Goran Radunović
Faculty of Science
Department of Mathematics
University of Zagreb, Croatia

Research Statement

The theory fractal strings and their complex dimensions has been developed by Professor M. L. Lapidus and his collaborators in the last two decades. Fractal strings can be viewed as objects that are generated by certain fractal sets on the real line. More precisely, for $A \subset \mathbb{R}$ of zero Lebesgue measure, a fractal string is defined as the sequence of the finite lengths of the complementary intervals of the set A; see [Lap–vFr1–3] and the relevant references therein. My research is focused on generalizing this vast theory and its consequences to the higher-dimensional case. This research started as a collaboration between my two supervisors, Professor Michel L. Lapidus and Professor Darko Žubrinić in 2009, with the later addition of myself. The collaboration resulted in an extensive research monograph [LapRaŽu1] which is a significant extension of the theory of fractal zeta functions for fractal strings, as well as two survey articles and a number of research articles in various stages of preparation; see [LapRaŽu2–9].

When referring to a fractal set, we actually mean any nonempty bounded subset A of the N-dimensional Euclidean space $\mathbb{R}^N$, with $N \geq 1$. The attribute 'fractal' actually means that the basic tool, when studying the set A, is the notion of fractal dimension. As it turns out, the one[1] that best suits this theory is the (upper) box dimension (also called the Minkowski dimension, Bouligand dimension, limit capacity, etc.). Furthermore, the value of the Minkowski content of a bounded subset A of $\mathbb{R}^N$ can be used as one of the equivalent ways to define the box dimension. More precisely, for a bounded subset A of $\mathbb{R}^N$ and $0 \leq r \leq N$ we denote its r-dimensional Minkowski content by

$$\mathcal{M}^r(A) := \lim_{\delta \to 0^+} \frac{|A_\delta|}{\delta^{N-r}}, \tag{1}$$

whenever this limit exists as a value in $[0, \infty]$. Here, $|\cdot|$ denotes the N-dimensional Lebesgue measure in $\mathbb{R}^N$ and

$$A_\delta := \{x \in \mathbb{R}^N : d(x, A) < \delta\} \tag{2}$$

is the δ-*neighborhood* (or the δ-*parallel set*) of A with $d(x, A) := \inf\{|x - a| : a \in A\}$ being the Euclidean distance from x to A. The set A is said to be Minkowski measurable (of dimension r) if $\mathcal{M}^r(A)$ exists and satisfies $0 < \mathcal{M}^r(A) < \infty$.

It has been of considerable interest in the past to determine whether a set A is Minkowski measurable. One of the motivations is Mandelbrot's suggestion in [Man2] to use the Minkowski content as a characteristic for the texture of sets (see [Man1, §X]). Mandelbrot called the quantity $1/\mathcal{M}^r(A)$ the *lacunarity* of the set A and made an observation that for subsets of $\mathbb{R}^N$ small lacunarity corresponds to spatial homogeneity of the

[1] There are several different notions of fractal dimension, e.g., Hausdorff dimension, packing dimension, etc. (see [Fal1]).

set, i.e., the set has small, uniformly distributed holes. On the other hand, large lacunarity corresponds to clustering of the set and large holes between different clusters. More on this can be found in [BedFi, Fr, Lap–vFr1] and in [Lap–vFr3, §12.1.3].

Particular attention to the notion of Minkowski content arose in connection to the (modified) Weyl–Berry conjecture[2] (see the formulation in [Lap1]) which was proved for subsets of $\mathbb{R}$ in 1993 by M. L. Lapidus and C. Pomerance [LapPo1]. This conjecture relates the spectral asymptotics of the Laplacian on a bounded open set and the Minkowski content of its boundary. A crucial part of this result was the characterization of Minkowski measurability of bounded subsets of $\mathbb{R}$ obtained in [LapPo1].[3] In particular, this led to an important reformulation of the Riemann hypothesis in terms of an inverse spectral problem for fractal strings; see [LapMa].

As already mentioned, the theory of complex dimensions has been generalized to higher-dimensions in the research monograph [LapRaŽu1] and in [LapRaŽu2–9]; that is, to the case of arbitrary compact subsets in Euclidean spaces of any dimension. The fractal zeta function on which this generalization is based was introduced in 2009 by M. L. Lapidus and its definition was inspired by a work of D. Žubrinić on singular sets of some spaces of functions (see [Žu1–4]). For another higher-dimensional generalization of the theory of complex dimensions see [LapRoŽu], where the notion of *box-counting zeta function* is introduced.

More specifically, for a given bounded subset $A \subseteq \mathbb{R}^N$ we define its *distance zeta function* as

$$\zeta_A(s;\delta) := \int_{A_\delta} d(x,A)^{s-N}\,\mathrm{d}x, \tag{3}$$

for $\operatorname{Re} s$ sufficiently large and a fixed $\delta > 0$. It turns out that the dependence of ζ_A on δ is inessential since in the theory of complex dimensions we are, generally, interested in the poles of a meromorphic extension of $\zeta_A(s;\delta)$ and changing the parameter δ does not affect the poles or the principal part of a (possible) meromorphic extension of the distance zeta function in any way.

