\right\} 3A=2 \Rightarrow A=\frac{2}{3},\ B=-\frac{2}{3},\ C=-\frac{4}{3}$$
$$A-C=2$$

$$\Rightarrow \frac{2}{x^3-1}=\frac{2/3}{x-1}+\frac{-2/3\,x-4/3}{x^2+x+1}$$

- $\dfrac{1}{x^3-2x^2+x}$

$x^3-2x^2+x=x(x-1)^2$

$$\Rightarrow \frac{1}{x^3-2x^2+x}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}=\frac{A(x-1)^2+Bx(x-1)+Cx}{x^3-2x^2+x}$$

$$=\frac{x^2(A+B)+x(-2A-B+C)+A}{x^3-2x^2+x}$$

$$\begin{array}{l} A+B=0 \\ -2A-B+C=0 \\ A=1 \end{array} \Rightarrow B=-1,\ C=1$$

$$\Rightarrow \frac{1}{x^3-2x^2+x}=\frac{1}{x}-\frac{1}{x-1}+\frac{1}{(x-1)^2}$$

Odredite integrale

- $$\int \frac{1}{x^2+4}\,dx = \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx = \left\{\begin{array}{l} t = \frac{x}{2} \\ dt = \frac{1}{2}\,dx \end{array}\right\} =$$
$$= \frac{1}{2}\int \frac{1}{t^2+1}\,dt = \frac{1}{2}\,\mathrm{arctg}(t) + C = \frac{1}{2}\,\mathrm{arctg}\left(\frac{x}{2}\right) + C$$

- $$\int \frac{1}{x^2-5}\,dx =$$
$$\frac{1}{x^2-5} = \frac{A}{x-\sqrt{5}} + \frac{B}{x+\sqrt{5}} = \frac{Ax + A\sqrt{5} + Bx - B\sqrt{5}}{x^2-5}$$
$$\left.\begin{array}{l} A+B=0 \\ A\sqrt{5}-B\sqrt{5}=1 \end{array}\right\} 2A\sqrt{5}=1 \Rightarrow A = \frac{1}{2\sqrt{5}}, \quad B = -\frac{1}{2\sqrt{5}}$$
$$= \frac{1}{2\sqrt{5}}\int \frac{1}{x-\sqrt{5}}\,dx - \frac{1}{2\sqrt{5}}\int \frac{1}{x+\sqrt{5}}\,dx = \frac{1}{2\sqrt{5}}\ln|x-\sqrt{5}| - \frac{1}{2\sqrt{5}}\ln|x+\sqrt{5}| + C$$

- $$\int \frac{x-1}{x^2+2x+2}\,dx = \int \frac{x-1}{(x+1)^2+1}\,dx = \left\{\begin{array}{l} t = x+1, \quad x = t-1 \\ dt = dx \end{array}\right\} =$$
$$= \int \frac{t-2}{t^2+1}\,dt = \int \frac{t}{t^2+1}\,dt - 2\int \frac{1}{t^2+1}\,dt = \frac{1}{2}\ln|t^2+1| - 2\,\mathrm{arctg}(t) + C$$
$$= \frac{1}{2}\ln|(x+1)^2+1| - 2\,\mathrm{arctg}(x+1) + C$$

- $$\int_0^1 \frac{5-x}{x^2-5x+6}\,dx =$$
$$\frac{5-x}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3} = \frac{Ax - 3A + Bx - 2B}{x^2-5x+6}$$
$$\begin{array}{l} A+B=-1 \\ -3A-2B=5 \end{array} \Rightarrow -A = 3 \Rightarrow A=-3, \quad B=2$$

$$= \int_0^1 \frac{-3}{x-2}\,dx + \int_0^1 \frac{2}{x-3}\,dx = -3 \ln|x-2|\Big|_0^1 + 2\ln|x-3|\Big|_0^1$$

$$= -3(\ln 1 - \ln 2) + 2(\ln 2 - \ln 3) = 5\ln 2 - 2\ln 3$$

$$\cdot \int_{-1}^{1} \frac{x^3}{x^2+x+1}\,dx = \int_{-1}^{1} \frac{(x^2+x+1)\cdot x + (x^2+x+1)\cdot(-1) + 1}{x^2+x+1}\,dx =$$

$$= \int_{-1}^{1} x - 1 + \frac{1}{x^2+x+1}\,dx = \underbrace{\left(\frac{x^2}{2} - x\right)\Big|_{-1}^{1}}_{=-2} + \int_{-1}^{1} \frac{1}{(x+\frac{1}{2})^2 + \frac{3}{4}}\,dx =$$

$$\left\{\begin{array}{ll} t = x + \frac{1}{2} & 1 \to \frac{3}{2} \\ dt = dx & -1 \to -\frac{1}{2} \end{array}\right\} = -2 + \int_{-1/2}^{3/2} \frac{1}{t^2 + \frac{3}{4}}\,dt =$$

$$-2 + \frac{1}{3/4}\int_{-1/2}^{3/2} \frac{1}{\left(\frac{t}{\sqrt{3/4}}\right)^2 + 1}\,dt = \left\{\begin{array}{ll} s = \frac{t}{\sqrt{3/4}} & \frac{3}{2} \to \sqrt{3} \\ ds = \frac{dt}{\sqrt{3/4}} & -\frac{1}{2} \to -\frac{1}{\sqrt{3}} \end{array}\right\} =$$

$$= -2 + \frac{4}{3}\int_{-1/\sqrt{3}}^{\sqrt{3}} \frac{1}{s^2+1}\sqrt{\frac{3}{4}}\,ds = -2 + \frac{2}{\sqrt{3}}\,\mathrm{arctg}(s)\Big|_{-1/\sqrt{3}}^{\sqrt{3}} =$$

$$= -2 + \frac{2}{\sqrt{3}}\Big(\underbrace{\mathrm{arctg}(\sqrt{3})}_{\pi/3} - \underbrace{\mathrm{arctg}\left(-\frac{1}{\sqrt{3}}\right)}_{-\pi/6}\Big) = \frac{\pi}{\sqrt{3}} - 2$$

$$\cdot \int_{-3}^{3} \sqrt{9-x^2}\,dx = \left\{\begin{array}{ll} x = 3\sin t & 3 \to \pi/2 \\ dx = 3\cos t\,dt & -3 \to -\pi/2 \end{array}\right\} =$$

$$= 3\int_{-\pi/2}^{\pi/2} \sqrt{9 - 9\sin^2 t}\cdot\cos t\,dt = 9\int_{-\pi/2}^{\pi/2} (\cos t)^2\,dt = 9\int_{-\pi/2}^{\pi/2} \frac{1+\cos 2t}{2}\,dt$$

$$= 9\left(\frac{t}{2} + \frac{\sin 2t}{4}\right)\Big|_{-\pi/2}^{\pi/2} = 9\left(\frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) + 0\right) = \frac{9\pi}{2}$$

$$\bullet \int_0^1 \ln(1+x^2)\,dx = \left\{ \begin{matrix} u = \ln(1+x^2) & du = \frac{2x}{1+x^2}\,dx \\ dv = dx & v = x \end{matrix} \right\} =$$

$$x\ln(1+x^2)\Big|_0^1 - \int_0^1 \frac{2x^2}{1+x^2}\,dx = \ln 2 - 2\int_0^1 \frac{1+x^2-1}{1+x^2}\,dx =$$

$$= \ln 2 - 2\int_0^1 dx + 2\int_0^1 \frac{1}{1+x^2}\,dx = \ln 2 - 2x\Big|_0^1 + 2\,\mathrm{arctg}\,x\Big|_0^1$$

$$= \ln 2 - 2 + 2\left(\frac{\pi}{4} - 0\right) = \ln 2 - 2 + \frac{\pi}{2}$$

$$\bullet \int e^{\sqrt{x}}\,dx = \left\{ \begin{matrix} t = \sqrt{x} \\ dt = \frac{1}{2\sqrt{x}}\,dx \end{matrix} \right\} = \int e^t \cdot 2t\,dt$$

$$= 2\int te^t\,dt \quad \left\{ \begin{matrix} u = t & du = dt \\ dv = e^t dt & v = e^t \end{matrix} \right\} = 2te^t - 2\int e^t\,dt =$$

$$= 2te^t - 2e^t + C = e^{\sqrt{x}}(2\sqrt{x} - 2) + C$$

Koliki je rad potreban da oprugu konstante elastičnosti k rastegnemo za L iz stanja mirovanja? Hookov zakon kaže da opruga rastegnuta za x djeluje silom kx.

