IRREDUCIBILITY OF STANDARD REPRESENTATIONS FOR IWAHORI-SPHERICAL REPRESENTATIONS

Goran Muić and Freydoon Shahidi

Abstract. In this paper the authors generalize a result of Vogan on irreducibility of standard representations for generic representations from real groups to split $p$-adic groups whenever the inducing data is Iwahori-spherical. In particular they give all the reducibility points for such standard representations as well as the degenerate principal series obtained from them by Aubert involution.

Introduction

Let $G$ be a split connected reductive group over a local nonarchimedean field $F$ of characteristic zero and let $B = TU$ be a Borel subgroup of $G$, where $T$ is a maximal split torus of $B$ and $U$ is its unipotent radical. Fix a parabolic subgroup $P$ of $G$ defined over $F$ with a Levi decomposition $P = MN$, with $T \subset M$ and $N \subset U$. Let $\sigma$ be an irreducible tempered representation of $M = M(F)$ and choose $\nu \in \mathfrak{a}_G^*$, the complex dual of the real Lie algebra of the split component $A$ of $M$. (See Section 1.) Let $I(\nu, \sigma)$ be the representation induced from $\nu$ and $\sigma$. Assume $\nu$ is in the positive Weyl chamber (Section 1). Then $I(\nu, \sigma)$ is called a standard representation. Let $J(\nu, \sigma)$ be the (unique) Langlands quotient of $I(\nu, \sigma)$ ([BW], [Si]). Up to conjugation of the data $(\nu, \sigma)$, every irreducible admissible representation of $G = G(F)$ is uniquely equivalent to a $J(\nu, \sigma)$. Moreover, every irreducible admissible generic (having a Whittaker model) representation of $G$ is a $J(\nu, \sigma)$ with $\sigma$ an irreducible generic tempered representation of $M$.

When $F = \mathbb{R}$, it was proved by Vogan in ([V]) that if $J(\nu, \sigma)$ is generic, then $I(\nu, \sigma)$ is irreducible. The main result of this paper is (Theorem 3.1)

Theorem 1. Suppose $\sigma$ is an Iwahori-spherical tempered generic representation of $M$ and fix $\nu \in \mathfrak{a}_G^*$ in the positive Weyl chamber. Then $I(\nu, \sigma)$ is irreducible if and only if $J(\nu, \sigma)$ is generic. Points of reducibility can be computed explicitly (Propositions 3.2 and 3.3).

In the special case that $P = B$, i.e. is a minimal parabolic subgroup and $\sigma$ is an unramified quasicharacter of $M = T$, Theorem 1 was proved in ([BM1]), ([Re1]), and ([Li]), each using a different method. We refer to ([CaSh]) for the general conjectures about

1991 Mathematics Subject Classification. Primary 22E35.

The second named author was partially supported by NSF Grant DMS9622585.

Typeset by \textsc{AMS-TEX}
reducibility of standard representations in which they proved Theorem 1 for supercuspidal
inducing data.

Our proof of Theorem 1 follows the same strategy as the one proving the same result
for the two series of classical groups \( \mathbf{G} = Sp(n) \) or \( SO(2n + 1) \) and arbitrary \( \sigma \) ([Mu2]).
As in ([Mu2]), the crucial step of the proof is Proposition 3.1 of the present paper. Here
we use the results of M. Reeder (see Section 3).

As a corollary of Theorem 1 we obtain a characterization for reducibility of standard
representations in terms of poles of \( L \)-functions (Propositions 3.2 and 3.3). These results
are conjectured in ([CaSh]). Moreover, when combined with the fact that in the unram-
ified case the local factors are Artin factors ([Sh3], Theorem 3.5, and Theorem 2.1 here),
Propositions 3.2 and 3.3 can be used to calculate the reducibility of generalized and de-
generate principal series. This can also be done without using the parametrization and
Artin factors. (See Remarks 3.3 and 3.4.) As an example, we calculate the reducibility
of generalized (degenerate, respectively) principal series of the simply-connected group of
type \( E_6 \), induced from the Steinberg (trivial, respectively) representation of the maximal
parabolic subgroup whose Levi factor has \( Spin(10) \) as its derived group.

The first author would like to thank M. Tadić for helping him to understand several
aspects of representation theory involved here.

1. Preliminaries

Let \( F \) be a nonarchimedean field of characteristic zero. Denote by \( \mathcal{O} \) the ring of integers
of \( F \) and \( \wp \) its maximal ideal. Let \( q \) be the number of elements of \( \mathcal{O}/\wp \). We fix a nontrivial
additive character \( \psi_F \) of \( F \). Let \( \mathbb{Z} \), \( \mathbb{R} \) and \( \mathbb{C} \) denote the ring of rational integers, the field
of real numbers and, the field of complex numbers, respectively.

Throughout this paper \( \mathbf{G} \) denotes an arbitrary split connected reductive algebraic group
over \( F \). Fix a Borel subgroup \( \mathbf{B} \) and write \( \mathbf{B} = \mathbf{T} \mathbf{U} \), where \( \mathbf{T} \) is a maximal split torus and
\( \mathbf{U} \) denotes the unipotent radical of \( \mathbf{B} \).

Fix a parabolic subgroup \( \mathbf{P} = \mathbf{M} \mathbf{N} \) of \( \mathbf{G} \) defined over \( F \) with \( \mathbf{N} \subset \mathbf{U} \) and \( \mathbf{T} \subset \mathbf{M} \),
a Levi decomposition. Denote by \( \mathbf{W} \) the Weyl group of \( \mathbf{T} \) in \( \mathbf{G} \). Let \( \bar{w}_0 \) be the longest
element in \( \mathbf{W} \) modulo that of the Weyl group of \( \mathbf{T} \) in \( \mathbf{M} \). Let \( \psi \) be a generic character of
\( \mathbf{U} = \mathbf{U}(F) \) ([CaSi], [Sh3]) and set \( \psi_M = \psi|\mathbf{U} \cap \mathbf{M} \). Suppose \( \sigma \) is an irreducible admissible
\( \psi_M \)-generic representation of \( \mathbf{M} = \mathbf{M}(F) \). Changing the splitting in \( \mathbf{U} \) we may assume
that \( \psi \) and \( \bar{w}_0 \) are compatible ([Sh3]).