One generalization of the notion of Minkowski content and box dimension can be made for objects that we call *relative fractal drums*. A relative fractal drum is an ordered pair of subsets (A,Ω) of $\mathbb{R}^N$ such that Ω is Lebesgue measurable and of finite N-dimensional Lebesgue measure. Furthermore, for a relative fractal drum (A,Ω) in $\mathbb{R}^N$ and $r \in \mathbb{R}$ we denote its r-dimensional relative Minkowski content by

$$\mathcal{M}^r(A,\Omega) = \lim_{\delta\to 0^+} \frac{|A_\delta \cap \Omega|}{\delta^{N-r}}, \tag{4}$$

whenever this limit exists as a value in $[0,\infty]$. Now, by using this notion, we can define the *relative box dimension of* (A,Ω) in a standard way, with values in $[-\infty, N]$. The novelty here is that we now let $r \in \mathbb{R}$, which is not a coincidence, since there exist relative fractal drums with negative box dimension as was demonstrated in [LapRaŽu1]. Furthermore, for

[2]For the original Weyl–Berry conjecture and its physical applications see Berry's papers [Berr1–2]. Furthermore, early mathematical work on this conjecture and its applications can be found in [BroCar, FlVa, Lap1, Lap2, LapPo1, LapPo2]. For a more extensive list of later work see [Lap–vFr3, §12.5].

[3]A new proof of this is given in [Fal2] and more recently in [RatWi2].

a relative fractal drum (A, Ω) in $\mathbb{R}^N$, one defines its relative distance zeta function as

$$\zeta_{A,\Omega}(s) := \int_\Omega d(x, A)^{s-N}\,\mathrm{d}x, \tag{5}$$

for $\operatorname{Re} s$ sufficiently large. This, in turn, allows one to develop a theory of complex dimensions of relative fractal drums in a much the same manner as it is done for bounded subsets in $\mathbb{R}^N$.

The notion of RFDs gives us a unified category under which fractal strings, bounded subsets of Euclidean spaces (of arbitrary dimension) and open subsets of Euclidean spaces with fractal boundary (also known as fractal drums) fall into. By developing the theory of complex dimensions in this generality we can apply it to all of these settings without the need to distinguish them separately and this is one of the powers and flexibilities of the notion of RFDs.

Some results obtained

Fractal tube formulas for relative fractal drums

These results (announced in [LapRaŽu7] and fully exposed in [LapRaŽu8–9], see also [Ra1]) concern the problem of obtaining *fractal tube formulas*, for a class of relative fractal drums in terms of sums over the residues of their relative distance or tube zeta functions. By a fractal tube formula of the relative fractal drum (A, Ω) we mean an exact or asymptotic expansion of the relative tube function $t \mapsto |A_t \cap \Omega|$ when $t \to 0^+$. These formulas hold pointwise or distributionally, depending on the growth properties of the corresponding relative zeta function. Furthermore, these results extend the corresponding ones obtained in [Lap–vFr1–3] for fractal strings to sets and RFD in higher-dimensional Euclidean spaces. Roughly speaking, under suitable growth conditions imposed on the relative distance zeta function of an relative fractal drum (A, Ω) in $\mathbb{R}^N$, we have that the following asymptotic formula for its relative tube function holds:

$$|A_t \cap \Omega| = \sum_{\omega \in \mathcal{P}(\zeta_{A,\Omega}, W)} \operatorname{res}\left(\frac{t^{N-s}}{N-s}\zeta_{A,\Omega}(s), \omega\right) + R^{[0]}(t). \tag{6}$$

Here, $\mathcal{P}(\zeta_{A,\Omega}, W)$ denotes the set of *visible complex dimensions* of (A, Ω) defined as the poles of (a meromorphic continuation) of $\zeta_{A,\Omega}$ contained in the domain W and $R^{[0]}(t)$ is the error term which is of strictly higher asymptotic order as $t \to 0^+$ than the sum above. As an application of these fractal tube formulas and a Tauberian theorem due to Wiener and Pitt we obtain a Minkowski measurability criterion for a large class of relative fractal drums (and, a posteriori, compact subsets) of $\mathbb{R}^N$. We refer to [Lap–vFr3, §13.1] for many additional references on tube formulas in various settings, including, [Gra, HuLaWe, Schn, Zäh, LapPe, LapPeWi1, LapPeWi2].

Fractal analysis of unbounded sets at infinity

In [Ra1, Ra2] we extend the theory of complex dimensions (with appropriate definitions) to the case of unbounded subsets of $\mathbb{R}^N$. There are two different (but, in a way, related) approaches to this problem. One of them was explored in the paper [RaŽuŽup], where for an unbounded subset $A \subseteq \mathbb{R}^N$ the fractal properties of its image $\Phi(A)$ under the geometric inversion $\Phi(x) := x/|x|^2$ in $\mathbb{R}^N$ were applied in investigating bifurcations of some polynomial vector fields in $\mathbb{R}^2$. There it was also shown that this approach is equivalent to studying the fractal properties of the image of $\Psi(A)$ on the Riemann sphere $\mathbb{S}^2$ under the stereographic projection $\Psi\colon \mathbb{R}^2 \to \mathbb{S}^2$.