„Rad = sila · put" – preciznije „Rad je integral sile duž puta"

$$W = \int_0^L F(x)\,dx = \int_0^L kx\,dx = k\int_0^L x\,dx = k\,\frac{x^2}{2}\Big|_0^L = \frac{kL^2}{2}$$

(to je „elastična energija spremljena u oprugi")

Koliki je rad potreban da naboj q_2 pomaknemo s udaljenosti d_2 od naboja q_1 na udaljenost d_1? Coulombov zakon kaže da q_1 djeluje silom $k_e \frac{q_1 \cdot q_2}{r^2}$ na naboj q_2, kad su oni udaljeni za r. k_e je poznata Coulombova konstanta.

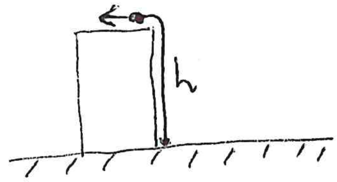
„Rad = integral sile duž puta"

$$W = \int_{d_2}^{d_1} k_e \frac{q_1 \cdot q_2}{r^2} dr = k_e q_1 q_2 \int_{d_2}^{d_1} \frac{1}{r^2} dr =$$

$$= k_e q_1 q_2 \cdot \left(-\frac{1}{r}\right)\Big|_{d_2}^{d_1} = k_e q_1 q_2 \left(\frac{1}{d_2} - \frac{1}{d_1}\right)$$

($k_e \frac{q_1 q_2}{r}$ je „potencijalna električna energija između naboja q_1 i q_2")

Koliki rad je potreban za povući uže mase m koje visi sa zgrade visine h, do poda? Uže povlačimo preko ruba zgrade, trenje zanemarujemo.



Na visini x od poda, masa užeta koje povlačimo je $m(x) = m \cdot \frac{h-x}{h}$. Zato je

$$W = \int_0^h m(x) g\, dx = \int_0^h m \frac{h-x}{h} g\, dx = \int_0^h mg - \frac{m}{h} g x\, dx =$$

$$mgx\Big|_0^h - \frac{m}{h} g \frac{x^2}{2}\Big|_0^h = mgh - \frac{m}{h} g \frac{h^2}{2} = \frac{mgh}{2}$$

Izračunajte integrale

- $\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{B\to\infty} \int_0^B \frac{1}{1+x^2}\,dx = \lim_{B\to\infty} \operatorname{arctg} x \Big|_0^B =$

$$= \lim_{B\to\infty} \operatorname{arctg} B = \frac{\pi}{2}$$

- $\int_e^\infty \frac{1}{x\cdot(\ln x)^3}\,dx = \lim_{B\to\infty} \int_e^B \frac{1}{x(\ln x)^3}\,dx =$

$$\left\{\begin{array}{ll} t = \ln x & e \to 1 \\ dt = \frac{1}{x}\,dx & B \to \ln B \end{array}\right\} = \lim_{B\to\infty} \int_1^{\ln B} \frac{1}{t^3}\,dt =$$

$$\lim_{B\to\infty} \left(-\frac{1}{2}t^{-2}\right)\Big|_1^{\ln B} = \lim_{B\to\infty} \left(-\frac{1}{2(\ln B)^2} + \frac{1}{2}\right) = \frac{1}{2}$$

- $\int_0^{1^-} \frac{x}{\sqrt{1-x^2}}\,dx = \lim_{B\to1^-} \int_0^B \frac{x}{\sqrt{1-x^2}}\,dx = \left\{\begin{array}{ll} x = \sin t & 0 \to 0 \\ dx = \cos t\,dt & B \to \arcsin B \end{array}\right\}$

$$= \lim_{B\to1^-} \int_0^{\arcsin B} \frac{\sin t}{\sqrt{1-\sin^2 t}} \cos t\,dt = \lim_{B\to1^-} \int_0^{\arcsin B} \sin t\,dt =$$

$$\lim_{B\to1^-} (-\cos t)\Big|_0^{\arcsin B} = \lim_{B\to1^-} \left(-\cos(\arcsin B) + 1\right) =$$

$$= \lim_{B\to1^-} \left(-\sqrt{1-\sin^2(\arcsin B)} + 1\right) = \lim_{B\to1^-} \left(-\sqrt{1-B^2} + 1\right) =$$

$$= 1$$

TM (Apsolutna konvergencija nepravog integrala povlači konvergenciju nepravog integrala)

Neka je $f: [a,b\rangle \to \mathbb{R}$ integrabilna na svakom $[a,B] \subset [a,b\rangle$ (može biti $b=\infty$). Tada, ako konvergira $\int_a^b |f(x)|\,dx$, konvergira i $\int_a^b f(x)\,dx$.

TM (Usporedni kriterij konvergencije nepravih integrala)

Neka su $f, g: [a,b\rangle \to \mathbb{R}$ integrabilne na svakom $[a,B] \subset [a,b\rangle$ (može biti $b=\infty$), te neka je $0 \le f(x) \le g(x)$, $\forall x \in [a,b\rangle$. Tada, ako konvergira $\int_a^b g(x)\,dx$, konvergira i $\int_a^b f(x)\,dx$.

TM (Granični usporedni kriterij konv. nepravih integrala)

Neka su $f, g: [a,b\rangle \to \mathbb{R}$ integrabilne na svakom $[a,B] \subset [a,b\rangle$ (može biti $b=\infty$), te neka postoji

$$L = \lim_{x \to b} \frac{f(x)}{g(x)} \in [-\infty, \infty].$$

Ako je $L \in \langle -\infty, \infty\rangle$ i $\int_a^b g(x)\,dx$ konvergira, tada i $\int_a^b f(x)\,dx$ konvergira.

Ako je $L \in [-\infty, 0\rangle \cup \langle 0, \infty]$ i $\int_a^b g(x)\,dx$ divergira, tada i $\int_a^b f(x)\,dx$ divergira.

Ako je $L \in \langle -\infty, 0\rangle \cup \langle 0, \infty\rangle$, tada se integrali $\int_a^b f(x)\,dx$ i $\int_a^b g(x)\,dx$ ponašaju isto po pitanju konvergencije.

TM $\int_1^\infty \frac{1}{x^p}\,dx$ konvergira za $p \in \langle 1, \infty \rangle$, a divergira za $p \in \langle 0, 1]$.

$\int_{0 \leftarrow}^1 \frac{1}{x^p}\,dx$ –‖– $p \in \langle 0, 1 \rangle$, –‖– $p \in [1, \infty \rangle$

Ispitajte konvergenciju nepravih integrala

- $\int_1^\infty \frac{\cos x}{x^2}\,dx$ .... Dokažimo da konvergira

(aps. konv. $\Rightarrow$ konv.) $\Rightarrow$ dovoljno vidjeti da $\int_1^\infty \frac{|\cos x|}{x^2}\,dx$ konv.

$\frac{|\cos x|}{x^2} \le \frac{1}{x^2}$ i usporedni kriterij $\Rightarrow$

$\Rightarrow$ dovoljno je vidjeti da $\int_1^\infty \frac{1}{x^2}\,dx$ konv. ✓

Dakle, konvergira i početni integral.

- $\int_1^\infty \frac{3+\sin x}{\sqrt[3]{x}}\,dx$ ... Dokažimo da divergira

$\sin x \ge -1 \Rightarrow \frac{3+\sin x}{\sqrt[3]{x}} \geq \frac{2}{\sqrt[3]{x}}$ i usporedni kriterij $\Rightarrow$

$\Rightarrow$ dovoljno je vidjeti da $\int_1^\infty \frac{2}{x^{1/3}}\,dx$ divergira. ✓

- $\int_5^\infty \frac{x\sin x}{\sqrt{x^5+4}}\,dx$ ... Dokažimo da konvergira.

(aps. konv. $\Rightarrow$ konv.) $\Rightarrow$ dovoljno je vidjeti da $\int_5^\infty \frac{x\,|\sin(x)|}{\sqrt{x^5+4}}\,dx$ konv.

