ON THE STRUCTURE OF THE FULL LIFT
FOR THE HOWE CORRESPONDENCE OF
\((Sp(n), O(V))\) FOR RANK-ONE REDUCIBILITIES

Goran Muić

INTRODUCTION

In this paper we study the Howe correspondence for the dual pair \((Sp(n), O(V_r))\), where \(n \geq 0\), \(V_r\) is a quadratic space in a fixed, but arbitrary even dimensional Witt tower (cf. Section 1) over a non–archimedean local field \(F\) of the characteristic different from 2. To explain our results, let us write \(\omega_{n,r} = \omega^\psi_{n,r}\), for the oscillator representation associated to that pair and a fixed additive character \(\psi\) of \(F\). Let \(\chi\) be a quadratic character associated to \(V_r\). Let \(\sigma \in \text{Irr}(Sp(n))\). Then the \(\sigma\)–isotypic component \(\omega_{n,r}\) we denote by \(\Theta(\sigma, r)\) (see Section 2). This is a smooth representation of \(O(V_r)\). We call it the full lift of \(\sigma\). The basic problems in the theory are: to determine when \(\Theta(\sigma, r) \neq 0\), to prove Howe duality conjecture (see [W] when residue characteristic is different than two), and to describe the structure of the unique irreducible quotient of \(\Theta(\sigma, r) \neq 0\). In our paper [M], we succeeded solving all three problems for discrete series representations. Although methods of [M] are rather powerful for discrete series, they are not sufficient for tempered (but not in discrete series) representations and non–tempered representations. In this paper we present a different approach based on a deep result of Bernstein [Be] about the existence of right–adjoint functor to the functor of induction in the category of smooth representations [Be]. This method also gives the information about the structure of the full lift \(\Theta(\sigma, r)\).

To describe our main results, let \(\sigma \in \text{Irr}(Sp(n))\) be a supercuspidal representation. Then it is known that at the first occurrence \(r\), the lift \(\Theta(\sigma, r) = \tau\) is a supercuspidal irreducible representation [MWV]. Assume that \(\rho \in \text{Irr}(GL(j, F))\) is supercuspidal. Let \(P_j\) (resp. \(Q_j\)) be a maximal parabolic subgroup of \(O(V_r+j)\) (resp. \(Sp(n+j)\)) with a Levi factor isomorphic to \(GL(j, F) \times O(V_r)\) (resp. \(GL(j, F) \times Sp(n)\)). The main results of this paper, Theorems 2.1 and 2.2, describe the structure of the full Howe lift to \(Sp(n+j)\) (resp. \(O(V_r+j)\)) of all irreducible subquotients of \(\text{Ind}_{P_j}^{O(V_r+j)}(\chi \rho \otimes \tau)\) (resp. \(\text{Ind}_{Q_j}^{Sp(n+j)}(\rho \otimes \sigma)\)). As a result, the reduction and composition series of these induced representations are described in terms of the Howe correspondence. Also, the results of the present paper (and the results about the lift of supercuspidal representations [MVW]) suggest the following conjecture:

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Conjecture. Assume that $\sigma$ is in the discrete series, then $\Theta(\sigma, r)$ is irreducible or zero.

Theorems 2.1 and 2.2 are proved in Section 2. The basic method is to consider Jacquet modules of $\omega_{n+j, r+j}$ (cf. [Ku]), but computing certain isotypic components in Jacquet modules of $\omega_{n+j, r+j}$ (Proposition 2.1) using [Be]. Section 3 is devoted to these computations.

In Section 4 we apply our results to get some explicit and new reducibility results for the inner form of the split $SO(2n)$ (Theorem 4.1), although more can be done using chains of supercuspidal representations as discussed in [Ku1].

This paper is an outgrowth of our old manuscript [M1]. We are planning to pursue approach to Howe correspondence adopted here further in the sequel of this paper [M2]. The present paper is the first in that direction and is fundamental since it treats the generalized rank–one case.

I would like to thank Joseph Bernstein for help with his work.