The second approach deals with the notion of a "fractal set at infinity". More precisely, D. Žubrinić suggested to try to analyze an unbounded Lebesgue measurable set Ω of finite N-dimensional Lebesgue measure by means of its *tube function at infinity* which is defined as

$$t \mapsto |B_t(0)^c \cap \Omega|, \tag{7}$$

where $B_t(0)^c$ denotes the complement of the open ball in $\mathbb{R}^N$ of radius t with center at the origin. Also, a suggestion by D. Žubrinić was to define a *Lapidus* (*distance*) *zeta function of* Ω *at infinity* by replacing the integrand in (5) by some suitably chosen power of $|x|$. As it turned out, the right way to define the Lapidus zeta function of Ω at infinity was

$$\zeta_{\infty,\Omega}(s) := \int_\Omega |x|^{-s-N}\,\mathrm{d}x, \tag{8}$$

for $\mathrm{Re}\, s$ sufficiently large. We show that his definition is perfectly in accordance with the (also new) notion of the *r-dimensional Minkowski content of* Ω *at infinity* defined by

$$\mathcal{M}^r(\infty,\Omega) := \lim_{t\to+\infty} \frac{|B_t(0)^c \cap \Omega|}{t^{N+r}} \tag{9}$$

for $r \in \mathbb{R}$ whenever it exists and also with the notion of *Minkowski dimension of* Ω *at infinity* which it induces. Using these definitions, we extend the theory of [LapRaŽu1–9] to the case of unbounded Lebesgue measurable subsets Ω of $\mathbb{R}^N$.

The notation of (9) suggests that a fractal set Ω at infinity may be understood as a special case of a relative fractal drum (A, Ω) where the set A has degenerated to a point at infinity. This is indeed the case and the fractal properties of this relative fractal drum will be closely related to the fractal properties of its 'inverted relative fractal drum'; that is, of $(\{\mathbf{0}\}, \Phi(\Omega))$. In light of this, the box dimensions of unbounded sets at infinity will always be nonpositive[4] or, more precisely, less than or equal to $-N$.

We present examples of relative fractal drums of type (∞, Ω) that provide some interesting insights into the notion of 'fractality' or rather, 'relative fractality'.[5] Namely, although the 'fractal set' A has degenerated to a point at infinity (and thus, one would not expect it to be fractal in any way; that is, to have nontrivial fractal properties), we show that the set Ω will be the source of fractality in this case. More precisely, we demonstrate this by constructing quasiperiodic sets at infinity and even a set Ω that is *maximally hyperfractal at infinity*.[6] The idea of this construction can also be applied to the case of

[4] The box dimension of a relative fractal drum of type $(\{\mathbf{0}\}, \Omega)$ is at most equal to 0 since the set A here consists of a single point.

[5] For a discussion of the notion of fractality see [Lap–vFr3] and the relevant references therein.

[6] The notion of a (maximally) hyperfractal set was introduced in [LapRaŽu1] in terms of the corresponding fractal zeta function associated to that set. In a way, such sets exhibit the most complicated geometrical nature.

ordinary relative fractal drums of form $(\{\mathbf{0}\},\Omega)$ in order to demonstrate the existence of such a complicated objects even though the set A consists only of a single point and thus, again, the source of fractality of $(\{\mathbf{0}\},\Omega)$ in this case is the set Ω.

By looking at Equation (8); that is, at the definition of the distance zeta function at infinity, one can see that for it to make sense, it is not necessary for the set Ω to have finite Lebesgue measure, but rather, to be just Lebesgue measurable. This turned out to be nicely related to a new notion of a parametric Minkowski content introduced in [Ra1] called the *r-dimensional ϕ-shell Minkowski content at infinity*, where, $\phi > 1$ is a parameter and $r \in \mathbb{R}$. For a Lebesgue measurable subset $\Omega \in \mathbb{R}^N$ we define it as

$$\mathcal{M}_\phi^r(\infty,\Omega) := \lim_{t\to+\infty} \frac{|B_t(0)^c \cap B_{\phi t}(0) \cap \Omega|}{t^{N+r}} \tag{10}$$

whenever this limit exists. The idea to introduce this notion originally came into existence as a side effect of studying the connection between fractal properties of the relative fractal drum (∞,Ω) and the fractal properties of its image on the N-dimensional Riemann sphere under the stereographic projection Ψ. Furthermore, the notion of *ϕ-shell box dimension of Ω at infinity*, which the ϕ-shell Minkowski content at infinity induces, generalizes the already introduced definition of box dimension at infinity for sets of finite Lebesgue measure. Moreover, the sets of infinite Lebesgue measure will have their ϕ-shell box dimension (if it exists) always in the interval $[-N,0]$ which fills out the 'dimensional gap' left over by the sets of finite Lebesgue measure.[7]

We point out that one can also define an analog of the ϕ-shell Minkowski content for relative fractal drums and study its properties. In particular, it would be interesting to fully relate this notion to the notion of *surface Minkowski content* studied in [RatWi1] and [RatWi2]. Some preliminary results about this problem (in the case of fractal sets at infinity) are also obtained in [Ra1].