$|\sin x| \le 1$ + usporedni kriterij $\Rightarrow$ dovoljno je vidjeti

da $\int_5^\infty \frac{x}{\sqrt{x^5+4}}$ konvergira.

$$\lim_{x\to\infty} \frac{\frac{x}{\sqrt{x^5+4}}}{\frac{1}{x^{3/2}}} = \lim_{x\to\infty} \sqrt{\frac{x^2\cdot x^3}{x^5+4}} = \lim_{x\to\infty} \sqrt{\frac{1}{1+\frac{4}{x^5}}} = 1$$

granični usp. kriterij $\Rightarrow$ dovoljno je vidjeti $\int_5^\infty \frac{1}{x^{3/2}}$ konvergira ✓

- $\int_2^\infty \operatorname{arctg} \frac{1}{x}\, dx$

$$\lim_{x\to\infty} \frac{\operatorname{arctg}\frac{1}{x}}{\frac{1}{x}} = \left\{\begin{matrix} y=\frac{1}{x} \\ y\to 0 \end{matrix}\right\} = \lim_{y\to 0} \frac{\operatorname{arctg} y}{y} \overset{L'H}{=} \lim_{y\to 0} \frac{\frac{1}{1+y^2}}{1} = 1$$

Dakle, po graničnom usporednom kriteriju se početni integral ponaša kao $\int_2^\infty \frac{1}{x}dx$, to jest, divergira.

- $\int_0^{\to 1} \frac{\cos(x^2)}{\sqrt[4]{(1-x)^3}}\, dx$ ... Dokazimo da konvergira.

(aps. konvergencija $\Rightarrow$ konv.) $\Rightarrow$ dovoljno je vidjeti da

$\int_0^{\to 1} \frac{|\cos(x^2)|}{\sqrt[4]{(1-x)^3}}\, dx$ konvergira.

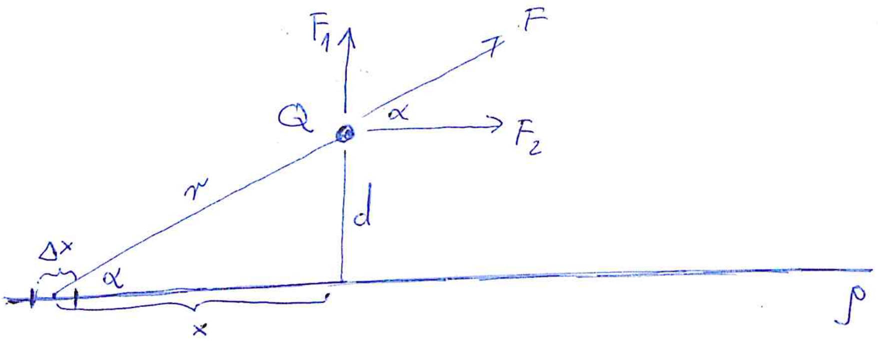
$|\cos(x^2)| \le 1$ + usporedni kriterij $\Rightarrow$ dovoljno je vidjeti

da $\int_0^{\to 1} \frac{1}{\sqrt[4]{(1-x)^3}}$ konvergira.

$$\int_0^{\to 1} \frac{1}{\sqrt[4]{(1-x)^3}} = \left\{\begin{matrix} t = 1-x & 0\to 1 \\ dt = -dx & 1^+\to 0^- \end{matrix}\right\} = \int_1^{0\leftarrow} \frac{-1}{\sqrt[4]{t^3}}\, dt = \int_{0\leftarrow}^1 \frac{1}{\sqrt[4]{t^3}}\, dt =$$

$= \int_{0\leftarrow}^1 \frac{1}{t^{3/4}}\, dt$ koji je konvergentan, pa je i početni integral konvergentan.

Kolika električna sila djeluje na naboj Q u prisustvu dugačke nabijene žice, linijske gustoće naboja ρ, na udaljenosti d od žice? Coulombov zakon kaže da naboj q djeluje na naboj Q silom $k_e \frac{qQ}{r^2}$, ako su oni međusobno udaljeni r. k_e = Coulombova konstanta.



Komad žice Δx sa slike ima ukupan naboj $\Delta x \cdot \rho$, pa djeluje silom $F \approx k_e \frac{\Delta x \rho Q}{r^2}$, za mali Δx.

$$\Rightarrow F_1 \approx k_e \frac{\Delta x \rho Q}{r^2} \sin\alpha = k_e \frac{\Delta x \rho Q}{r^2} \cdot \frac{d}{r} = k_e \frac{\Delta x \rho Q d}{(x^2+d^2)^{3/2}}$$

$$F_2 \approx \quad -\|- \quad \cos\alpha = \quad -\|- \quad \frac{x}{r} = k_e \frac{\Delta x \rho Q x}{(x^2+d^2)^{3/2}}$$

$$\Rightarrow F_{1,\text{ukupno}} = \lim_{\Delta x \to 0} \sum_{\substack{x=-\infty \\ (x \in \{\dots, -2\Delta x, -\Delta x, \\ 0, \Delta x, 2\Delta x, \dots\})}}^{\infty} k_e \frac{\Delta x \rho Q d}{(x^2+d^2)^{3/2}} = \int_{-\infty}^{\infty} k_e \frac{\rho Q d}{(x^2+d^2)^{3/2}} dx$$

$$= k_e \rho Q d \int_{-\infty}^{\infty} \frac{1}{(x^2+d^2)^{3/2}} dx$$

$$F_{2,\text{ukupno}} = k_e \rho Q \int_{-\infty}^{\infty} \frac{x}{(x^2+d^2)^{3/2}} dx$$

Treba odrediti $\int \frac{1}{(x^2+d^2)^{3/2}} dx$. Pokušajmo s

$$\left(\frac{x}{(x^2+d^2)^{1/2}}\right)' = \frac{1}{(x^2+d^2)^{1/2}} - \frac{1}{2} \frac{x}{(x^2+d^2)^{3/2}} \cdot 2x = \frac{x^2+d^2}{(x^2+d^2)^{3/2}} - \frac{x^2}{(x^2+d^2)^{3/2}} = \frac{d^2}{(x^2+d^2)^{3/2}}$$

$$\Rightarrow \int \frac{1}{(x^2+d^2)^{3/2}} dx = \frac{x}{d^2(x^2+d^2)^{1/2}} + C$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{(x^2+d^2)^{3/2}}\,dx = \{\text{parna funkcija}\} = 2\int_0^{\infty} \frac{1}{(x^2+d^2)^{3/2}}\,dx =$$

$$= 2 \lim_{B\to\infty} \frac{x}{d^2(x^2+d^2)^{1/2}}\Bigg|_0^B = 2\lim_{B\to\infty} \frac{B}{d^2(B^2+d^2)^{1/2}} = 2\cdot\frac{1}{d^2}$$

$$\Rightarrow F_{1,\text{ukupno}} = k_e \rho Q d \cdot 2 \cdot \frac{1}{d^2} = \frac{2k_e\rho Q}{d}$$

$F_{2,\text{ukupno}}$ .... sa slike očito $=0$, jer „će se pokratiti zbog simetrije"!

Preciznije, $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+d^2)^{3/2}}\,dx = 0$, jer je funkcija neparna.

Moramo se samo uvjeriti da integral postoji, jer ako postoji, onda je $=0$, jer je funkcija neparna. Kad bi ~~.. ..~~ $\displaystyle\int_0^{\infty} \frac{x}{(x^2+d^2)^{3/2}}\,dx$ bio $=+\infty$, onda bismo imali $\infty-\infty$ što nije definirano. Fizikalno, sila u lijevo bi bila = sila u desno $=+\infty$.