Let \( X(\mathbf{M}_F) \) be the group of \( F \)-rational characters of \( \mathbf{M} \). Set
\[
\mathfrak{a}^* = X(\mathbf{M}_F) \otimes_{\mathbb{Z}} \mathbb{R} \text{ and } \mathfrak{a}^*_\mathbb{C} = \mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}.
\]

Let \( H_M \) be the homomorphism
\[
H_M : \mathbf{M} \to \mathfrak{a} = \text{Hom}(X(\mathbf{M}_F), \mathbb{R}),
\]
defined by
\[
q^{\langle x, H_F(m) \rangle} = |\chi(m)|_F
\]
for all $\chi \in X(M_F)$. We denote by $\chi_\nu$ the character of $M$ defined by $\chi_\nu(m) = q^{\langle \nu, H_M(m) \rangle}$, for any $\nu \in \mathfrak{a}_C^*$. We let

$$I(\nu, \sigma) = \text{Ind}_{\mathcal{M}_N}^G(\sigma \otimes \chi_\nu)$$

where $\nu \in \mathfrak{a}_C^*$. Further, if $P = B$ and $\sigma$ is the trivial representation of $T$ then we set $I_G(\chi_\nu) = I(\nu, \sigma)$, and call it an unramified principal series.

Suppose $P = MN$ is maximal, $M \supset T$. Let $r$ be the adjoint action of $LM$, the $L$-group of $M$, on the Lie algebra $^L\mathfrak{n}$ of the $L$-group of $N$. Then

$$r = \bigoplus_{i=1}^m r_i,$$

with $r_i$’s irreducible, ordered as in [Sh3], i.e. according to the order of eigenvalues of $^L\mathfrak{a}$ in $^L\mathfrak{n}$. Here $^L\mathfrak{a}$ is the $L$-group of $A$, the split component of $M$ which is contained in $T$. Finally, let $L(s, \sigma, r_i)$ be the local $L$-function, $\epsilon(s, \sigma, r_i, \psi_F)$ the $\epsilon$-factor, and $\gamma(s, \sigma, r_i, \psi_F)$ the $\gamma$-factor attached to $\sigma$ and $r_i$ in [Sh3]. (See Theorem 3.5 and Section 7 of [Sh3].)

Next, assume $M$ is generated by a subset $\theta$ of simple roots $\Delta$ of $T$ in $U$. Fix $w \in W$ such that $\bar{w}(\theta) \subset \Delta$ and let $w \in G$ be a representative for $\bar{w}$. Let $N_{\bar{w}} = U \cap w\overline{N}w^{-1}$, where $\overline{N}$ is unipotent subgroup opposed to $N$. Given $f$ in the space of $I(\nu, \sigma)$, let

$$A(\nu, \sigma, w)f(g) = \int_{N_{\bar{w}}} f(w^{-1}ng)dn \quad (g \in G)$$

denote the standard intertwining operator from $I(\nu, \sigma)$ into $I(w(\nu), w(\sigma))$. It converges absolutely in some cone and extends to a meromorphic function of $\nu \in \mathfrak{a}_C^*$. When $\sigma$ is tempered the cone of convergence for $\nu \in \mathfrak{a}_C^*$ equals to what one usually calls the positive Weyl chamber $(\mathfrak{a}_C^*)^+$ for $\mathfrak{a}$. Every $\nu \in (\mathfrak{a}_C^*)^+$ satisfies $\text{Re}\langle \nu, H_\alpha \rangle > 0$ for every $\alpha \in \Delta - \theta$ and conversely, where $H_\alpha$ is the standard coroot attached to $\alpha$ and $\nu$ is realized as an element of $\mathfrak{a}_0^\mathfrak{c}$. Here $\mathfrak{a}_0$ is the real Lie algebra of $T$.

Suppose $\sigma$ is tempered and $\nu \in (\mathfrak{a}_C^*)^+$. Then $I(\nu, \sigma)$ has a unique quotient $J(\nu, \sigma)$, called the Langlands quotient of $I(\nu, \sigma)$ ([BW], [Si]). Given an irreducible admissible representation $\pi$ of $G$, there exists a parabolic subgroup $P = MN$, $N \subset U$, $M \subset T$, an irreducible tempered representation $\sigma$ of $M$, and a $\nu \in (\mathfrak{a}_C^*)^+$, such that $\pi = J(\nu, \sigma)$. Moreover, by Rodier’s Theorem, $\pi$ is generic only if $\sigma$ is.

If $\sigma$ is $\psi_M$-generic then $I(\nu, \sigma)$ is $\psi$-generic and to it is attached a canonical Whittaker functional $\lambda_\psi(\nu, \sigma)$ ([Sh1]). Further, there exists a meromorphic function $C_\psi(\nu, \sigma, \bar{w}_0)$ such that

$$\lambda_\psi(\nu, \sigma) = C_\psi(\nu, \sigma, \bar{w}_0)\lambda_\psi(\bar{w}_0(\nu), \bar{w}_0(\sigma))A(\nu, \sigma, \bar{w}_0).$$

Assume $P$ is maximal and let $\alpha$ be the unique simple root in $N$. As in ([Sh3]), let $\bar{\alpha} = \langle \rho, \alpha \rangle^{-1} \cdot \rho$, where $\rho$ is half the sum of roots in $N$. Further, $C_\psi(s\bar{\alpha}, \sigma, \bar{w}_0)$ is, up to units of the ring $\mathbb{C}[q^s, q^{-s}]$, equal to ([Sh3], Theorem 3.5)

$$\prod_{i=1}^m L(1 - is, \bar{\sigma}, r_i)/L(is, \sigma, r_i)$$ (1.1)
Remark 1.1. We should warn the reader that in (1.1) the role of $\sigma$ and $\bar{\sigma}$ are interchanged as opposed to the usual definition of local coefficients. It agrees with our parametrization in Section 2.