The motivation to study the fractal properties of unbounded sets comes from a variety of sources. In particular, the notion of "unbounded" or "divergent" oscillations appears in problems in oscillation theory (see, e.g. [Džu, Karp]), automotive industry (see, e.g., [SBOPQD]), civil engineering (see, e.g, [Pou]) and mathematical applications in biology (see, e.g., [May]). Unbounded (divergent) oscillations are oscillations the amplitude of which increases with time. For instance, the oscillations of an airplane that has positive static stability but negative dynamic stability is an example of divergent oscillations that appears in aerodynamics (see, e.g. [Dol]).

Furthermore, unbounded domains themselves are also interesting in the theory of elliptic partial differential equations. More precisely, the question of solvability of the Dirichlet problem for quasilinear equations in unbounded domains is addressed in [Maz1] and [Maz2, Section 15.8.1]. Also, unbounded domains can be found in other aspects of the theory of partial differential equations; see, for instance [An, Hur, Lan, Rab] and [VoGoLat]. Research dealing with unbounded domains of infinite volume can be found in [GeWe], and connected with that is the research dealing with cusp-shaped domains (see, e.g., [ExBa1–2]), which also appear in examples in [Ra1]. Furthermore, the new notion of the ϕ-shell Minkowski content could possibly have a connection to certain comparison principles for

[7] Recall that the sets of finite Lebesgue measure always have (if it exists) their box dimension at infinity less than or equal to $-N$.

the p-Laplacian (see, e.g., [Ag, MarMizPin, PolSha] and the relevant references therein). Fractal properties of unbounded domains, studied here, could therefore have a future impact and lead to a new approach to these problems.

Current and future research problems

My current research is a continuation of work on the theory of complex dimensions, fractal zeta functions, fractal geometry and fractal analysis of differential equations and dynamical systems. The theory of fractal strings, their geometric zeta functions and the complex dimensions which these zeta functions generate has been a topic of extensive research in the last few decades and has, in its own, a wide variety of applications. The foundations for generalizing this theory to higher dimensions are laid in the research monograph [LapRaŽu1] which is completed and due to appear in 2016.

The fractal tube formulas obtained in [Ra1] and [LapRaŽu1,8–9] and the Minkowski measurability criterion given as their application are important results that generalize the corresponding ones for the case of one-dimensional fractal strings obtained in [Lap–vFr1–3] and give a justification of the notion of *complex dimensions* as a new tool to measure fractal properties of subsets of Euclidean spaces and, more generally, of relative fractal drums. Although it seems that a fairly large class of sets satisfy the conditions which enables one to derive their fractal tube formulas as a sum of residues over their complex dimensions and, hence, the theory from [Ra1] may be applied to them, it remains to investigate this in detail and obtain some general results.

Furthermore, the results from [Ra1] about embedded relative fractal drums give a strategy of computing complex dimensions of a class of higher-dimensional fractal sets by decomposing them into their lower-dimensional 'relative fractal subdrums' like it was shown in the example of the Cantor dust. Since it is not, in general, easy to compute the distance zeta function of a given relative fractal drum and, hence, its complex dimensions, we propose further investigation into other types of embeddings of relative fractal drums in higher dimensions and their fractal zeta functions. For instance, one could consider relative fractal drums $(\partial\Omega, \Omega)$ where the boundary $\partial\Omega$ is a subset of a piecewise smooth curve but also has lower-dimensional fractal properties. This situation appears, for instance in the well-known fractal sets such as the von Koch snowflake and the Menger sponge. Furthermore, also concerning the computation of fractal zeta functions, it would be of interest to obtain zero-free regions for these zeta functions as well as general results about stability of complex dimensions under perturbations of the integrand appearing in the definition of the distance zeta function.

Furthermore, a new direction of research has been started recently. Namely, we have found examples of fractal sets that generate nontrivial Riemann surfaces via their fractal zeta functions. This phenomena is related to the non-power like asymptotics of their tube functions. We plan to further investigate this phenomena and connect it to the notion of *gauge functions* and *gauge Minkowski measurability* of RFDs.

The generalization of the idea of complex dimensions and fractal zeta functions to the case of unbounded sets at infinity gives means of applying fractal analysis to unbounded regions of finite or infinite Lebesgue measure in Euclidean spaces and the examples provided demonstrate that such regions can have a very complex fractal structure at infinity exhibiting quasiperiodicity and even maximal hyperfractality. Since unbounded regions are of interest in the theory of partial differential equations we propose to study if our approach of fractal analysis can be applied to any problems in this area of research.