1. Preliminaries

Let $F$ be a nonarchimedean field of characteristic different from 2. Let $\mathbb{Z}_+, \mathbb{R}$, and $\mathbb{C}$ be the set of non-negative rational integers, the field of real numbers, and the field of complex numbers, respectively.

Let $G$ be an $l$–group (cf. [BZ]). Then, we will write $\mathcal{A}(G)$ for the category of all smooth representations of $G$. Let $P = MN$ be a closed subgroup of $G$, given as the semi–direct product of closed subgroups $M$ and $N$, $M$ normalizes $N$. Assume that $N$ is a union of its open compact subgroups. Then we have normalized induction and localization functors $\text{Ind}^G_P : \mathcal{A}(M) \to \mathcal{A}(G)$ and $\text{R}_P : \mathcal{A}(G) \to \mathcal{A}(M)$. They are related by Frobenius reciprocity $\text{Hom}_G(\pi, \text{Ind}^G_P(\pi')) \cong \text{Hom}_M(\text{R}_P(\pi), \pi')$. $\text{Ind}^G_P$ and $\text{R}_P$ are exact functors.

Assume that $G$ and $G'$ are $l$–groups. Let $V \in \mathcal{A}(G \times G')$. If $\rho \in \text{Irr}(G)$ is an admissible representation, then we write $\Theta(\rho, V) \in \mathcal{A}(G')$ for the $\rho$–isotypic part of $V$. (cf. [MVW], Chapitre II, Lemme III.4). More precisely, set $V' = \cap_f \ker(f)$, $f \in \text{Hom}_G(V, \rho)$, then $V/\text{V'} \cong \rho \otimes \Theta(\rho, V).

For convenience, let us state the next simple lemma.

**Lemma 1.1.** The dual $G'$–module $\Theta(\rho, V)^*$ is isomorphic to the obvious (not necessarily smooth) $G'$–module $\text{Hom}_G(V, \rho)$. Hence, we have an isomorphism of the corresponding smooth modules $\widehat{\Theta(\rho, V)} = \Theta(\rho, V)^* \cong \text{Hom}_G(V, \rho)^*$. 

Now, we are going to describe the groups that we will consider. We follow ([MVW], Chapitre I). Fix an anisotropic even dimensional inner product $F$–space $(V_0, (,)_0)$. Then its dimension can only be 0, 2, or 4. (For more details see [MVW], Chapitre I.) Let $\chi = \chi_{V_0}$ be the quadratic character of $F^\times$ associated to the quadratic space $V_0$. (See [Ku], pg. 240, (2.5), or [Ku1], Proposition 4.3.) If $V_0$ is trivial or four dimensional space, then $\chi$ is the trivial character.
For each \( r \in \mathbb{Z}_+ \), let \( V_r \) be the orthogonal direct sum of \( V_0 \) with \( r \) hyperbolic planes. We will fix a Witt decomposition

\[
V_r = V_r^{(1)} \oplus V_0 \oplus V_r^{(2)},
\]

where \( V^{(i)} = Fv_1^{(i)} \oplus \cdots \oplus Fv_r^{(i)} \), \( i = 1, 2 \), satisfying \((v_k^{(i)}, v_l^{(i)}) = 0 \) and \((v_k^{(1)}, v_l^{(2)}) = \delta_{kl} \). Let \( O(V_r) \) (resp. \( SO(V_r) \)) be the corresponding orthogonal (resp. special orthogonal) group. If \( V_0 = 0 \), then we let \( O(V_0) = SO(V_0) \) be the trivial group. Let \( \nu = \nu_r \) be the determinant character of \( O(V_r) \).