2. Unramified principal series

Denote by $W_F$ the Weil group of $F$ relative to a fixed algebraic closure $\overline{F}$ ([De]). Let $I_F \subset W_F$ be the inertia group of $W_F$, and let $Fr$ denote an arbitrary Geometric Frobenius element in $W_F$. Let $r_F$ denote the class field epimorphism $W_F \to F^\times$, normalized by $|r_F(Fr)|_F = q^{-1}$. Then there exists a $1 - 1$ correspondence between characters $\chi$ of $F^\times$ and characters of $W_F$ given by $\chi \mapsto \phi \circ r_F$.

Let $\chi$ be an unramified character of $T$. Denote by

$$\phi_{\chi} : W_F \to {}^L T,$$

the corresponding admissible homomorphism ([Bo]) of the Weil group $W_F$ of $F$. It is defined in the following way

$$\gamma^{\vee} \circ \phi_{\chi} = \chi \circ \gamma^{\vee} \circ r_F,$$

for all cocharacters $\gamma^{\vee} \in X^*_T(T) = X^{(L)}(T)$. In fact, $\chi \mapsto \phi_{\chi}$ gives a $1 - 1$ correspondence between unramified characters of $T$ and admissible homomorphisms into ${}^L T$, trivial on $I_F$. Moreover, any admissible homomorphism $\phi$, trivial on $I_F$, is completely determined by

$$s = \phi(Fr) \in {}^L T.$$  

If we consider ${}^L T \subseteq {}^L G$, then (2.1) gives a $1 - 1$ correspondence between unramified characters of $T$ and the set of all semisimple conjugacy classes of $^L G$ ([Bo]).

According to a conjecture of Langlands ([Bo]), to a $^L G$-conjugacy class of a admissible homomorphism $\phi : W_F \times SL(2, \mathbb{C}) \to {}^L G$, should correspond a finite set $\Pi(\phi)$ of irreducible representations of $G$. Furthermore, as one considers all classes, then the packets $\Pi(\phi)$ should partition classes of irreducible representations of $G$.

Iwahori–spherical representations of $G$ should belong to $L$-packets $\pi(\phi)$, where

$$\phi : W_F / I_F \times SL(2, \mathbb{C}) \to {}^L G.$$  

(We will call an admissible homomorphism of the form (2.2) $I_F$-spherical ([BM2]).) Furthermore, all Iwahori–spherical representations in $\Pi(\phi)$ should be subquotients of the unramified principal series $I_G(\chi)$, determined by the conjugacy class of the semisimple element

$$\phi_{\chi}(Fr) = \phi(Fr, \text{diag}(q^{-1/2}, q^{1/2})).$$
When $G$ has connected center Kazhdan and Lusztig ([KL]) classified Iwahori–spherical representations, and constructed a finite to one correspondence $\pi \mapsto \phi_\pi$, between them and the set of all classes of $I_F$–spherical admissible homomorphisms. Decompose

$$s = \phi_\pi(F) = s_c \cdot s_h,$$

into its compact and hyperbolic parts. Then ([KL], 8.2, 8.3)

(i) $\pi$ is tempered if and only if $s_h = 1$.
(ii) $\pi$ is square integrable if and only if $s_h = 1$, and there is no noncentral torus in $L_G$

which centralises both $s$ and $\phi(1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})$.

Let $\rho$ be a finite dimensional algebraic representation of $L_G$, and $\phi : W_F \times SL(2, \mathbb{C}) \to L_G$ an admissible homomorphism. Then $\rho \circ \phi$ is a semisimple representation of $W_F \times SL(2, \mathbb{C})$. Hence, we can write

$$\rho \cong \bigoplus_j \rho_j \otimes V_{a_j},$$

where $\rho_j$ is an irreducible representation of $W_F$, and each $V_{a_j}$ is the unique irreducible representation of $SL(2, \mathbb{C})$ of dimension $a_j$. The local factors for $\rho \circ \phi$ are defined as follows

$$L(s, \rho \circ \phi) = \prod_j L(s, \rho_j \otimes V_{a_j}),$$

$$\epsilon(s, \rho \circ \phi, \psi_F) = \prod_j \epsilon(s, \rho_j \otimes V_{a_j}, \psi_F).$$

Moreover, for an irreducible representation $\rho' \otimes V_a$ local factors are defined as follows. First, define a new representation of $W_F$

$$\rho''(w) = \rho'(w) \otimes V_a(\text{diag}(\sqrt{|w|}, \sqrt{|w|^{-1}})),$$

where $|w| = |\psi_F(w)|_F$, and set

$$N = Id \otimes V_a(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}).$$

Then

$$\rho'(w)N \rho'(w)^{-1} = |w| \cdot N,$$

and, in fact, it defines an indecomposable representation of the Deligne–Weil group ([De]). Then $L$–factor $L(s, \rho' \otimes V_a)$ and $\epsilon$–factor $\epsilon(s, \rho' \otimes V_a, \psi_F)$ are those defined by Deligne in ([De]). For example,

$$L(s, \rho' \otimes V_a) = \det(Id - q^{-s} \cdot \rho''(Fr)|_{\rho' \otimes \mathfrak{n}_a})^{-1},$$

where $v_a \neq 0$ is highest weight vector in $V_a$ for the standard Borel subgroup of $SL(2, \mathbb{C})$.

Next, we prove ([Sh3], Theorem 3.5)
Theorem 2.1. Assume that $P = MN$ is maximal, and let $r = \oplus_{i=1}^{m} r_i$ be the decomposition of the adjoint action of $L\mathcal{M}$ on $L\mathfrak{n}$. Assume that $\sigma$ is a generic irreducible subquotient of the unramified principal series $I_M(\chi)$. Then we have

(i) If $\phi : W_F / I_F \times SL(2, \mathbb{C}) \rightarrow L\mathcal{M}$ is an admissible homomorphism, such that (2.3) holds, then

$$\gamma(s, \sigma, r_i, \psi_F) = \gamma(s, r_i \circ \phi, \psi_F) = \gamma(s, r_i \circ \phi_{\chi}, \psi_F),$$

for all $i = 1, \ldots, m$.