No, $\displaystyle\int_0^{\infty} \frac{x}{(x^2+d^2)^{3/2}}$ konvergira po usporednom kriteriju s $\frac{1}{x^2}$, jer

$$\lim_{x\to\infty} \frac{\frac{x}{(x^2+d^2)^{3/2}}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x^3}{(x^2+d^2)^{3/2}} = \lim_{x\to\infty} \sqrt{\left(\frac{x^2}{x^2+d^2}\right)^3} = 1.$$

Alternativno, drugi način za izračunati $\displaystyle\int_0^{\infty} \frac{1}{(x^2+d^2)^{3/2}}$

$$\int_0^{\infty} \frac{1}{(x^2+d^2)^{3/2}}\,dx = \left\{\begin{array}{ll} x = d\,\mathrm{tg}\,t & 0\to 0 \\ dx = d\frac{1}{\cos^2 t}\,dt & \infty \to \frac{\pi}{2} \end{array}\right\} =$$

$$= \int_0^{\pi/2} \frac{1}{d^3(\underbrace{\mathrm{tg}^2 t+1}_{=1/\cos^2 t})^{3/2}} \cdot \frac{d}{\cos^2 t}\,dt = \int_0^{\pi/2} \frac{1}{d^2}\cos t\,dt = \frac{1}{d^2}\sin t\Big|_0^{\pi/2} = \frac{1}{d^2}$$

Primjenom trapezne formule približno izračunajte $\int_0^1 e^{-x^2}\,dx$ uz podjelu na 6 podintervala.

Trapezna formula: $\int_a^b f(x)\,dx \approx \frac{h}{2}\left(f(a) + 2\sum_{k=1}^{n-1} f(x_k) + f(b) \right)$

$$\int_0^1 e^{-x^2}\,dx \approx \frac{1}{12}\left(1 + 2e^{-(1/6)^2} + 2e^{-(2/6)^2} + 2e^{-(3/6)^2} + 2e^{-(4/6)^2} + 2e^{(-5/6)^2} + e^{-1} \right) \approx 0.745$$

Izračunajte sumu redova

- $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

(rastav na parcijalne razlomke) $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$$S_N = \sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \frac{1}{n} - \frac{1}{n+1} =$$

$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{N} - \frac{1}{N-1}\right) = 1 - \frac{1}{N-1}$$

$$\lim_{N\to\infty} S_N = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

- $\sum_{n=0}^{\infty} \frac{1}{2^n}$

$$S_N = \sum_{n=0}^{N} \frac{1}{2^n} = \frac{1}{2^0} + \frac{1}{2^1} + \dots + \frac{1}{2^N} = \frac{\left(1 - \frac{1}{2}\right)\left(\frac{1}{2^0} + \dots + \frac{1}{2^N}\right)}{1 - \frac{1}{2}}$$

$$= \frac{1 - \frac{1}{2^{N+1}}}{\frac{1}{2}} \Rightarrow \lim_{N\to\infty} S_N = \frac{1}{\frac{1}{2}} = 2 \Rightarrow \sum_{n=0}^{\infty} \frac{1}{2^n} = 2$$

<u>TM</u> (nužan uvjet konvergencije)

Ako je $\sum_{n=1}^{\infty} a_n$ konvergentan, tad je $\lim_{n\to\infty} a_n = 0$

<u>TM</u> (usporedni test konvergencije redova)

Neka su $\sum_{n=1}^{\infty} a_n$ i $\sum_{n=1}^{\infty} b_n$ redovi s nenegativnim članovima takvi da je $0 \le a_n \le b_n, \ \forall n \in \mathbb{N}$.

Ako $\sum_{n=1}^{\infty} b_n$ konvergira, tada i $\sum_{n=1}^{\infty} a_n$ konvergira.

(Ako $\sum_{n=1}^{\infty} a_n$ divergira, tada i $\sum_{n=1}^{\infty} b_n$ divergira).

<u>TM</u> (granični usporedni test konvergencije redova)

Neka su $\sum_{n=1}^{\infty} a_n$ i $\sum_{n=1}^{\infty} b_n$ redovi s nenegativnim članovima, te $\lim_{n\to\infty} \frac{a_n}{b_n} = g \in [0,\infty]$.

Ako je $g \in [0,\infty\rangle$ i $\sum_{n=1}^{\infty} b_n$ konvergira, tada i $\sum_{n=1}^{\infty} a_n$ konvergira.

Ako je $g \in \langle 0,\infty]$ i $\sum_{n=1}^{\infty} b_n$ divergira, tada i $\sum_{n=1}^{\infty} a_n$ divergira.

Ako je $g \in \langle 0,\infty\rangle$, tada se redovi $\sum_{n=1}^{\infty} a_n$ i $\sum_{n=1}^{\infty} b_n$ ponašaju isto po pitanju konvergencije.

Konvergira li

- $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ... Vidjeli smo da $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ konvergira, a kako je $\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n(n+1)}} = \lim_{n\to\infty} \frac{n^2+n}{n^2} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1} = 1$, po graničnom usporednom testu i $\sum_{n=1}^{\infty} \frac{1}{n^2}$ konvergira.

• $\sum_{n=1}^{\infty} \frac{1}{n^3}$

$\frac{1}{n^3} \leq \frac{1}{n^2} \quad \forall n \in \mathbb{N}$, a kako znamo da $\sum_{n=1}^{\infty} \frac{1}{n^2}$ konvergira, onda konvergira i $\sum_{n=1}^{\infty} \frac{1}{n^3}$, po usporednom kriteriju.

TM $\sum_{n=1}^{\infty} \frac{1}{n^p}$ konvergira za $p > 1$, a divergira za $0 < p \leq 1$

Konvergira li

• $\sum_{n=1}^{\infty} \frac{n^2}{n^4+n^2}$

$$\lim_{n \to \infty} \frac{\frac{n^2}{n^4+n^2}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{n^4}{n^4+n^2} = \lim_{n \to \infty} \frac{1}{1+\frac{1}{n^2}} = 1,$$

a obzirom da $\sum_{n=1}^{\infty} \frac{1}{n^2}$ konvergira, konvergira i $\sum_{n=1}^{\infty} \frac{n^2}{n^4+n^2}$, po graničnom usporednom kriteriju.

• $\sum_{n=1}^{\infty} \frac{\sqrt{n+1} + \sqrt{n^3+1}}{\sqrt{n^4+1}}$

$$\lim_{n \to \infty} \frac{\sqrt{n+1} + \sqrt{n^3+1}}{\sqrt{n^4+1}} : \frac{1}{\sqrt{n}} = \lim_{n \to \infty} \frac{\sqrt{n^2+n} + \sqrt{n^4+n}}{\sqrt{n^4+1}} \cdot \frac{\sqrt{\frac{1}{n^4}}}{\sqrt{\frac{1}{n^4}}} =$$

$$= \lim_{n \to \infty} \frac{\sqrt{\frac{1}{n^2} + \frac{1}{n^3}} + \sqrt{1 + \frac{1}{n^3}}}{\sqrt{1 + \frac{1}{n^4}}} = \frac{0 + \sqrt{1}}{\sqrt{1}} = 1,$$

a obzirom da $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{0.5}}$ divergira, po graničnom usporednom kriteriju, divergira i početni red.

- $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$

$$\lim_{n\to\infty} \frac{(\ln n)^2}{n^3} : \frac{1}{n^2} = \lim_{n\to\infty} \frac{(\ln n)^2}{n} \overset{L'H}{=} \lim_{n\to\infty} \frac{2\ln n \cdot \frac{1}{n}}{1} = 2 \lim_{n\to\infty} \frac{\ln n}{n}$$

$$\overset{L'H}{=} 2 \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 2 \cdot 0 = 0.$$

Kako $\sum_{n=1}^{\infty} \frac{1}{n^2}$ konvergira, po graničnom usporednom testu, konvergira i početni red.

- $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3 \ln n}$

$\frac{1}{n^3 \ln n} \leq \frac{1}{n^3}$, za $n \geq 3$, pa po usporednom kriteriju početni red konvergira, jer ~~znamo~~ da $\sum_{n=1}^{\infty} \frac{1}{n^3}$ konvergira.

- $\displaystyle\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2+n-5}$

$\frac{|\cos n|}{n^2+n-5} \leq \frac{1}{n^2+n-5}$, ~~[illegible]~~ $\forall n \geq 3$, pa ako dokažemo da je $\sum_{n=1}^{\infty} \frac{1}{n^2+n-5}$ konvergentan, isto će vrijediti za početni red.

$$\lim_{n\to\infty} \frac{1}{n^2+n-5} : \frac{1}{n^2} = \lim_{n\to\infty} \frac{n^2}{n^2+n-5} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}-\frac{5}{n^2}} = 1.$$

Kako je $\sum_{n=1}^{\infty} \frac{1}{n^2}$ konvergentan, po usporednom kriteriju znamo da je i $\frac{1}{n^2+n-5}$ konvergentan, pa je i početni red konvergentan.