The decomposition (1.1) gives us the set of a standard parabolic subgroups in \( O(V_r) \) and \( SO(V_r) \). We will describe only maximal parabolic subgroups. For \( j, 1 \leq j \leq r \), let \( V_j^{(i,r)} = Fv_{r-j+1}^{(i)} \oplus \cdots \oplus Fv_r^{(i)} \), \( i = 1, 2 \). Then we have a Witt decomposition

\[
V_r = V_j^{(1,r)} \oplus V_{r-j} \oplus V_j^{(2,r)}. \tag{1.2}
\]

Let \( P_j \) be the parabolic subgroup of \( O(V_r) \) which stabilizes \( V_j^{(1,r)} \). There is a Levi decomposition \( P_j = M_j N_j \), where \( M_j \cong GL(V_j^{(1,r)}) \times O(V_{r-j}) \). (Beware of the difference between this choice of a Levi factor and that of ([Ku], pg. 233). There is considered \( GL(V_j^{(2,r)}) \) instead of \( GL(V_j^{(1,r)}) \).) Fix the isomorphism \( GL(n, F) \cong GL(V_j^{(1,r)}) \) using the above fixed basis of \( V_j^{(1,r)} \).

Let \( \epsilon \) be the element of \( O(V_r) \) defined as follows. First, \( \epsilon(v_j^{(i)}) = v_j^{(i)} \), for \( 1 \leq j \leq r - 1 \). Then, if \( V_0 \neq 0 \), we let \( \epsilon(v_r^{(i)}) = v_r^{(i)} \), and let \( \epsilon \) be any element \( \alpha \) of \( O(V_0) \) on \( V_0 \), \( \alpha^2 = 1 \), \( \nu(\alpha) = -1 \). If \( V_0 = 0 \), then we let \( \epsilon(v_r^{(1)}) = v_r^{(2)} \) and \( \epsilon(v_r^{(2)}) = v_r^{(1)} \). Clearly, \( \nu(\epsilon) = -1 \).

A set of standard maximal parabolic subgroups of \( SO(V_r) \) may be described as follows. For each \( j, 1 \leq j \leq r \), set \( P_j^0 = P_j \cap SO(V_r) \) and \( M_j^0 = M_j \cap SO(V_r) \). \( P_j^0 \) is a standard parabolic subgroup of \( SO(V_r) \) with a Levi decomposition \( P_j^0 = M_j^0 N_j \). If \( V_0 \neq 0 \), then \( P_j^0, 1 \leq j \leq r \), exhaust the set of all standard maximal parabolic subgroups. If \( V_0 = 0 \), then we need to add one more parabolic subgroup \( P_j^{0-} \), with a Levi decomposition \( P_j^{0-} = M_j^{0-} N_j^{0-} \), where \( M_j^{0-} = \epsilon M_j^0 \epsilon^{-1} \) and \( N_j^{0-} = \epsilon N_j^0 \epsilon^{-1} \). This will be used to identify \( M_j^{0-} \cong GL(r, F) \).

Now, we will discuss the representation theory of \( O(V_r) \). Since the algebraic group \( O(V_r) \) is not connected (in the algebraic sense), we need to discuss some general results of [Ca]. First, we will recall the relationship between irreducible representations of \( O(V_r) \), and those of \( SO(V_r) \) using the next simple lemma (cf. [MVW], Chapitre III).

**Lemma 1.2.**

(i) If \( \pi \in \text{Irr}(O(V_r)) \), then \( \pi' = \pi|_{SO(V_r)} \) is irreducible if and only if \( \pi \) is not equivalent to \( \nu \pi \). If \( \pi' \) is irreducible, then \( \text{Ind}_{SO(V_r)}^{O(V_r)}(\pi') \cong \pi \oplus \nu \pi \), and \( \pi' \cong \pi' \). (Here \( \pi'(g) = \pi'(eg \epsilon^{-1}) \), for all \( g \in SO(V_r) \).

(ii) If \( \pi' \in \text{Irr}(SO(V_r)) \), then \( \pi = \text{Ind}_{SO(V_r)}^{O(V_r)}(\pi') \) is irreducible if and only if \( \pi' \not\equiv \pi' \).

Furthermore, \( \nu \pi \cong \pi \).
Lemma 1.2 enables us to define supercuspidal, square integrable and tempered representations of $O(V_r)$ by considering their restrictions to $SO(V_r)$. Usual characterizations in terms of Jacquet modules (cf. [Ca], [Si]) hold. (See also [MVW], pages 61-62.) For our purpose the next theorem is sufficient.