(ii) Assume that $G$ has connected center (then $M$ also has connected center) and $\sigma$ is tempered. Then

$$L(s, \sigma, r_i) = L(s, r_i \circ \phi_{\sigma})$$

and $\epsilon(s, \sigma, r_i, \psi_F) = \epsilon(s, r_i \circ \phi_{\sigma}, \psi_F)$,

where $\phi_{\sigma}$ is the Langlands parameter of $\sigma$, $i = 1, \ldots, m$. Furthermore, $L(s, \sigma, r_i)$ is holomorphic for $Re(s) > 0$.

Proof. Part (i) is Theorem 3.5 (i) and Proposition 3.4 of ([Sh3]). Part (ii) follows from (i) and the definition of local factors given in ([Sh3], Section 7) as soon as we prove that $L(s, r_i \circ \phi_{\sigma})$ is holomorphic for $Re(s) > 0$, $i = 1, \ldots, m$.

Decompose $s = \phi_{\sigma}(Fr) = s_c \cdot s_h$, into its compact and hyperbolic part. Since $\sigma$ is tempered, we have $s_h = 1$. This implies

$$(2.4) \quad \alpha^\vee(s) = \alpha^\vee(s_c) = q^{-t_\alpha}, \quad t_\alpha \in \sqrt{-1}\mathbb{R},$$

for any root $\alpha^\vee$ of $L\mathfrak{t}$. Further, any irreducible summand of $r_i \circ \phi_{\sigma}$ is of the form $|t| \otimes V_a$, for some $a \geq 1$, and $t = t_\alpha$ is defined by (2.4). Now, it is clear that

$$L(s, |t| \otimes V_a) = (1 - q^{-(s+t+\frac{a-1}{2}})^{-1}$$

is holomorphic for $Re(s) > 0$. Since local $L$–functions never vanishes, we see that $L(s, r_i \circ \phi_{\sigma})$ is holomorphic for $Re(s) > 0$. \quad \Box

Remark 2.1. It is not hard to prove that $L(s, \sigma, r_i)$ are holomorphic for $Re(s) > 0$ without appealing to the corresponding Artin $L$–functions.

We would like to prove the holomorphicity from Theorem 2.1 (ii) for a split group with not necessarily connected center. To accomplish that goal we start with the next lemma. This lemma must be well–known, but we include the proof for the sake of completeness.

Lemma 2.1. Assume $G$ is a reductive $F$–split algebraic group with center $Z_G$. Then there exists a reductive $F$–split algebraic group $G'$, with connected center $Z_{G'}$, and a $F$–embedding $\phi : G \rightarrow G'$ such that:

1. $\phi(G_{ss}) = G'_{ss}$.
2. $\phi(Z_G) \subseteq Z_{G'}$.
(Here, for a reductive group $H$ we denote by $H_{ss}$ its derived group.)

Proof. Throughout the proof we fix an algebraic closure $\overline{F}$ of $F$, and denote by $X(H)$ the group of characters of algebraic group $H$ defined over $\overline{F}$.

Since $G$ is split over $F$, the center $Z_G$ is a diagonalizable group such that $X(Z_G) = X(Z_G)_F$ (that is, the Galois group $\Gamma = \text{Gal}(\overline{F}/F)$ acts trivially on $X(Z_G)$). Fix a finitely generated free abelian group $Q$, equipped with the trivial action of $\Gamma$, such that there exists an epimorphism

\[ Q \longrightarrow X(Z_G) \longrightarrow 1. \]

Denote by $Z$ a $F$-split torus such that $X(Z) = Q$. Then (2.5) implies that there exists an $F$-embedding $Z_G \hookrightarrow Z$. Fix any of these embeddings. Define a finite algebraic $F$-group in the following way

\[ Z' = \{ (z, z^{-1}) \mid z \in G_{ss} \cap Z_G \}. \]

Furthermore, define a reductive algebraic $F$-group by

\[ G' = (Z \times G_{ss})/Z'. \]

One checks directly that $G'$ is split over $F$, and $Z_G' \cong Z$. Now, if we consider $G$ as the quotient

\[ G = (Z_G \times G_{ss})/Z', \]

then we can define a $F$-embedding $\phi$, with required properties. □

Now, we will prove

**Proposition 2.1.** Assume that $P = MN$ is a maximal parabolic subgroup, and let $r = \oplus_{i=1}^{m} r_i$ be the decomposition of the adjoint action of $^L M$ on $^L n$. Assume that $\sigma$ is a tempered generic irreducible subquotient of the unramified principal series $I_M(\chi)$. Then the $L$-function $L(s, \sigma, r_i)$ is holomorphic for $\text{Re}(s) > 0$, for all $i = 1, \ldots, m$.

Proof. Choose group $G'$ as in Lemma 2.1, and identify $G$ with the image of $\phi$. In particular $P' = M'N$, where $M' = Z_G' M$, is a maximal parabolic subgroup of $G$. The group $M'$ has a connected center.

Moreover, by ([Bo]) the dual morphism $^L \phi : ^L G' \longrightarrow ^L G$ has a central kernel. Also, if we denote by $r'$ the adjoint action of $^L M'$ on $^L n$, then we have

\[ r' = \oplus_{i=1}^{m} r'_i, \quad r'_i = r_i \circ ^L \phi, \]

is the decomposition discussed in Section 1.