TM (D'Alembertov kriterij konvergencije redova)

Neka je $\sum_{n=1}^{\infty} a_n$ red takav da postoji $q = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \in [0,\infty]$

Ako je $q<1$, tad $\sum_{n=1}^{\infty} a_n$ konvergira,

ako je $q>1$, —"— divergira.

TM (Cauchyjev kriterij konvergencije redova)

Neka je $\sum_{n=1}^{\infty} a_n$ red takav da postoji $q = \lim_{n\to\infty} \sqrt[n]{|a_n|} \in [0,\infty]$

Ako je $q<1$, tad $\sum_{n=1}^{\infty} a_n$ konvergira,

ako je $q>1$, —"— divergira.

Konvergira li

- $\sum_{n=1}^{\infty} \frac{n^{10}}{3^n}$

$$\lim_{n\to\infty} \frac{(n+1)^{10}}{3^{n+1}} : \frac{n^{10}}{3^n} = \lim_{n\to\infty} \frac{1}{3}\cdot\left(\frac{n+1}{n}\right)^{10} =$$

$$\frac{1}{3}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{10} = \frac{1}{3}\cdot 1^{10} = \frac{1}{3} < 1$$

Po D'A kriteriju, red je konvergentan.

(drugo rješenje)

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^{10}}{3^n}} = \lim_{n\to\infty} \frac{1}{3}\cdot\left(\sqrt[n]{n}\right)^{10} = \frac{1}{3}\Big(\underbrace{\lim_{n\to\infty}\sqrt[n]{n}}_{=1}\Big)^{10} = \frac{1}{3}\cdot 1^{10} = \frac{1}{3} < 1$$

Po Cauchyjevom kriteriju, red je konvergentan.

- $\sum_{n=1}^{\infty} \frac{n^5}{e^{n^2}}$

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^5}{e^{n^2}}} = \lim_{n\to\infty} \left(\sqrt[n]{n}\right)^5\cdot\left(\frac{1}{e^{n^2}}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\sqrt[n]{n}\right)^5\cdot\frac{1}{e^n} = 1^5\cdot 0 = 0 < 1$$

Cauchy $\Rightarrow$ red je konvergentan.

- $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

$$\lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} : \frac{2^n}{n!} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$$

Po D'A kriteriju, red je konvergentan.

- $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n}$

$$\lim_{n\to\infty} \sqrt[n]{\frac{e^n}{n}} = \lim_{n\to\infty} \frac{e}{\sqrt[n]{n}} = \frac{e}{1} = e > 1$$

Po Cauchyjevom kriteriju, red je divergentan.

- $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} : \frac{n^n}{n!} = \lim_{n\to\infty} \frac{(n+1)(n+1)^n}{(n+1)} : n^n =$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1$$

Po D'A kriteriju, red je divergentan.

- $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

$$\lim_{n\to\infty} \frac{((n+1)!)^2}{(2(n+1))!} : \frac{(n!)^2}{(2n)!} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} =$$

$$= \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2} = \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{1}{n^2}}{4+\frac{6}{n}+\frac{2}{n^2}} = \frac{1}{4} < 1$$

Po D'A kriteriju, red je konvergentan.

$$\cdot \sum_{n=1}^{\infty} \left(\frac{n-1}{n+1}\right)^{n(n+1)}$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{n-1}{n+1}\right)^{n+1} = \lim_{n\to\infty} \left(\left(1-\frac{2}{n+1}\right)^{\frac{n+1}{-2}}\right)^{-2} =$$

$$= \left(\underbrace{\lim_{n\to\infty}\left(1-\frac{2}{n+1}\right)^{-\frac{n+1}{2}}}_{=e}\right)^{-2} = e^{-2} = \frac{1}{e^2} < 1$$

Po Cauchyjevom kriteriju red je konvergentan.

$$\cdot \sum_{n=1}^{\infty} \frac{1}{2^n}\left(\frac{n}{n+1}\right)^{-n^2}$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n}{n+1}\right)^{-n} = \frac{1}{2}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n =$$

$$= \frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \frac{1}{2}e > 1$$

Po Cauchyjevom kriteriju, red je divergentan

$$\cdot \sum_{n=1}^{\infty} \frac{n^n}{n!\,3^n}$$

$$\lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!\,3^{n+1}} : \frac{n^n}{n!\,3^n} = \frac{1}{3}\lim_{n\to\infty} \frac{(n+1)(n+1)^n}{(n+1)} : n^n =$$

$$= \frac{1}{3}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \frac{1}{3}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \frac{1}{3}e < 1$$

Po D'A kriteriju, red je konvergentan.

TM (o apsolutnoj konvergenciji redova)

Neka je $\sum_{n=1}^{\infty} a_n$ red takav da je $\sum_{n=1}^{\infty} |a_n|$ konvergentan.

Tad je i $\sum_{n=1}^{\infty} a_n$ konvergentan.

TM (Leibnizov kriterij konvergencije redova)

Neka je $\sum_{n=1}^{\infty} a_n$ red takav da niz $(a_n)_{n\in\mathbb{N}}$ alternira po predznaku, $|a_{n+1}| \leq |a_n|, \forall n \in \mathbb{N}$, te $\lim_{n\to\infty} |a_n| = 0$.

Tada je $\sum_{n=1}^{\infty} a_n$ konvergentan.

Konvergira li

- $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

alternira po predznaku ✓ $\left|\frac{(-1)^{n+1}}{n+1}\right| \leq \left|\frac{(-1)^n}{n}\right|, \forall n \in \mathbb{N}$ ✓

$\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0$ ✓

Po Leibnizovom kriteriju, red je konvergentan.

- $\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$

Red je čak i apsolutno konvergentan jer $\frac{|\sin n|}{n^3} \leq \frac{1}{n^3}$, pa je po usporednom kriteriju $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^3}$ konvergentan.

Kako apsolutna konvergencija povlači običnu, početni red je konvergentan.

$$\sum_{n=2}^{\infty} \frac{(-1)^{n-2}}{\ln(n)}$$

alternira po predznaku za $n \geq 3$ ✓

$$\left|\frac{(-1)^{n+1-2}}{\ln(n+1)}\right| = \frac{1}{\ln(n+1)} < \frac{1}{\ln(n)} = \left|\frac{(-1)^{n-2}}{\ln(n)}\right| \text{ za } n \geq 3 \checkmark$$

$$\lim_{n\to\infty}\left|\frac{(-1)^{n-2}}{\ln(n)}\right| = \lim_{n\to\infty}\frac{1}{\ln n} = 0 \checkmark$$

Po Leibnizovom kriteriju, red je konvergentan.

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \dots + \frac{1}{16} + \dots$$

Neka je $\sum_{n=1}^{\infty} b_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{16} + \frac{1}{32} + \dots$

$$\Rightarrow \sum_{n=1}^{\infty} b_n = 1 + \frac{1}{2} + 2\cdot\frac{1}{4} + 4\cdot\frac{1}{8} + 8\cdot\frac{1}{16} + \dots$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots = +\infty$$

Kako je po konstrukciji $\frac{1}{n} \geq b_n, \forall n \in \mathbb{N}$, a red $\sum_{n=1}^{\infty} b_n$ je divergentan, divergentan je i $\sum_{n=1}^{\infty} \frac{1}{n}$

<u>TM</u> (Integralni kriterij konvergencije redova)

Neka je $\sum_{n=1}^{\infty} a_n$ red s pozitivnim članovima. Neka je $f: [a, \infty\rangle \to [0, \infty\rangle$ neprekidna i padajuća takva da je $a_n = f(n), \forall n \in \mathbb{N}$. Tada red $\sum_{n=0}^{\infty} a_n$ konvergira ako i samo ako nepravi integral $\int_a^{\infty} f(x)dx$ konvergira.

3 Konvergira li

- $\sum_{n=1}^{\infty} \frac{1}{n}$ (drugo rješenje)

$f(x) = \frac{1}{x}$ je padajuća i neprekidna na $[1, \infty\rangle$

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{B\to\infty} \int_1^{B} \frac{1}{x}\,dx = \lim_{B\to\infty} \ln x \Big|_1^B =$$

$$\lim_{B\to\infty} (\ln B - \underbrace{\ln 1}_{=0}) = \infty$$

Po integralnom kriteriju, $\sum_{n=1}^{\infty} \frac{1}{n}$ divergira.