**Theorem 1.1.** Let $\rho \otimes \tau \in \text{Irr}(GL(j,F) \times O(V_{r-j}))$ be a supercuspidal representation. Then we have

(i) $\text{R}_p \text{Ind}_P^{O(V_r)}((\rho \otimes \tau)) = \rho \otimes \tau + \tilde{\rho} \otimes \tau$, in the corresponding Grothendieck group. Hence, the length of $\text{Ind}_P^{O(V_r)}(\rho \otimes \tau)$ is at most two.

(ii) If $\text{Ind}_P^{O(V_r)}(\rho \otimes \tau)$ (\rho unitary) reduces, then it is the direct sum of two irreducible non-equivalent tempered representations. If it is reducible, for nonunitary $\rho$, then it has the unique irreducible subrepresentation (with $\text{R}_P$-localization $\rho \otimes \tau$) and the unique irreducible quotient (with $\text{R}_P$-localization $\tilde{\rho} \otimes \tau$). Moreover, we can decompose $\rho = |det|^{e(\rho)}\rho^u$, where $e(\rho) \in \mathbb{R}$, and $\rho^u$ is unitary. If $e(\rho) > 0$, the unique irreducible subrepresentation is square integrable, and the other constituent is not tempered.

(iii) Let $\tau_0$ be any irreducible subrepresentation of $\tau|_{SO(V_{r-j})}$. Then $\text{Ind}_{P_0}^{SO(V_r)}(\rho \otimes \tau_0)$ reduces if and only if $\text{Ind}_{P_j}^{O(V_r)}(\rho \otimes \tau)$ reduces, unless one of the following holds

1. $V_0 = 0$, $r > 1$, $r = j$ is odd, and $\rho \cong \tilde{\rho}$
2. $j < r$, $j$ is odd, $\rho \cong \tilde{\rho}$, and $\tau_0 \not\cong \tau_0$

In both cases, $\text{Ind}_{P_0}^{SO(V_r)}(\rho \otimes \tau_0)$ is irreducible and $\text{Ind}_{P_r}^{O(V_r)}(\rho \otimes \tau)$ is the direct sum of two irreducible non-equivalent tempered representations.

We will also need the results from Remark 1.1.

**Remark 1.1.**

(i) Note that $N_{SO(V_r)}(M^0_j)$ has two elements, unless $V_0 = 0$, $r > 1$ is odd, and $r = j$ (cf. [Go]). If $w_0$ is the nontrivial element of that group, then

$w_0(\rho \otimes \tau_0) = \tilde{\rho} \otimes \tau_0^e$ (j is odd) and $w_0(\rho \otimes \tau_0) = \tilde{\rho} \otimes \tau_0$ (j is even).

If $V_0 = 0$, $r > 1$ is odd, and $j = r$, then Goldberg (cf. [Go]) has shown $P_r^0$ and $P_r^{1-}$ are associated in $SO(V_r)$.

(ii) For each $\pi \in \text{Irr}(O(V_r))$, $\pi \cong \tilde{\pi}$. (cf. [MVW], Chapitre III, Théorème II.1).

(iii) The only group, among $O(V_r)$, $r \geq 0$, which does not have supercuspidal representations, is $O(V_1)$, but only when $V_0 = 0$.

**Proof of Theorem 1.1.** The theorem, except the part of (iii), related to the case $V_0 = 0$, $r > 1$, $r = j$ is odd, is an easy consequence of Lemma 1.2 and the representation theory of $SO(V_r)$, and is left to reader. Let us prove that part of (iii). First, $\pi_1 = \text{Ind}_{P_r}^{SO(V_r)}(\rho \otimes 1)$ is irreducible because a necessary condition for reducibility is $N_{SO(V_r)}(M^0_r) \neq \{1\}$ ([Ca], Theorem 7.1.4) does not hold. Also, $\pi_1$ is equivalent to $\pi_2 = \text{Ind}_{P_0}^{SO(V_r)}(\tilde{\rho} \otimes 1)$. We have $\pi^*_1 = \text{Ind}_{P_r}^{SO(V_r)}(\rho \otimes 1)$, and