Since $\chi$ is unramified, it is, in fact, the character of the finitely generated free abelian group $T/T'(O)$. But, this group is a subgroup of another finitely generated free abelian group $T'/T'(O)$. Now, it is clear that there exists an unramified character $\chi'$ of $T'$ such
that $\chi'|_T = \chi$. Since, $\chi$ is unitary on $Z_M$, we can always arrange for $\chi'$ to be unitary on $Z_{M'}$ $(|T|)$. Now, by definition of parabolic induction we have

$$I_M (\chi')|_M \cong I_M (\chi).$$

Since $Z_G, G$ is of finite index in $G'$, the results of $|T|$ imply that there exists a tempered representation $\sigma'$ of $M'$ such that $\sigma$ is a subrepresentation of $\sigma'|_M$. Since $\sigma$ is generic, $\sigma'$ is also generic $(|T|)$. (If $\sigma$ is square integrable then one can choose $\sigma'$ to be square integrable.) Furthermore, we can (and will) choose $\sigma'$ to be an irreducible subrepresentation of the unitary representation

$$(2.7) \quad \text{Ind}_{Z_{M'}}^M (\chi'|_{Z_M}, \sigma').$$

(All irreducible subrepresentations of (2.7) are tempered (square integrable) if $\sigma$ is tempered (square integrable) $(|T|)$.) Now, we will prove that there exists an irreducible subquotient of $I_M (\chi')$ which is a subrepresentation of (2.7). We can assume that $\sigma$ is an irreducible quotient of $I_M (\chi)$. Then by Frobenius reciprocity

$$\text{Hom}_{M'} (I_M (\chi'), \text{Ind}_{Z_{M'}}^M (\chi'|_{Z_M}, \sigma)) \cong \text{Hom}_M (I_M (\chi), \sigma) \neq 0.$$ 

Now, we can assume $\sigma'$ to be a subquotient of $I_M (\chi')$. Further, it is easy to see $\chi = \chi'|_T$ implies

$$\phi_\chi = L \phi \circ \phi_\chi'.$$

Then, by Theorem 2.1 (i) and (2.6) we have

$$\gamma(s, \sigma, r_i, \psi_F) = \gamma(s, \sigma', r'_i, \psi_F),$$

for all $i = 1, \ldots, m$. Hence, by definition of local $L$-functions for tempered representations, we have $L(s, \sigma, r_i) = L(s, \sigma', r'_i)$, for all $i = 1, \ldots, m$. Now, the proposition is a consequence of Theorem 2.1 (ii).  

3. Reducibility result

Main result of this section is the following theorem.

**Theorem 3.1.** Assume that $P = MN \subseteq G$ is a standard parabolic subgroup, $\sigma$ an Iwahori–spherical tempered generic representation of $M$, and $\nu \in (a^*_G)^+$. Then the standard representation $I (\nu, \sigma)$ is irreducible if and only if the corresponding Langlands quotient is generic.

The proof of this result is similar to that given for classical groups in ([Mu2]). We start by recalling some results from ([Re2]) and ([Re3]). Assume that $G$ is adjoint. Denote by $D_G (\chi)$ the set of all Iwahori-spherical square integrable subquotients of $I_G (\chi)$. Assume that $D_G (\chi)$ is not empty, and denote by

$$\phi : W_F \times SL(2, \mathbb{C}) \rightarrow I_G$$
the Langlands parameter of representations in $D_G(\chi)$. Let

$$\tau = \phi(\Fr, \text{diag}(q^{-1/2}, q^{1/2})) \text{ and } N = \phi(1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}).$$

Kazhdan and Lusztig classified $D_G(\chi)$. To explain their results, we follow ([Re2], p.465). We denote by $Z(\tau, N)$ the mutual centralizer in $^L G$ of $\tau$ and $N$, and denote by $Z$ the center of $^L G$. Further, let $X = ^L G/^L B$ denote the variety of all Borel subgroups of $^L G$. Denote by $X(\tau, N)$ the subvariety of those which contain both $\tau$ and $N$. Then $Z(\tau, N)/Z$ acts on $X(\tau, N)$, and induces a linear action of the corresponding component group $A(\tau, N)$ on the homology group $H_* (X(\tau, N))$. Square integrable representations in $D(G)_\chi$ are parametrised by inequivalent irreducible subrepresentations of this action. If $\rho$ is such an irreducible representation of $A(\tau, N)$, then we denote by $\pi(\chi, \rho)$ the corresponding discrete series. The dimension of the space of Iwahori invariants in $\pi(\chi, \rho)$ is equal to the multiplicity of $\rho$ in $H_* (X(\tau, N))$. Finally, the trivial representation $1_{A(\tau, N)}$ always appears in $H_* (X(\tau, N))$, and we have ([Re3]).

**Theorem 3.2.** ([Re3]) Suppose that $G$ is adjoint. Then $\pi(\chi, 1_{A(\tau, N)})$ is generic.

**Remark 3.1.** When $G$ is adjoint then there is only one $T$–orbit of generic characters of $U'$. The same is true for an arbitrary split $G$ with connected center. To see the last statement it is enough to note that by Hilbert’s Theorem 90 we have the exact sequence

$$1 \longrightarrow Z_G \longrightarrow G \longrightarrow G_{ad} \longrightarrow 1,$$

where $G_{ad}$ denotes the group of $F$–points of the adjoint group of $G$.

Theorem 3.2 and Remark 3.1 immediately imply

**Corollary 3.1.** Suppose that $G$ is a reductive $F$–split algebraic group with connected center, and $\chi$ is an unramified character of $T$. If $D(G)_\chi$ is not empty, then the unique irreducible generic subquotient of $I_G(\chi)$ belongs to $D(G)_\chi$.

**Proof.** Denote by $\chi_0$ the restriction of $\chi$ to the center $Z_G$. Since $Z_G$ is a $F$–split torus, $Z_G(\mathcal{O})$ is defined. Further, the quotient $Z_G/Z_G(\mathcal{O})$ is a finitely generated free abelian group, which is a subgroup of another finitely generated free abelian group $G/\mathfrak{g}_G$, where $\mathfrak{g}_G$ is the intersection of kernels of all positive valued unramified characters of $G$ ([BW], Chapter XI). Now, it is clear that there exists a unitary unramified character $\mu$ of $G$ which extends $\chi_0$. Furthermore, $(\mu|_T)^{-1}\chi$ is trivial on $Z_G$, and

$$\mu^{-1}I_G(\chi) \cong I_G((\mu|_T)^{-1}\chi)$$

is in fact an unramified principal series for the adjoint group. \qed
Proposition 3.1. Suppose that $G$ is an arbitrary $F$–split reductive group, and $\chi$ is an unramified character of $T$. Assume that the unramified principal series $I_G(\chi)$ contains a square integrable subquotient. Then every generic irreducible subquotient of $I_G(\chi)$ is square integrable.