Our effort to apply fractal analysis to unbounded sets of infinite Lebesgue measure led to the introduction of new notions, at least to our knowledge, of parametric ϕ-shell Minkowski content and the corresponding parametric ϕ-shell Minkowski (or box) dimension. Although introduced in the context of unbounded sets at infinity these notions are also well defined for bounded subsets and relative fractal drums. Preliminary results obtained show that these new notions are connected with notions of the (usual) Minkowski content and the *surface Minkowski content* studied by Rataj and Winter in [RatWi1–2]. We suggest to study this connection in detail in a future work and obtain general results as well as to investigate possible applications.

The paper [RaŽuŽup] demonstrates how fractal analysis of unbounded sets may be applied to investigate dynamical systems and their bifurcations at infinity. Fractal analysis of dynamical systems and differential equations has been an ongoing investigation for the past decade by our research group resulting in the publication of a number of articles (see, e.g., [Hor1–2, MaResŽup, PaŽuŽup1–2, Res1–2, Vl, ŽuŽup1–3, ŽupŽu]). We propose to continue this research by applying the new theory developed in this thesis and in the research monograph [LapRaŽu1].

References

[Ag] S. Agmon, Bounds on exponential decay of eigenfunctions, in *Schrödinger Operators*, ed. S. Graffi, Lecture Notes in Math., Vol. 1159, Springer-Verlag, Berlin, 1985, pp. 1–38.

[An] R. Andersson, Unbounded Soboleff regions, *Math. Scand.* **13** (1963), 75–89.

[BedFi] T. Bedford and A. M. Fisher, Analogues of the Lebesgue density theorem for fractal sets of reals and integers, *Proc. London Math. Soc.* (3) **64** (1992), 95–124.

[Berr1] M. V. Berry, Distribution of modes in fractal resonators, in: *Structural Stability in Physics* (W. Güttinger and H. Eikemeier, eds.), Springer-Verlag, Berlin, 1979, pp. 51–53.

[Berr2] M. V. Berry, Some geometric aspects of wave motion: Wavefront dislocations, diffraction catastrophes, diffractals, in: *Geometry of the Laplace Operator*, Proc. Sympos. Pure Math., vol. **36**, Amer. Math. Soc., Providence, R. I., 1980, pp. 13–38.

[BroCar] J. Brossard and R. Carmona, Can one hear the dimension of a fractal?, *Commun. Math. Phys.* **104** (1986), 103–122.

[Dol] C. E. Dole, *Flight Theory and Aerodynamics*, John Wiley & Sons, Inc, New York, 1981.

[Džu] J. Džurina, Unbounded oscillation of the second-order neutral differential equations, *Mathematica Slovaca* No. 4, **51** (2001), 441–447.

[ExBa1] P. Exner and D. Barseghyan, Spectral analysis of Schrödinger opertaors with unusual semiclassical behavior, *Acta Polytechnica* **53** (2013), 271–279.

[ExBa2] P. Exner and D. Barseghyan, Spectral estimates for Dirichlet Laplacians and Schrödinger operators on geometrically nontrivial cusps, *Journal of Spectral Theory* **3** (2013), 465–484.

[Fal1] K. J. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley and Sons, Chichester, 1990. (Second edition, 2003.)

[Fal2] K. J. Falconer, On the Minkowski measurability of fractals, *Proc. Amer. Math. Soc.* **123** (1995), 1115–1124.

[FlVa] J. Fleckinger and D. Vassiliev, An example of a two-term asymptotics for the "counting function" of a fractal drum, *Trans. Amer. Math. Soc.* **337** (1993), 99–116.

[Fr] M. Frantz, Lacunarity, Minkowski content, and self-similar sets in $\mathbb{R}$, in: *Fractal Geometry and Applications: A Jubilee of Benoit Mandelbrot* (M. L. Lapidus and M. van Frankenhuijsen, eds.), Proc. Sympos. Pure Math. vol. **72**, Part 1, Amer. Math. Soc., Providence, R. I., 2004, pp. 77–91.

[GeWe] L. Geisinger and T. Weidl, Sharp spectral estimates in domain of infinite volume, *Rev. Math. Phys.* **23** (2011), 615–641.

[Gra] A. Gray, *Tubes*, 2nd edn., Progress in Math., vol. 221, Birkhäuser, Boston, 2004.

[Hor1] L. Horvat Dmitrović, Box dimension and bifurcations of one-dimensional discrete dynamical systems, *Discrete and continuous dynamical systems* **32** (2012), 1287–1307.

[Hor2] L. Horvat Dmitrović, Box dimension of Neimark-Sacker bifurcation, *Journal of difference equations and applications* **20** (2014), 1033–1054.

[HuLaWe] D. Hug, G. Last and W. Weil, A local Steiner-type formula for general closed sets and applications, *Math. Z.* **246** (2004), 237–272.

[Hur] R. Hurri, The weighted Poincaré inequalities, *Ann. Acad. Sci. Fenn. Ser. A. I. Math.* **17** (1992), 409–423.