$\text{Ind}_{P_r}^{O(V_r)}(\rho \otimes 1)|_{SO(V_r)} \cong \pi_1 \oplus \pi^*_1$. 
Again, by ([Ca], Theorem 7.1.4), $\pi'_1 \cong \pi'_2$ ($\cong \pi_1$) if and only if $\rho \cong \tilde{\rho}$. Now, Lemma 1.2 implies the last part of (iii). \qed

We will finish this section by briefly discussing symplectic groups $Sp(n) = Sp(n, F)$. (For more details see [Ku], page 235, [MVW] and [T].) By the analogous geometric description, the groups $Sp(n)$ ($n$ is the semisimple rank) have proper maximal parabolic subgroups parametrized by numbers $i$, $1 \leq i \leq n$. Write $Q_i = M'_i N'_i$ for the corresponding parabolic subgroup, where $M'_i = GL(i, F) \times Sp(n-i)$. (i) and (ii) from Theorem 1.1 have the similar form for $Sp(n)$ (cf. [T]). Finally, set $Sp(0) = \{1\}.$

2. CORRESPONDENCE

In this section we prove our main results. To explain the results, we need to introduce more notation. Put

$$m_r = \dim_F (V_r)/2.$$ 

The pair $(Sp(n), O(V_r))$ is a dual pair in $Sp(n \cdot (2m_r), F)$ (cf. [MVW], [Ku1]). We write

$$\omega_{n,r} = \omega_{n,r}^\psi,$$

for the oscillator representation associated to that pair and a fixed additive character $\psi$ of $F$. (Here $\omega_{0,r}$ is the trivial representation of $O(V_r)$, and if $V_0 = 0$, then $\omega_{0,0}$ is the trivial representation of $Sp(n)$.)

For each $\sigma \in \text{Irr}(Sp(n))$ and $r \geq 0$, write $\Theta(\sigma, r)$, for a smooth representation of $O(V_r)$, defined as the $\sigma$-isotypic part of $\omega_{n,r}$ (cf. [MVW], Chapitre II, Lemme III.4). If $\tau \in \text{Irr}(O(V_r))$, we will write $\Theta(\tau, n)$, for the analogously defined smooth representation of $Sp(n)$, $n \geq 0$.

It is known (cf. [MVW]) that if $j$ is large enough, then $\Theta(\sigma, j) \neq 0$. We write $r$, for the smallest $j$, such that $\Theta(\sigma, j) \neq 0$. (It depends on $\sigma$ and the tower $V_r$, $r \geq 0$.) We call $r$ the first occurrence of $\sigma$ in the tower $V_j$, $j \geq 0$. We have, $\Theta(\sigma, j) \neq 0$, for $j \geq r$ ([MVW], Chapitre III). If $\sigma$ is supercuspidal, then $\Theta(\sigma, j)$ is irreducible, for $j \geq r$, and supercuspidal only for $j = r$. The analogous discussion is also valid for $\tau$ and the symplectic tower.

In what follows we shall assume that $\sigma$ and $\tau$ are supercuspidal irreducible representations of $Sp(n)$ and $O(V_r)$, respectively, such that $\Theta(\sigma, r) = \tau$. (Hence $\Theta(\tau, n) = \sigma$.)

**Theorem 2.1.** Let $\rho \in \text{Irr}(GL(j, F))$ be a supercuspidal representation; if $j = 1$, then $\rho \not\in \{\chi|^{\pm(m_r-n)}, \chi|^{\pm(m_r-n-1)}\}$. Then, $\text{Ind}^{Sp(n+j)}_{Q_j}(\rho \otimes \sigma)$ reduces if and only if $\text{Ind}^{O(V_{r+j})}_{P_j}(\chi \rho \otimes \tau)$ reduces. More precisely, we have

(i) Assume that $\rho$ is not unitary and $\text{Ind}^{Sp(n+j)}_{Q_j}(\rho \otimes \sigma)$ reduces. Write $\pi_1$ and $\pi_2$ for its irreducible subrepresentation and irreducible quotient, respectively. Then $\Theta(\pi_i, r+j)$ is not zero, $i = 1, 2$. Furthermore, $\Theta(\pi_1, r+j)$ and $\Theta(\pi_2, r+j)$ are the unique irreducible subrepresentation and quotient of $\text{Ind}^{O(V_{r+j})}_{P_j}(\chi \rho \otimes \tau)$, respectively.