Proof. This follows from Corollary 3.1 and Lemma 2.1, using the same arguments as ones used in the proof of Proposition 2.1. □

Corollary 3.2. Assume that the unramified principal series $I_G(\chi)$ contains a tempered subquotient. Then every generic irreducible subquotient of $I_G(\chi)$ is tempered.

Proof. Denote by $\pi$ a tempered irreducible subquotient of $I_G(\chi)$. Fix a non-degenerate character $\psi$ of $U$. We have to show that the unique $\psi$–generic subquotient $\pi_\psi$ of $I_G(\chi)$ is tempered.

Let $P = MN$ be a parabolic subgroup of $G$ and $\delta$ a square integrable representation of $M$ such that $\pi$ is a subrepresentation of $\text{Ind}^G_{MN}(\delta)$. It is clear that $\delta$ is Iwahori–spherical, and, consequently, there exists an unramified character $\chi'$ of $T$ such that $\delta$ is subquotient of $I_M(\chi')$. By induction in stages, every irreducible subquotient of $\text{Ind}^G_{MN}(\delta)$ is a subquotient of $I_G(\chi') = \text{Ind}^G_{MN}(I_M(\chi'))$. Now, $I_G(\chi)$ and $I_G(\chi')$ have an irreducible subquotient in common. Hence, there exists $w \in W$ such that $\chi' = w(\chi)$ ([BZ]).

Now, by Proposition 3.1 every generic irreducible subquotient of $I_M(\chi')$ is square integrable. It is clear that we can always choose a generic discrete series subquotient $\delta'$ of $I_M(\chi)$ such that $\text{Ind}^G_{MN}(\delta')$ is $\psi$–generic. By the first part of the proof, the unique $\psi$–generic subquotient of $I_G(\chi)$ is a subquotient of $\text{Ind}^G_{MN}(\delta')$. □

Proof of Theorem 3.1. Denote by $\pi$ the Langlands quotient of $J(\nu, \sigma)$. We can consider (and will) $\nu_\pi = \nu$ as an element of $X(T)_F \otimes_{\mathbb{Z}} \mathbb{R}$. We denote by $< \cdot, \cdot >$ the partial order on $X(T)_F \otimes_{\mathbb{Z}} \mathbb{R}$ explained in ([BW], Chapter XI, 2.1) (for our purpose it is not important to write it explicitly).

Assume that the induced representation $I(\nu, \sigma)$ reduces. Suppose that $\sigma$ is $\psi_M$–generic. Then we have to show that $\pi = J(\nu, \sigma)$ is not $\psi$–generic (see Section 1).

Let $\pi'$ be any irreducible subquotient of the induced representation $I(\nu, \sigma)$, which is not isomorphic to $\pi$. Assume that $\pi'$ is realized as a unique irreducible quotient of the standard representation

\begin{equation}
I(\nu', \sigma') = \text{Ind}^G_{M'N'}(\sigma' \chi_{\nu'}), \quad \nu' = \nu_{\pi'}.
\end{equation}

Since $\pi'$ is Iwahori–spherical, $\sigma'$ is also Iwahori–spherical.

Furthermore, ([BW], Chapter XI, Lemma 2.13)

\begin{equation}
\nu_{\pi'} < \nu_\pi.
\end{equation}

Observe that representations $\pi$ and $\pi'$ are the subquotients of the same unramified principal series. More precisely, if $\sigma'$ is an irreducible subquotient of $I_M(\chi')$, then $\pi$ and $\pi'$ are subquotients of $I_G((\chi_{\nu'} | T)\chi')$. In fact, all irreducible subquotients of $I(\nu, \sigma)$ and
$I(\nu', \sigma')$ are its subquotients. Denote by $\pi_\psi$ the unique $\psi$-generic irreducible subquotient of this principal series.

By Corollary 3.2, every irreducible generic subquotient of $I_{M'}(\chi')$ is tempered. Furthermore, we can take its tempered $\psi_{M'}$-generic subquotient $\sigma''$ such that the standard module (see (3.1))

$$I(\nu', \sigma'') = \text{Ind}_{M' \cap N}^G(\sigma'' \chi_{\nu'})$$

admits a nontrivial $\psi$-generic functional. Moreover, the unique $\psi$-generic subquotient of (3.3) is $\pi_\psi$. Set $\pi'' = J(\nu', \sigma'')$. By definition, $\nu_{\pi''} = \nu = \nu_{\pi'}$. (See the first paragraph of the proof.) Now, (3.2) implies

$$\nu_{\pi''} = \nu_{\pi'} < \nu_{\pi}.$$

This implies that $\pi$ cannot be an irreducible subquotient of (3.3). Hence $\pi$ is not isomorphic to $\pi_\psi$ and therefore $\pi$ is not $\psi$-generic. $\square$

**Remark 3.2.** There exists a unique minimal $\nu_0$ among all $\nu_{\pi'}$, where $\pi'$ is an irreducible subquotient of the standard representation $I(\nu, \sigma)$. Every generic subquotient $\pi''$ of $I(\nu, \sigma)$ satisfies $\nu_{\pi''} = \nu_0$. This is a corollary of the proof of Theorem 3.1.

Now we have the following general reducibility criteria for representations induced from irreducible generic Iwahori–spherical essentially tempered representations of Levi factors of arbitrary parabolic subgroups. Propositions 3.2 and 3.3 are very useful in determining points of reducibility explicitly (cf. Section 4).

**Proposition 3.2.** Let $P = MN$ be an arbitrary parabolic subgroup of $G$, $P \supset B$, and fix an irreducible $\psi_M$-generic Iwahori–spherical tempered representation $\sigma$ of $M$. Let $\nu \in \mathfrak{a}_G^*$ be in the positive Weyl chamber of the split component of $M$. Then $I(\nu, \sigma)$ is irreducible if and only if $C_\psi(\nu, \sigma, w_0)^{-1} \neq 0$.