[Karp] B. Karpuz, Unbounded oscillation of higher-order nonlinear delay dynamic equations of neutral type with oscillating coefficients, *Electronic Journal of Qualitative Theory of Differential Equations* **34** (2009), 1–14.

[Lan] E. M. Landis, On the behaviour of solutions of higher-order elliptic equations in unbounded domains, *Tr. Mosk. Mat. Obŝ.* **31** (1974), 35–58 (in Russian). English translation: *Trans. Mosc. Math. Soc.* **31** (1974), 30–54.

[Lap1] M. L. Lapidus, Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the Weyl–Berry conjecture, *Trans. Amer. Math. Soc.* **325** (1991), 465–529.

[Lap2] M. L. Lapidus, Vibrations of fractal drums, the Riemann hypothesis, waves in fractal media, and the Weyl–Berry conjecture, in: *Ordinary and Partial Differential Equations* (B. D. Sleeman and R. J. Jarvis, eds.), vol. IV, Proc. Twelfth Internat. Conf. (Dundee, Scotland, UK, June 1992), Pitman Research Notes in Math. Series, vol. **289**, Longman Scientific and Technical, London, 1993, pp. 126–209.

[LapMa] M. L. Lapidus and H. Maier, The Riemann hypothesis and inverse spectral problems for fractal strings, *J. London Math. Soc.* (2) **52** (1995), 15–34.

[LapPe] M. L. Lapidus and E. P. J. Pearse, Tube formulas and complex dimensions of self-similar tilings, *Acta Applicandae Mathematicae* No. 1, **112** (2010), 91–137. (Springer Open Access: DOI 10.1007/S10440-010-9562-x.) (Also: e-print, `arXiv:math.DS/0605527v5`, 2010; IHES preprint, IHES/M/08/27, 2008.)

[LapPeWi1] M. L. Lapidus, E. P. J. Pearse and S. Winter, Pointwise tube formulas for fractal sprays and self-similar tilings with arbitrary generators, *Adv. in Math.* **227** (2011), 1349–1398. (Also: e-print, `arXiv:1006.3807v3` [math.MG], 2011.)

[LapPeWi2] M. L. Lapidus, E. P. J. Pearse and S. Winter, Minkowski measurability results for self-similar tilings and fractals with monophase generators, in: *Fractal Geometry and Dynamical Systems in Pure and Applied Mathematics I*: *Fractals in Pure Mathematics* (D. Carfi, M. L. Lapidus, E. P. J. Pearse and M. van Frankenhuijsen, eds.), Contemporary Mathematics, vol. **600**, Amer. Math. Soc., Providence, R. I., 2013, pp. 185–203. (Also: e-print, `arXiv:1104.1641v3 [math.MG]`, 2012; IHES preprint, IHES/M/12/33, 2012.)

[LapPo1] M. L. Lapidus and C. Pomerance, The Riemann zeta-function and the one-dimensional Weyl–Berry conjecture for fractal drums, *Proc. London Math. Soc.* (3) **66** (1993), No. 1, 41–69.

[LapPo2] M. L. Lapidus and C. Pomerance, Counterexamples to the modified Weyl-Berry conjecture on fractal drums, *Math. Proc. Cambridge Philos. Soc.* **119** (1996), 167–178.

[LapRaŽu1] M. L. Lapidus, G. Radunović and D. Žubrinić, *Fractal Zeta Functions and Fractal Drums*: *Higher-Dimensional Theory of Complex Dimensions*, Springer Monographs in Mathematics, Springer, New York, 2016, in press, approx. 680 pages.

[LapRaŽu2] M. L. Lapidus, G. Radunović and D. Žubrinić, Distance and tube zeta functions of fractals and arbitrary compact sets, *Advances in Mathematics* **307** (2017), 1215–1267. (Also: e-print, `arXiv:1506.03525v3 [math-ph]`, 2016; IHES preprint, IHES/M/15/15, 2015.)

[LapRaŽu3] M. L. Lapidus, G. Radunović and D. Žubrinić, Complex dimensions of fractals and meromorphic extensions of fractal zeta functions, in preparation, 2015.

[LapRaŽu4] M. L. Lapidus, G. Radunović and D. Žubrinić, Zeta functions and complex dimensions of relative fractal drums: Theory, examples and applications, in preparation, 2015.

[LapRaŽu5] M. L. Lapidus, G. Radunović and D. Žubrinić, Fractal zeta functions and complex dimensions of relative fractal drums, appeared in the Festschrift volume of the *Journal of Fixed Point Theory and Applications* **15** (2014), 321–378 in honor of Haim Brezis' 70th birthday (DOI 10.1007/s11784-014-0207-y) (Also: e-print, `arXiv:1407.8094v3 [math-ph]`, 2014, 60 pp.)