(ii) Assume that $\rho$ is unitary and $\text{Ind}^{Sp(n+j)}_{Q_j}(\rho \otimes \sigma)$ reduces. Write $\pi_1$ and $\pi_2$ for its non-equivalent irreducible subrepresentations. Then $\Theta(\pi_i, r+j)$ is not zero, $i = 1, 2$.\n

and

\[ \text{Ind}_{P_j}^{O(V_{r+j})}(\chi \rho \otimes \tau) \cong \Theta(\pi_1, r+j) \oplus \Theta(\pi_2, r+j). \]

(iii) If \( \text{Ind}_{Q_j}^{\text{Sp}(n+j)}(\rho \otimes \sigma) \) is irreducible, then the lift is given by

\[ \Theta(\text{Ind}_{Q_j}^{\text{Sp}(n+j)}(\rho \otimes \sigma), r+j) = \text{Ind}_{P_j}^{O(V_{r+j})}(\chi \tilde{\rho} \otimes \tau), \] and is also irreducible.

Let us state the following interesting corollary of Theorem 2.1.

**Corollary 2.1.** Assume that \( V_0 = 0 \), and put \( O(2j) = O(V_j) \). Let \( \rho \in \text{Irr}(GL(j, F)) \), \( j > 1 \), be a supercuspidal representation. Then \( \text{Ind}_{Q_j}^{\text{Sp}(n+j)}(\rho) \) reduces if and only if \( \text{Ind}_{P_j}^{O(2j)}(\rho) \) reduces.

Corollary 2.1 is just the reformulation of a part of the results obtained by Shahidi in [Sh], using his theory of \( L \)-functions. (Actually, in [Sh] is considered \( SO(2j) \) instead of \( O(2j) \) (cf. Theorem 1.1).)

Now, we will explain the assumption on \( \rho \) in Theorem 2.1. By Théorème principal in ([MVW], page 69),

\[ \Theta(\sigma, r+1) \subset \text{Ind}_{P_1}^{O(V_{r+1})}(|^{n-m_r} \otimes \tau), \quad \text{and} \quad R_{P_1}(\Theta(\sigma, r+1)) = |^{n-m_r} \otimes \tau. \]

Since the reducibility point \( s_0 = n - m_r \) of \( \text{Ind}_{P_1}^{O(V_{r+1})}(|^{s} \otimes \tau), s \in \mathbb{R}, \) is unique up to a sign ([Si1], Lemma 1.2), \( \text{Ind}_{P_1}^{O(V_{r+1})}(|^{n-m_r+1} \otimes \tau) \cong \text{Ind}_{P_1}^{O(V_{r+1})}(|^{-(n-m_r+1)} \otimes \tau) \) is irreducible. Also,

\[ \Theta(\tau, n+1) \subset \text{Ind}_{Q_1}^{\text{Sp}(n+1)}(\chi|^{m_r-n-1} \otimes \sigma), \quad \text{and} \quad R_{Q_1}(\Theta(\tau, n+1)) = \chi|^{m_r-n-1} \otimes \sigma. \]

Again, \( \text{Ind}_{Q_1}^{\text{Sp}(n+1)}(\chi|^{m_r-n} \otimes \sigma) \cong \text{Ind}_{Q_1}^{\text{Sp}(n+1)}(\chi|^{-(m_r-n)} \otimes \sigma) \) is irreducible. The next theorem is a complement to Theorem 2.1.

**Theorem 2.2.**

(i) The lift \( \Theta(\text{Ind}_{Q_1}^{\text{Sp}(n+1)}(\chi|^{m_r-n} \otimes \sigma), r+1) \) has the unique proper maximal submodule. The corresponding quotient is isomorphic to \( \Theta(\sigma, r+1) \). Furthermore,

\[ \Theta(\Theta(\sigma, r+1), n+1) = \text{Ind}_{Q_1}^{\text{Sp}(n+1)}(\chi|^{m_r-n} \otimes \sigma). \]

Finally, if \( \pi \) is the other irreducible constituent of \( \text{Ind}_{P_1}^{O(V_{r+1})}(|^{n-m_r} \otimes \tau) \), then \( \Theta(\pi, n+1) = 0 \).