**Proof.** This follows from ([CaSh], Proposition 5.4), as soon as Theorem 3.1 is proved. $\square$

Finally, we have the characterization of reducibility in terms of $L$-functions.

**Proposition 3.3.** Assume that $P = MN \subseteq G$ is maximal parabolic, $\sigma$ Iwahori–spherical tempered generic representation of $M$, and $s > 0$. Let $\alpha$ be the unique simple root in $N$. As in ([Sh3]), let $\bar{\alpha} = \langle \rho, \alpha \rangle^{-1} \rho$, where $\rho$ is half the sum of roots in $N$. Then the standard representation $I(s\bar{\alpha}, \sigma)$ is irreducible if and only if

$$\prod_{i=1}^m L(1 - is, \bar{\sigma}, r_i)^{-1} = 0.$$

**Proof.** This a direct consequence of Proposition 2.1 and Theorem 3.1, using ([CaSh], Proposition 5.3). (See also Remark 1.1.) $\square$
Remark 3.3. If $s > 0$, then the contragredient of the generalized principal series $I(-s\tilde{\alpha}, \sigma)$ is isomorphic to the standard representation $I(s\tilde{\alpha}, \sigma)$. Consequently, $I(-s\tilde{\alpha}, \sigma)$ is irreducible if and only if
\[ \prod_{i=1}^{m} L(1 - is, \sigma, \nu_i)^{-1} = 0. \]

The case of $s = 0$ is covered by the theory of $R$-groups.

Remark 3.4. Denote by $D_G$ the involution on the Grothendieck group $R(G)$ of finite length representations of $G$, studied by Aubert in ([A1], see also [A2]). More precisely, let $\Delta$ be the set of simple roots of $T$ with respect to the fixed Borel subgroup $B$. Then for any $\Theta \subseteq \Delta$ we denote by $P_{\Theta} = M_{\Theta}N_{\Theta}$ the corresponding standard parabolic subgroup of $G$. Let $i_{Me}^{G}$ and $r_{Me}^{G}$ be the functors of induction and restriction, respectively ([BZ], 2.3). They define homomorphisms
\[ i_{Me}^{G} : R(M_{\Theta}) \rightarrow R(G) \quad \text{and} \quad r_{Me}^{G} : R(G) \rightarrow R(M_{\Theta}). \]

The involution $D_G$ is defined by ([A1], Definition 1.5)
\[ D_G(\pi) = \sum_{\Theta \subseteq \Delta} (-1)^{|\Theta|} i_{Me}^{G}(r_{Me}^{G}(\pi)). \]

(Here we denote by $|X|$ the number of elements of the set $X$.) The involution maps irreducible representations, up to sign, to irreducible ones. For example, if $\pi$ is an irreducible subquotient of some principal series then $D_G(\pi)$ is an irreducible subquotient of the same principal series ([A1], Corollary 3.9).

Now, under assumptions of Proposition 3.3, using ([A1], Theorem 1.7 (2)), we have
\[ I(s\tilde{\alpha}, D_M(\sigma)) = i_{M}^{G}(\chi s\tilde{\alpha} D_M(\sigma)) = i_{M}^{G}(D_M(\chi s\tilde{\alpha} \sigma)) = D_G(I(s\tilde{\alpha}, \sigma)). \]

This implies that the generalized principal series $I(s\tilde{\alpha}, \sigma)$ and the degenerate principal series $I(s\tilde{\alpha}, D_M(\sigma))$ have the same points of reducibility.

4. An example

Let $G$ be a simply-connected split group of type $E_6$. Let $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_6\}$ be the set of simple roots of $T$ with respect to the Borel subgroup $B$, which are labeled on Dynkin diagram in the standard way (see [Sh2], or [Bou]). Denote by $P = MN$ ($P' = M'N'$, respectively) the maximal parabolic subgroup of $G$ which corresponds to the set of simple roots $\Theta = \Delta \setminus \{\alpha_1\}$ ($\Theta' = \Delta \setminus \{\alpha_6\}$, respectively). Then the derived groups $M_{ss}$ and $M'_{ss}$ are simply-connected (thanks to Alan Roche for a short proof of this) and of type $D_5$, i.e.
\[ M_{ss} \cong M'_{ss} \cong Spin(10). \]

(This is true even if $G$ is adjoint $E_6$.) As in Section 1, denote by $\tilde{w}_0$ the longest element of the Weyl group $W$ modulo that of $T$ in $M$. Then $\tilde{w}_0(\Theta) = \Theta'$, and $M' = \tilde{w}_0 M \tilde{w}_0^{-1}$.
IRREDUCIBILITY OF STANDARD REPRESENTATIONS

Denote by $St_M$ ($St_{M'}$), respectively, and $1_M$ ($1_{M'}$, respectively) the Steinberg and the trivial representation of $M$ ($M'$, respectively), respectively. It is well known that $St_M$ is the unique irreducible subrepresentation and $1_M$ is the unique irreducible quotient of $I_M(\chi_{\rho_0})$, where $\rho_0 = \rho_0^M$ is a half sum of positive roots of $M$.