- $\sum_{n=1}^{\infty} \frac{1}{n^p}$, za $p>1$

$f(x) = \frac{1}{x^p}$ je padajuća i neprekidna na $[1, \infty\rangle$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{B\to\infty} \int_1^{B} x^{-p}\,dx = \lim_{B\to\infty} \frac{x^{-p+1}}{-p+1} \Big|_1^B =$$

$$\frac{1}{1-p} \lim_{B\to\infty} \Big(\underbrace{\frac{1}{B^{p-1}}}_{\to 0,\ \text{za } p>1} - 1 \Big) = -\frac{1}{1-p}$$

Po integralnom kriteriju, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ konvergira za $p>1$.

- $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

$f(x) = \frac{1}{x \ln x}$ je padajuća i neprekidna na $[2, \infty\rangle$

$$\int_2^{\infty} \frac{1}{x \ln x}\,dx = \lim_{B\to\infty} \int_2^{B} \frac{1}{x \ln x}\,dx = \left\{ \begin{array}{l} t = \ln x \\ dt = \frac{1}{x}\,dx \end{array} \right\} = \quad \begin{array}{l} 2 \to \ln 2 \\ B \to \ln B \end{array}$$

$$= \lim_{B\to\infty} \int_{\ln 2}^{\ln B} \frac{1}{t}\,dt = \lim_{B\to\infty} \ln t \Big|_{\ln 2}^{\ln B} = \lim_{B\to\infty} \left(\ln \ln B - \ln \ln 2\right)$$

$= \infty$ Po integralnom kriteriju, $\sum\limits_{n=2}^{\infty} \frac{1}{n \ln n}$ je divergentan.

- $\sum\limits_{n=1}^{\infty} \frac{1+\cos(n\pi)}{n!}$

$$1+\cos(n\pi) = \begin{cases} 2 & , n \text{ paran} \\ 0 & , n \text{ neparan} \end{cases}$$

$\Rightarrow$ red ima sve članove ne negativne,

$$\frac{1+\cos(n\pi)}{n!} \leq \frac{2}{n!} \leq \frac{2}{n^2}, \quad \forall n \geq 4$$

$$\left\{ \begin{array}{l} \text{za } n \geq 4 \text{ je } n! \geq n(n-1)(n-2) \geq n(n-1)\cdot 2 = n(2n-1) = \\ n(n+(n-1)) > n\cdot n = n^2, \text{ pa je } \frac{2}{n!} \leq \frac{2}{n^2} \end{array} \right\}$$

Kako je $\sum\limits_{n=1}^{\infty} \frac{2}{n^2}$ konvergentan, po usporednom kriteriju je

i početni red konvergentan.

- $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{n^2+2}{n^3+n-1}$

alternirajući predznaci ✓

$$\lim_{n\to\infty} \left| \frac{n^2+2}{n^3+n-1} (-1)^{n+1} \right| = \lim_{n\to\infty} \frac{\frac{1}{n} + \frac{2}{n^2}}{1+\frac{1}{n^2}-\frac{1}{n^3}} = \frac{0}{1} = 0 \checkmark$$

Neka je $f(x) = \frac{x^2+2}{x^3+x-1}$

$$f'(x) = \frac{2x(x^3+x-1) - (x^2+2)(3x^2+1)}{(x^3+x-1)^2} = \frac{2x^4+2x^2-2x - 3x^4 - 7x^2 - 2}{(x^3+x-1)^2} =$$

$$\frac{-x^4-5x^2-2x-2}{(x^3+x-1)^2} < 0 \quad , \forall x \geq 1$$

$\Rightarrow f(x)$ je padajuća na $[1, \infty\rangle$

$\Rightarrow f(n+1) \leq f(n) \ , \forall n \in \mathbb{N}$

$\Rightarrow |a_{n+1}| \leq |a_n| \ , \forall n \in \mathbb{N}$ ✓

Po Leibnizovom kriteriju, red je konvergentan.

- $\displaystyle\sum_{n=1}^{\infty} \sin\left(\sqrt{n+1}-\sqrt{n}\right)$

$$\left\{ \sqrt{n+1}-\sqrt{n} = \sqrt{n+1}-\sqrt{n} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \right\}$$

$$\lim_{n\to\infty} \frac{\sin\frac{1}{\sqrt{n+1}+\sqrt{n}}}{\frac{1}{\sqrt{n+1}+\sqrt{n}}} = \left\{ \begin{array}{l} t = \frac{1}{\sqrt{n+1}+\sqrt{n}} \\ t \to 0 \end{array} \right\} = \lim_{t\to 0} \frac{\sin t}{t} = 1$$

$\Rightarrow$ početni red se po graničnom usporednom kriteriju ponaša kao $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$

$$\frac{1}{\sqrt{n+1}+\sqrt{n}} \geq \frac{1}{2\sqrt{n+1}} \quad , \forall n \in \mathbb{N}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n+1}} = \frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ je divergentan, pa je po usporednom kriteriju divergentan i $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$, pa je i početni red divergentan.

- $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \cdot \ln n \cdot 2^n \cdot n!}$

$\displaystyle\frac{1}{n \cdot \ln n \cdot 2^n \cdot n!} \leq \frac{1}{2^n} \ , \forall n \geq 3$, pa kako je $\displaystyle\sum_{n=2}^{\infty} \frac{1}{2^n}$ konvergentan, i početni red je konvergentan.

TM Neka je $\sum_{n\geq 0} a_n x^n$ red potencija, $\rho = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.

Tada je radijus konvergencije $r = 1/\rho$ (uz $\frac{1}{0}=\infty$, $\frac{1}{\infty}=0$)

Dodatno, ako postoji $\rho_1 = \lim_{n\to\infty} \sqrt[n]{|a_n|}$, tada je $r = 1/\rho_1$.

Dodatno ako postoji $\rho_2 = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$, tada je $r = \rho_2$.

Izračunajte radijus konvergencije za

- $\sum_{n=0}^{\infty} x^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{1} = 1 \Rightarrow r = \frac{1}{1} = 1$$

- $\sum_{n=1}^{\infty} n^n x^n$

$$\lim_{n\to\infty} \sqrt[n]{n^n} = \lim_{n\to\infty} n = \infty \Rightarrow r = \frac{1}{\infty} = 0$$

- $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{1}{n!} : \frac{1}{(n+1)!} = \lim_{n\to\infty} \frac{1}{1} \cdot \frac{1}{n+1} = \lim_{n\to\infty} n+1 =$$

$$= \infty \Rightarrow r = \infty$$

- $\sum_{n=1}^{\infty} \frac{\ln n}{2^n} x^n$

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{\ln n}{2^n} : \frac{\ln(n+1)}{2^{n+1}} = \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \cdot 2$$

$$= 2 \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \overset{L'H}{=} 2 \lim_{n\to\infty} \frac{1/n}{1/(n+1)} = 2 \lim_{n\to\infty} \frac{n+1}{n} =$$

$$= 2 \lim_{n\to\infty} \frac{1+1/n}{1} = 2 \cdot 1 = 2 \Rightarrow r = 2$$

- $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{1/n}{1/(n+1)} = \lim_{n\to\infty} \frac{n+1}{n} = 1 \Rightarrow r = 1$$

- $\displaystyle\sum_{n=1}^{\infty} \frac{(3n)!}{n^{3n}} x^n$

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{(3n)!}{n^{3n}} : \frac{(3(n+1))!}{(n+1)^{3(n+1)}} =$$

$$= \lim_{n\to\infty} \frac{1}{n^{3n}} \cdot \frac{(n+1)^3 (n+1)^{3n}}{(3n+3)(3n+2)(3n+1)} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{3n} \cdot$$

$$\cdot \underbrace{\frac{n+1}{(3n+3)}}_{\to 1/3} \cdot \underbrace{\frac{n+1}{3n+2}}_{\to 1/3} \cdot \underbrace{\frac{n+1}{3n+1}}_{\to 1/3} = \frac{1}{27} \lim_{n\to\infty} \Big(\underbrace{\left(1+\frac{1}{n}\right)^n}_{\to e}\Big)^3 =$$

$$= \frac{1}{27} \cdot e^3 \Rightarrow r = \frac{e^3}{27}$$

- $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n^2)}{n^n} x^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{\sqrt[n]{|\cos(n^2)|}}{n}$$

$$0 \leq \frac{\sqrt[n]{|\cos(n^2)|}}{n} \leq \frac{1}{n}, \quad \forall n \in \mathbb{N},$$

pa puštanjem $\lim\limits_{n\to\infty}$ i primjenom teorema o sendviču,

dobivamo $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 0 \Rightarrow r = \infty$