(ii) The lift \( \Theta(\text{Ind}_{P_1}^{O(V_{r+1})}(|^{n-m_r+1} \otimes \tau), n+1) \) has the unique proper maximal submodule. The corresponding quotient is isomorphic to \( \Theta(\tau, n+1) \). Furthermore,

\[ \Theta(\Theta(\tau, n+1), r+1) = \text{Ind}_{P_1}^{O(V_{r+1})}(|^{n-m_r+1} \otimes \tau). \]

Finally, if \( \pi \) is the other irreducible constituent of \( \text{Ind}_{Q_2}^{\text{Sp}(n+1)}(|^{m_r-n-1} \otimes \sigma) \), then \( \Theta(\pi, r+1) = 0 \).

Now, we are going to prove Theorems 2.1 and 2.2. Their proofs depend on the next proposition.
Proposition 2.1.

(i) If $\rho \neq \chi |_{m_r-n-1}$, then $\Theta(\chi\rho \otimes \tau, R_P(\omega_{n+j}, r+j)) \cong \text{Ind}_{Q_j}^{\text{Sp}(n+j)}(\tilde{\rho} \otimes \sigma)$.

(ii) If $\rho \neq \chi |_{m_r-n}$, then $\Theta(\rho \otimes \sigma, R_Q(\omega_{n+j}, r+j)) \cong \text{Ind}_{P_j}^{O(V_{r+j})}(\chi\tilde{\rho} \otimes \tau)$.

We will postpone the proof of Proposition 2.1, and prove Theorems 2.1 and 2.2 first.

Proof of Theorem 2.1. We will prove the theorem in several steps. First, to simplify formulae, we write

$$\Pi = \text{Ind}_{Q_j}^{\text{Sp}(n+j)}(\rho \otimes \sigma), \quad \Pi' = \text{Ind}_{Q_j}^{\text{Sp}(n+j)}(\tilde{\rho} \otimes \sigma), \quad \Upsilon = \text{Ind}_{P_j}^{O(V_{r+j})}(\chi\tilde{\rho} \otimes \tau).$$

Since $\tilde{\tau} \cong \tau$ and $\chi^2 = 1$, we have $\tilde{\Upsilon} \cong \text{Ind}_{P_j}^{O(V_{r+j})}(\chi\tilde{\rho} \otimes \tau)$.

Step 1. Assume that $\pi$ is an irreducible quotient of $\Pi$. Then, by Frobenius reciprocity,

$$\text{Hom}_{\text{Sp}(n+j) \times O(V_{r+j})}(\omega_{n+j}, r+j, \pi \otimes \tilde{\Upsilon}) \cong \text{Hom}_{\text{Sp}(n+j) \times O(V_{r+j})}(\omega_{n+j}, r+j, \pi \otimes \chi\tilde{\rho} \otimes \tau).$$

Hence, by Proposition 2.1 (i),

$$\text{Hom}_{\text{Sp}(n+j) \times O(V_{r+j})}(\omega_{n+j}, r+j, \pi \otimes \tilde{\Upsilon}) \cong \text{Hom}_{\text{Sp}(n+j)}(\Pi, \pi).$$

The last intertwining space is one dimensional. Hence $\Theta(\pi, r+j) \neq 0$. Finally, each irreducible subquotient of $\Pi$ is a quotient of $\Pi$ or $\Pi'$. So, its lift is non-zero.