[LapRaŽu6] M. L. Lapidus, G. Radunović and D. Žubrinić, Fractal zeta functions and complex dimensions: A general higher-dimensional theory, survey article, 2015. (For inclusion in the *Proceedings of the International Conference "Fractal Geometry and Stochastics V,"* Tabarz, Germany, March 2014, *Progress in Probability*, Birkhäuser, Basel, Boston and Berlin, 2015, C. Bandt, K. Falconer and M. Zähle, eds.; based on a plenary lecture given by the first author at that conference.) (Also: e-print, `arXiv:1502.00878v2 [math.CV]`, 2015, 29 pp.)

[LapRaŽu7] M. L. Lapidus, G. Radunović and D. Žubrinić, Fractal tube formulas and a Minkowski measurability criterion for compact subsets of Euclidean spaces, preprint, 2015. (Also: e-print, `arXiv:1411.5733v2 [math-ph]`, 2014, 9 pp.)

[LapRaŽu8] M. L. Lapidus, G. Radunović and D. Žubrinić, Fractal tube formulas for compact sets and relative fractal drums: Oscillations, complex dimensions and fractality, *Journal of Fractal Geometry*, in press, 2017, 104 pp. (Also: e-print, `arXiv:1604.08014v3 [math-ph]`, 2016.)

[LapRaŽu9] M. L. Lapidus, G. Radunović and D. Žubrinić, Minkowski measurabilty criteria for compact sets and relative fractal drums in Euclidean spaces, preprint, 2016.

[LapRoŽu] M. L. Lapidus, J. A. Rock and D. Žubrinić, Box-counting fractal strings, zeta functions, and equivalent forms of Minkowski dimension, in: *Fractal Geometry and Dynamical Systems in Pure and Applied Mathematics I*: *Fractals in Pure Mathematics* (D. Carfi, M. L. Lapidus, E. P. J. Pearse and M. van Frankenhuijsen, eds.), Contemporary Mathematics, vol. **600**, Amer. Math. Soc., Providence, R. I., 2013, pp. 239–271. (Also: e-print, `arXiv:1207.6681v2 [math-ph]`, 2013; IHES preprint, IHES/M/12/22, 2012.)

[Lap–vFr1] M. L. Lapidus and M. van Frankenhuijsen, *Fractal Geometry and Number Theory*: *Complex Dimensions of Fractal Strings and Zeros of Zeta Functions*, Birkhäuser, Boston, 2000.

[Lap–vFr2] M. L. Lapidus and M. van Frankenhuijsen, *Fractal Geometry, Complex Dimensions and Zeta Functions*: *Geometry and Spectra of Fractal Strings*, Springer Monographs in Mathematics, Springer, New York, 2006.

[Lap–vFr3] M. L. Lapidus and M. van Frankenhuijsen, *Fractality, Complex Dimensions, and Zeta Functions*: *Geometry and Spectra of Fractal Strings*, second revised and enlarged edition (of the 2006 edn.), Springer Monographs in Mathematics, Springer, New York, 2013.

[Man1] B. B. Mandelbrot, *The Fractal Geometry of Nature*, revised and enlarged edition (of the 1977 edn.), W. H. Freeman, New York, 1983.

[Man2] B. B. Mandelbrot, Measures of fractal lacunarity: Minkowski content and alternatives, in: *Fractal Geometry and Stochastics* (C. Bandt, S. Graf and M. Zähle, eds.), Progress in Probability, vol. **37**, Birkhäuser-Verlag, Basel, 1995, pp. 15–42.

[MaResŽup] P. Mardešić, M. Resman and V. Županović, Multiplicity of fixed points and growth of ε-neighborhoods of orbits, *Journal of Differential Equations* **253** (2012), 2493–2514.

[MarMizPin] M. Marcus, V. J. Mizel and Y. Pinchover, On the best constant for Hardy's inequality in $\mathbb{R}^N$, *Trans. A.M.S.* **350** (1998), 3237–3255.

[May] J. Maynard Smith, *Mathmatical Ideas in Biology*, Cambridge University Press, Cambridge 1968.

[Maz1] V. G. Maz'ya, The Dirichlet problem for elliptic equations of arbitrary order in unbounded regions, *Dokl. Akad. Nauk SSSR* **150** (1963), 1221–1224 (in Russian). English translation: *Sov. Math. Dokl.* **4** (1963), 860–863.

[Maz2] V. G. Maz'ya, *Sobolev Spaces: with Applications to Elliptic Partial Differential Equations*, second revised and augmented edition, Springer-Verlag, Berlin, 2011.

[PaŽuŽup1] M. Pašić, D. Žubrinić and V. Županović Oscillatory and phase dimensions of solutions of some second-order differential equations, *Bulletin des Sciences Mathématiques* **133** (2009), 859–874.

[PaŽuŽup2] M. Pašić, D. Žubrinić and V. Županović, Fractal properties of solutions of differential equations, *Classification and Application of Fractals*, ed. L. H. William, (Nova Science Publishers), pp. 1–62., 2011.