- $\sum\limits_{n=1}^{\infty} \left(1+(-1)^n\right) x^n$

$$\Rightarrow a_n = \begin{cases} 0, & n \text{ neparan} \\ 2, & n \text{ paran} \end{cases}$$

$$\Rightarrow \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{2} = 1 \Rightarrow r = 1$$

- $\sum\limits_{n=1}^{\infty} \left(1+(-2)^n\right) x^n$

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{1+(-2)^n}{1+(-2)^{n+1}} \right| =$$

$$= \lim_{n\to\infty} \left| \frac{\frac{1}{(-2)^n} + 1}{\frac{1}{(-2)^n} - 2} \right| = \left| \frac{0+1}{0-2} \right| = \frac{1}{2} \Rightarrow r = \frac{1}{2}$$

Odredite derivaciju i antiderivaciju reda član po član

- $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots$

derivacija: $= 0 + 1 + x + \frac{x^2}{2!} + \ldots = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$

antiderivacija: $= x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = \sum\limits_{n=1}^{\infty} \frac{x^n}{n!}$

- $\sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+\dots$

derivacija: $1+2x+3x^2+\dots = \sum_{n=0}^{\infty} (n+1)x^n$

antiderivacija: $x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\dots = \sum_{n=1}^{\infty} \frac{x^n}{n}$

Odredite područje konvergencije za

- $\sum_{n=0}^{\infty} x^n$

radijus ... $\lim_{n\to\infty} \sqrt[n]{1} = 1 \Rightarrow r=1$

za $x=1$, red $\sum_{n=0}^{\infty} 1^n$ divergira jer ne ispunjava nužan uvjet konvergencije.

za $x=-1$, red $\sum_{n=0}^{\infty} (-1)^n$ –||–

$\Rightarrow$ područje konvergencije je $\langle -1, 1\rangle$

- $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$

$\lim_{n\to\infty} \left|\frac{a_n}{a_{n-1}}\right| = \lim_{n\to\infty} \frac{n+1}{n} = 1 \Rightarrow r=1$

za $x=1$, red $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ konvergira po Leibnitzovom kriteriju

za $x=-1$, red $\sum_{n=1}^{\infty} \frac{1}{n}$ divergira $\left(\frac{1}{n^p}\right)$

$\Rightarrow$ područje konvergencije je $[-1, 1\rangle$

Odredite Taylorov red oko nule i njegov radijus konvergencije za funkciju

---

- $f(x) = e^x$

$f'(x) = e^x$ $\quad f(0) = f'(0) = \dots = f^{(n)}(0) = 1$

$f^{(n)}(x) = e^x$ $\quad \Rightarrow T(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

$$\lim_{n\to\infty} \left| \frac{1}{n!} : \frac{1}{(n+1)!} \right| = \lim_{n\to\infty} (n+1) = \infty \Rightarrow r = \infty$$

- $f(x) = \sin(x)$ $\quad f(0) = 0$

$f'(x) = \cos(x)$ $\quad f'(0) = 1$

$f''(x) = -\sin(x)$ $\quad f''(0) = 0$

$f'''(x) = -\cos(x)$ $\quad f'''(0) = -1$

$f^{(4)}(x) = \sin(x)$ $\quad f^{(4)}(0) = 0$

...

...

$$\Rightarrow T(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}$$

Red $\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^n$ ima isti radijus konvergencije kao $T(x)$ jer ima sve iste članove bez nula, koje i tako neće igrati ulogu u računanju $\limsup_{n\to\infty} \sqrt[n]{|a_n|}$.

Za taj red računamo $\lim_{n\to\infty} \left| \frac{(-1)^n}{(2n+1)!} : \frac{(-1)^{n+1}}{(2n+3)!} \right| =$

$= \lim_{n\to\infty} (2n+3)(2n+2) = \infty \cdot \infty = \infty \Rightarrow r = \infty$, pa je i radijus konvergencije $T(x) = \infty$.

$$\bullet \quad f(x) = \frac{1}{1-3x}$$

Neka je $g(y) = \frac{1}{1-y} = (1-y)^{-1} \Rightarrow g(3x) = f(x)$

$$g(y) = (1-y)^{-1}$$
$$g'(y) = -(1-y)^{-2} \cdot (-1) = (1-y)^{-2}$$
$$g''(y) = -2(1-y)^{-3}(-1) = 2(1-y)^{-3}$$
$$g'''(y) = 6(1-y)^{-4}$$
$$g^{(4)}(y) = 24(1-y)^{-5}$$
$$\dots$$
$$g^{(n)}(y) = n!\,(1-y)^{-n-1}$$

$$g(0) = 1$$
$$g'(0) = 1$$
$$g''(0) = 2$$
$$g'''(0) = 6$$
$$g^{(4)}(0) = 24$$
$$\dots$$
$$g^{(n)}(0) = n!$$

$$T_g(y) = \sum_{n=0}^{\infty} \frac{n!}{n!} y^n = \sum_{n=0}^{\infty} y^n$$

$$\Rightarrow T_f(x) = \sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} 3^n x^n$$

$$\lim_{n\to\infty} \left| \frac{3^n}{3^{n+1}} \right| = \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3} \Rightarrow r = \frac{1}{3}$$

$$\bullet \quad f(x) = x^4 - 2x^3 + 7x^2 - x - 6$$
$$f'(x) = 4x^3 - 6x^2 + 14x - 1$$
$$f''(x) = 12x^2 - 12x + 14$$
$$f'''(x) = 24x - 12$$
$$f^{(4)}(x) = 24$$
$$f^{(5)}(x) = 0$$
$$\dots$$
$$f^{(n)}(x) = 0$$

$$f(0) = -6$$
$$f'(0) = -1$$
$$f''(0) = 14$$
$$f'''(0) = -12$$
$$f^{(4)}(0) = 24$$
$$f^{(5)}(0) = 0$$
$$\vdots$$
$$f^{(n)}(0) = 0$$

$$T(x) = \frac{-6}{1} - \frac{1}{1}x + \frac{14}{2!}x^2 - \frac{12}{3!}x^3 + \frac{24}{4!}x^4 + 0 + \dots$$
$$= -6 - x + 7x^2 - 2x^3 + x^4$$

Taj „red potencija" ima $a_n = 0$ za $n > 5$, pa je zato

$$\rho = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = 0 \Rightarrow r = \infty$$

TM Neka je $n \in \mathbb{N}$, $I \subseteq \mathbb{R}$ otvoren interval, $0 \in I$, $f \in C^{n+1}(I)$. Tada $\forall x \in I$, $\exists c_x \in \langle -|x|, |x| \rangle$ t.d.

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k}_{T_n(x)} + \underbrace{\frac{f^{(n+1)}(c_x)}{(n+1)!} x^{n+1}}_{R_n(x)}$$

Izračunajte

- $\cos(0.1)$ s greškom manjom od 10^{-3}

$f(x) = \cos(x)$ $\quad f(0) = 1$

$f'(x) = -\sin(x)$ $\quad f'(0) = 0$

$f''(x) = -\cos(x)$ $\quad f''(0) = -1$

$f'''(x) = \sin(x)$ $\quad f'''(0) = 0$

$f^{(4)}(x) = \cos(x)$ $\quad f^{(4)}(0) = 1$

$\Rightarrow |f^{(n+1)}(x)| \leq 1, \ \forall n \in \mathbb{N}, \ \forall x \in \mathbb{R}$