$$\rho_0 = 5\alpha_2 + 5\alpha_3 + 9\alpha_4 + 7\alpha_5 + 4\alpha_6.$$ 

The adjoint representation of $L_M$ on $L_n$, when restricted to $Spin(10, \mathbb{C})$, the derived group of $L_M$, becomes the irreducible representation whose lowest weight is $\alpha_1^\vee$. Denote this adjoint representation by $r$. We observe that this restriction is one of the two 16-dimensional irreducible half spin representations of $Spin(10, \mathbb{C})$ ([SK]), and the weights are

$$\begin{align*}
\alpha_1^\vee, & \quad \alpha_1^\vee + \alpha_2^\vee, & \quad \alpha_1^\vee + \alpha_3^\vee + \alpha_4^\vee, & \quad \alpha_1^\vee + \alpha_2^\vee + \alpha_4^\vee + \alpha_5^\vee, \\
\alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee + \alpha_5^\vee, & \quad \alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee + \alpha_5^\vee + \alpha_6^\vee, & \quad \alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee + \alpha_5^\vee + 2\alpha_6^\vee, \\
\alpha_1^\vee + \alpha_2^\vee + 2\alpha_3^\vee + 2\alpha_4^\vee + \alpha_5^\vee, & \quad \alpha_1^\vee + \alpha_2^\vee + \alpha_4^\vee + \alpha_5^\vee + \alpha_6^\vee, & \quad \alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee + 2\alpha_3^\vee + 2\alpha_4^\vee + \alpha_5^\vee + \alpha_6^\vee, \\
\alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee + 2\alpha_4^\vee + 2\alpha_5^\vee + \alpha_6^\vee, & \quad \alpha_1^\vee + \alpha_2^\vee + 2\alpha_3^\vee + 2\alpha_4^\vee + \alpha_5^\vee + \alpha_6^\vee, & \quad \alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee + 2\alpha_3^\vee + 2\alpha_4^\vee + \alpha_5^\vee + \alpha_6^\vee.
\end{align*}$$

To apply Theorem 3.1 explicitly, we need to use Proposition 3.3. We therefore need to calculate the local $L$-function $L(s, St_M, r)$. First, by ([Sh3], Section 7) $L(s, St_M, r)$ is defined as $P(q^{-s})^{-1}$, where $P$ is a polynomial, $P(0) = 1$, such that $P(q^{-s})$ has the same zeros as $\gamma(s, St_M, \psi_F)$. Next, by the results recalled in Section 2, we have

$$\beta^\vee(\phi_{\rho_0}(w)) = |w|^{|\rho_0, \beta^\vee|}, \forall w \in W_F.$$

Further, by Theorem 2.1 (i)

$$\gamma(s, St_M, r, \psi_F) = \gamma(s, r \circ \phi_{\rho_0}, \psi_F) = \prod_{\beta^\vee} \gamma(s + \langle \rho_0, \beta^\vee \rangle, 1, \psi_F),$$

where the product ranges over all above weights. Here $\gamma(s, 1, \psi_F)$ is the $\gamma$-factor associated to the trivial representation of $W_F$. Since

$$\langle \rho_0, \alpha_1^\vee \rangle = -5, \quad \text{and} \quad \langle \rho_0, \alpha_i^\vee \rangle = 1, \quad i = 2, \ldots, 6,$$

we can rewrite (4.1) as follows

$$\gamma(s, St_M, r, \psi_F) = \gamma(s - 5, 1, \psi_F) \cdot \gamma(s - 4, 1, \psi_F) \cdot \gamma(s - 3, 1, \psi_F) \cdot \gamma(s - 2, 1, \psi_F) \cdot \gamma(s - 1, 1, \psi_F) \cdot \gamma(s, 1, \psi_F) \cdot \gamma(s + 1, 1, \psi_F) \cdot \gamma(s + 2, 1, \psi_F) \cdot \gamma(s + 3, 1, \psi_F) \cdot \gamma(s + 4, 1, \psi_F) \cdot \gamma(s + 5, 1, \psi_F).$$
Using $L(s, 1) = (1-q^{-s})^{-1}$, we see that $L(1-s-\alpha, 1)$ is, up to units of the ring $\mathbb{C}[q^s, q^{-s}]$, equal to $L(s + \alpha - 1, 1)$; we can therefore write (4.2), up to units of the ring $\mathbb{C}[q^s, q^{-s}]$, in the form

$$L(s - 6, 1)L(s - 3, 1)$$

$$L(s + 2, 1)L(s + 5, 1)$$

Hence

$$(4.3) \quad L(s, St_M, r) = L(s + 2, 1)L(s + 5, 1).$$

We have

**Proposition 4.1.** The generalized principal series $I(s\tilde{\alpha}_1, St_M)$ and $I(s\tilde{\alpha}_6, St_M)$, and the degenerate principal series $I(s\tilde{\alpha}_1, 1_M)$ and $I(s\tilde{\alpha}_6, 1_M)$, $s \in \mathbb{R}$, reduce if and only if

$$s \in \{-6, -3, 3, 6\}.$$ 

**Proof.** First, note that ([BDK], Lemma 5.4 (iii))

$$I(-s\tilde{\alpha}_6, \pi) = I(s\tilde{w}_0(\tilde{\alpha}_1), \tilde{w}_0(\pi)),$$ 

for any irreducible representation $\pi$ of $M$, in the Grothendieck group of representations of finite length of $G$.

Since $P$ and $P'$ are not self-conjugate, the generalized principal series $I(0\tilde{\alpha}_1, St_M)$ and $I(0\tilde{\alpha}_6, St_M)$ are irreducible. Consider $s > 0$. Then by Proposition 3.3, $I(s\tilde{\alpha}_1, St_M)$ reduces if and only if $L(1-s, \tilde{St}_M, r)^{-1} = 0$. Since the Steinberg representation is self contragredient, (4.3) implies that points of reducibility for $s > 0$ are $s = 3, 6$. Further, for the contragredient we have

$$I(s\tilde{\alpha}_1, St_M)^\sim \cong I(-s\tilde{\alpha}_1, St_M),$$

and the proposition is proved for the generalized principal series.

It is well known that $D_M(St_M) = 1_M$ and $D_M'(St_M') = 1_{M'}$. Now, Remark 3.4 implies the degenerate principal series part of the proposition. □

**Remark 4.1.** A similar strategy for finding reducibility points of generalized and degenerate principal series for $G_2$ was already used in ([Mu1]).

**References**


Irreducibility of Standard Representations


Department of Mathematics, University of Utah, Salt Lake City, UT 84112
E-mail address: gmuic@math.utah.edu

Department of Mathematics, Purdue University, West Lafayette, IN 47907
E-mail address: shahidi@math.purdue.edu