[PolSha] A. Poliakovsky and I. Shafrir, A Comparison Principle for the p-Laplacian, A Comparison principle for the p-Laplacian, *Proceedings of the 4th European Conference "Rolduc, Netherlands 18–22 June 2001: Gaeta, Italy 24–28 September 2001*, Rolduc, Netherlands, June 2001, *Elliptic and Parabolic problems: Rolduc and Gaeta 2001*, World Scientific Publishing Company, 2002, pp. 243–252, J. Bemelmans, ed.

[Pou] S. Pourzeynali and T. K. Datta, Control of suspension bridge flutter instability using pole-placement technique, *Journal of Sound and Vibration* **282** (2005), 89–109.

[Rab] V. S. Rabinovič, Pseudodifferential equations in unbounded regions with conical structure at infinity, *Math. USSR SB.* No. 1, **9** (1969), 73–92.

[Ra1] G. Radunović, Fractal analysis of unbounded sets in Euclidean spaces and Lapidus zeta functions, PhD Thesis, University of Zagreb, Croatia, 2015.

[Ra2] G. Radunović, Fractality and Lapidus zeta functions at infinity, *Mathematical Communications*, accepted for publication, 22 pages, 2016.

[RaŽuŽup] G. Radunović, D. Žubrinić and V. Županović, Fractal analysis of Hopf bifurcation at infinity, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **22** (2012), 1230043-1–1230043-15.

[RatWi1] J. Rataj and S. Winter, On volume and surface area of parallel sets, *Indiana Univ. Math. J.* **59** (2010), 1661–1685.

[RatWi2] J. Rataj and S. Winter, Characterization of Minkowski measurability in terms of surface area, *J. Math. Anal. Appl.* **400** (2013), 120–132. (Also: e-print, `arXiv 1111.1825v2 [math.CA]`, 2012.)

[Res1] M. Resman, ε-neighborhoods of orbits and formal classification of parabolic diffeomorphisms, *Discrete and continuous dynamical systems* **33** (2013), 3767–3790.

[Res2] M. Resman, Epsilon-neighborhoods of orbits of parabolic diffeomorphisms and cohomological equations, *Nonlinearity* **27** (2014), 3005–3029.

[SBOPQD] G. Sheng, L. Brown, J. Otremba, J. Pang, M. S. Quatu and R. V. Dukkipati, Chirp, squeal and dynamic instability of misaligned V-ribbed belts in automotive accessory belt drive systems, *Int. J. of Vehicle Noise and Vibration* No. 1, **3** (2007), 88–105.

[Schn] R. Schneider, *Convex Bodies*: *The Brunn–Minkowski Theory*, Cambridge Univ. Press, Cambridge, 2003.

[Vl] D. Vlah, *Phase dimension of oscillatory solutions of a class of ordinary differential equations with applications*, Ph.D. Thesis, University of Zagreb, Croatia, 2013.

[VoGoLat] S. K. Vodop'yanov, V. M. Gol'dshtein and T. G. Latfullin, Criteria for extension of functions of the class L_2^1 from unbounded plane domains, *Sib. Mat. Zh.* **20** (1979), 416–419 (in Russian). English translation: *Sib. Math. J.* **20** (1979), 298–301.

[Zäh] M. Zähle, Curvatures and currents for unions of sets with positive reach, *Geom. Dedicata* **23** (1987), 155–171.

[Žu1] D. Žubrinić, Singular sets of Sobolev functions, *C. R. Acad. Sci. Paris Sér. I. Math., Analyse mathématique* **334** (2002), 539–544.

[Žu2] D. Žubrinić, Minkowski content and singular integrals, *Chaos, Solitons & Fractals* No. 1, **17** (2003), 169–177.

[Žu3] D. Žubrinić, Singular sets of Lebesgue integrable functions, *Chaos, Solitons & Fractals* **21** (2004), 1281–1287.

[Žu4] D. Žubrinić, Hausdorff dimension of singular sets of Sobolev functions and applications, in: *More Progress in Analysis*, Proc. of the 5th Internat. ISAAC Congress (H. G. W. Begher and F. Nicolosi, eds.), World Scientific, 2009, pp. 793–802.

[ŽuŽup1] D. Žubrinić and V. Županović, Fractal analysis of spiral trajectories of some planar vector fields, *Bulletin des Sciences Mathématiques* No. 6, **129** (2005), 457–485.

[ŽuŽup2] D. Žubrinić and V. Županović, Box dimension of spiral trajectories of some vector fields in $\mathbb{R}^3$, *Qualitative Theory of Dynamical Systems* No. 2, **6** (2005), 251–272.

[ŽuŽup3] D. Žubrinić and V. Županović, Poincaré map in fractal analysis of spiral trajectories of planar vector fields, *Bull. Belg. Math. Soc. Simon Stevin* **15** (2008), 947-960.

[ŽupŽu] V. Županović and D. Žubrinić, Fractal dimensions in dynamics, *Encyclopedia of Mathematical Physics*, ed. J.-P. Françoise, G. L. Naber and S. T. Tsou, (Oxford: Elsevier), Chapter 2, pp. 394–402., 2006.