$\Rightarrow |R_n(x)| \leq \frac{1}{(n+1)!} x^{n+1}$

$|R_n(0.1)| \leq \frac{1}{(n+1)!} 0.1^{n+1}$

$|R_2(0.1)| \leq \frac{1}{3!} 0.1^3 = \frac{1}{6} \cdot 10^{-3}$

$T_2(0.1) = 1 - \frac{1}{2} 0.1^2$

$\Rightarrow \cos(0.1) \approx 1 - \frac{0.1^2}{2}$

- $\ln(1.2)$ s greškom manjom od 10^{-4}

$f(x) = \ln(1+x)$

$$f'(x) = \frac{1}{1+x} = (1+x)^{-1} \qquad f'(0) = 1$$

$$f''(x) = -(1+x)^{-2} \qquad f''(0) = -1$$

$$f'''(x) = 2(1+x)^{-3} \qquad f'''(0) = 2$$

$$f^{(4)}(x) = -6(1+x)^{-4} \qquad f^{(4)}(0) = -6$$

$$\ldots$$

$$f^{(n)}(x) = (-1)^{n+1}(n-1)!\,(1+x)^{-n}$$

$$\Rightarrow |R_n(x)| = \frac{n!\,(1+c_x)^{-n-1}}{(n+1)!}\, x^{n+1}$$

$$|R_n(0.2)| = \frac{1}{n+1}\underbrace{(1+c_{0.2})^{-n-1}}_{\leq 1} \cdot 0.2^{n+1}$$

$$\leq \frac{1}{n+1} \cdot \frac{1}{5^{n+1}}$$

$$\Rightarrow R_4(0.2) \leq \frac{1}{5} \cdot \frac{1}{5^5} = \frac{1}{5^6} = \frac{1}{25} \cdot \frac{1}{5^4} < \frac{1}{16} \cdot \frac{1}{5^4} = \frac{1}{2^4} \cdot \frac{1}{5^4} = \frac{1}{10^4}$$

$$T_4(0.2) = 0 + \frac{1}{1}\, 0.2 - \frac{1}{2}\, 0.2^2 + \frac{2}{3!}\, 0.2^3 - \frac{6}{4!}\, 0.2^4$$

$$\Rightarrow \ln(1.2) \approx 0.2 - \frac{1}{2}\, 0.2^2 + \frac{1}{3}\, 0.2^3 - \frac{1}{4}\, 0.2^4$$

Razvijte funkciju u Fourierov red na $[-\pi, \pi]$

- $f(x) = x$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx = \Big\{ \cos(nx) \text{ je parna funkcija}$$

za svaki $n \geq 0$, a x neparna, pa je $x\cos(nx)$ neparna,

pa je njen integral po simetričnoj domeni $= 0 \Big\} = 0.$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = \left\{ \begin{matrix} u = x & du = dx \\ dv = \sin(nx)\,dx & v = -\frac{\cos(nx)}{n} \end{matrix} \right\} =$$

$$\frac{1}{\pi}\left(-\frac{x\cos(nx)}{n}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos(nx)}{n}\,dx \right) =$$

$$\frac{1}{\pi}\left(-\frac{1}{n}\cdot 2\pi\cos(n\pi) + \frac{\sin(nx)}{n^2}\Big|_{-\pi}^{\pi} \right) =$$

$$-\frac{2}{n}(-1)^n = (-1)^{n+1}\cdot\frac{2}{n}$$

$$\Rightarrow S(x) = \sum_{n=1}^{\infty} (-1)^{n+1}\cdot\frac{2}{n}\sin(nx)$$

- $f(x) = \begin{cases} 1, & -\pi \leq x < 0 \\ 0, & 0 \leq x \leq \pi \end{cases}$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} 1\,dx + \frac{1}{\pi}\int_{0}^{\pi} 0\,dx = \frac{1}{\pi}\cdot x\Big|_{-\pi}^{0} + 0$$

$$= \frac{1}{\pi}\left(0 - (-\pi)\right) = \frac{1}{\pi}\cdot\pi = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} \cos nx = \frac{1}{\pi}\,\frac{\sin nx}{n}\Big|_{-\pi}^{0} = \frac{1}{n\pi}\left(\sin 0 - \sin n\pi\right)$$

$$= \frac{1}{n\pi}(0-0) = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} \sin nx = \frac{1}{\pi}\,\frac{-\cos nx}{n}\Big|_{-\pi}^{0} = \frac{1}{n\pi}\left(-\cos 0 + \cos n\pi\right)$$

$$= \frac{1}{n\pi}\left(-1 + (-1)^n\right) = \begin{cases} \frac{-2}{n\pi}, & n \text{ neparan} \\ 0, & n \text{ paran} \end{cases}$$

$$\Rightarrow S(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-2}{(2n-1)\pi} \sin\left((2n-1)x\right)$$

- $f(x) = \sin x$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(x)\cos(nx)\,dx = \Big\{ \sin(x) \text{ je neparan, a}$$

$\cos(nx)$ paran, za svaki $n \geq 0$, pa je njihov umnožak neparan i njegov integral po simetričnoj domeni $= 0 \Big\} = 0$.

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin x \cdot \sin x = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1-\cos 2x}{2} =$$

$$\frac{1}{2\pi}\left(x\Big|_{-\pi}^{\pi} - \frac{\sin 2x}{2}\Big|_{-\pi}^{\pi}\right) = \frac{1}{2\pi}(2\pi - 0) = 1$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin x \sin(nx) = \left\{ \sin a \sin b = \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right)\right\}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}\left(\cos((n-1)x) - \cos((n+1)x)\right) dx = \left\{ \begin{matrix} \text{za} \\ n > 1 \end{matrix} \right\}$$

$$\frac{1}{2\pi}\left(\frac{\sin((n-1)x)}{n-1}\Bigg|_{-\pi}^{\pi} - \frac{\sin((n+1)x)}{n+1}\Bigg|_{-\pi}^{\pi} \right) =$$

$$\frac{1}{2\pi}(0-0) = 0$$

$$\Rightarrow S(x) = \sin(x)$$

- $f(x) = e^x$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{\pi} e^x \Big|_{-\pi}^{\pi} = \frac{1}{\pi}\left(e^{\pi} - e^{-\pi}\right)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos(nx)\,dx = \left\{ \begin{array}{ll} u = \cos(nx) & du = -n\sin(nx)\,dx \\ dv = e^x\,dx & v = e^x \end{array} \right\} =$$

$$= \frac{1}{\pi}\left(e^x\cos(nx)\Big|_{-\pi}^{\pi} + n\int_{-\pi}^{\pi} e^x \sin(nx)\,dx \right) =$$

$$= \frac{1}{\pi}\left((e^{\pi} - e^{-\pi})(-1)^n + \quad -\!\!\parallel\!\!- \right) = \left\{ \begin{array}{ll} u = \sin(nx) & du = n\cos(nx)\,dx \\ dv = e^x\,dx & v = e^x \end{array} \right\}$$

$$= \frac{1}{\pi}\left((e^{\pi} - e^{-\pi})(-1)^n + n\left(e^x\sin(nx)\Big|_{-\pi}^{\pi} - n\underbrace{\int_{-\pi}^{\pi} e^x\cos(nx)\,dx}_{= a_n \cdot \pi} \right)\right) =$$

$$= \frac{1}{\pi}\left((e^{\pi} - e^{-\pi})(-1)^n - n^2 a_n \pi \right)$$

$$\Rightarrow \pi a_n = (e^{\pi} - e^{-\pi})(-1)^n - n^2 a_n \pi$$

$$\Rightarrow a_n = \frac{(e^{\pi} - e^{-\pi})(-1)^n}{\pi + n^2\pi}$$

$$\Rightarrow a_n = \frac{1}{\pi}(e^{\pi} - e^{-\pi})\frac{(-1)^n}{1+n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin(nx)\,dx = \left\{ \begin{array}{ll} u=\sin(nx) & du = n\cos(nx)\,dx \\ dv = e^x\,dx & v = e^x \end{array}\right\} =$$

$$= \frac{1}{\pi}\left(e^x \sin(nx)\Big|_{-\pi}^{\pi} - n\int_{-\pi}^{\pi} e^x\cos(nx)\,dx\right) =$$

$$= \frac{1}{\pi}\left(0 - n\, a_n\, \pi\right) = -n\, a_n =$$

$$= \frac{1}{\pi}\left(e^{\pi} - e^{-\pi}\right)(-1)^{n+1}\frac{n}{1+n^2}$$

$$\Rightarrow S(x) = \frac{1}{\pi}\left(e^{\pi}-e^{-\pi}\right)\left(\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{1+n^2}\cos(nx) + \frac{(-1)^{n+1}\, n}{1+n^2}\sin(nx)\right)\right)$$