Step 2. Assume that $\Pi$ is reducible and $\rho$ is not unitary. Write $\pi_1$ and $\pi_2$ for its unique irreducible subrepresentation and unique irreducible quotient, respectively. We have $R_{Q_j}(\pi_1) = \rho \otimes \sigma$ and $R_{Q_j}(\pi_2) = \tilde{\rho} \otimes \sigma$. By Step 1, $\Theta(\pi_i, r+j) \neq 0$, $i = 1, 2$. Since $\pi_1 \otimes \Theta(\pi_1, r+j) = \omega_{n+j}, r+j$, $\rho \otimes \sigma \otimes \Theta(\pi_1, r+j)$ is a quotient of $R_{Q_j}(\omega_{n+j}, r+j)$. Hence $\Theta(\pi_1, r+j)$ is a quotient of $\tilde{\Upsilon}$. Let us prove that it is not whole $\tilde{\Upsilon}$. Otherwise, $R_{P_j}(\omega_{n+j}, r+j)$ has a quotient $\pi_1 \otimes \chi\tilde{\rho} \otimes \tau$. So, by Proposition 2.1, $\pi_1$ is a quotient of $\Pi$. This is a contradiction. Hence, $\Theta(\pi_1, r+j)$ is the unique irreducible quotient of $\tilde{\Upsilon}$. Similarly, we conclude that $\Theta(\pi_2, r+j)$ is the unique irreducible quotient of $\Upsilon$. Now, by Remark 1.1 (ii), Theorem 2.1 (i) follows.

Step 3. Assume that $\Pi$ is irreducible and $\rho \not\cong \tilde{\rho}$. Now, by Frobenius reciprocity,

$$\text{Hom}_{\text{Sp}(n+j)}(\omega_{n+j}, r+j, \Pi) \cong \text{Hom}_{\text{GL}(j,F) \times \text{Sp}(n)}(R_{Q_j}(\omega_{n+j}, r+j), \rho \otimes \sigma).$$

The usual map that gives the isomorphism above is, in fact, the isomorphism of the corresponding (not necessarily smooth) $O(V_{r+j})$–modules. Now, by the last part of Lemma 1.1, and Proposition 2.1, we get

$$\Theta(\Pi, r+j) \cong \Theta(\rho \otimes \sigma, R_{Q_j}(\omega_{n+j}, r+j)) \cong \text{Ind}_{P_j}^{O(V_{r+j})}(\chi\tilde{\rho} \otimes \tau) = \tilde{\Upsilon}.$$

Similarly, we can prove $\Theta(\Pi, r+j) = \Upsilon$. Now, $\Upsilon \cong \tilde{\Upsilon}$. Hence, if $\rho$ is not unitary, applying Theorem 1.1 (ii), it is not difficult to see that $\Upsilon$ is irreducible. If $\rho$ is unitary, then $\Upsilon$
is also irreducible. More precisely, \( \text{Ind}_{Q_j}^{SO(V_{r+j})} (\rho \otimes \tau_0) \) is irreducible because a necessary condition for reducibility \( w_0 (\rho \otimes \tau_0) = \rho \otimes \tau_0 \) does not hold (cf. [Ca] and Remark 1.1 (i)). Since \( \tilde{\rho} \not\cong \rho \), we can apply Theorem 1.1 (iii) to see that \( \Upsilon \) is irreducible.

**Step 4.** Assume \( \Pi \) is irreducible and \( \rho \cong \tilde{\rho} \). We shall show that \( \Upsilon \) is also irreducible. Take \( s \in \mathbb{R} \), and set

\[
\Pi_s = \text{Ind}_{Q_j}^{Sp(n+j)} (|s| \rho \otimes \sigma).
\]

The family \( \Pi_s \), \( s \in \mathbb{R} \), can be considered as a continuous family of Hermitian representations. By usual complementary series’ argument, since \( \Pi_0 \) is irreducible, there exists \( s_0 \neq 0 \), such that \( \Pi_{s_0} \) is reducible. By Step 2, \( \text{Ind}_{P_j}^{O(V_{r+j})} (\chi |s_0| \rho \otimes \tau) \) is also reducible. Considering the restriction to \( SO(V_{r+j}) \) (cf. Theorem 1.1 (iii)), we see that \( \text{Ind}_{P_j}^{SO(V_{r+j})} (\chi |s_0| \rho \otimes \tau_0) \) is reducible. As before, by ([Si1], Lemma 1.2), \( \text{Ind}_{P_j}^{SO(V_{r+j})} (\chi \rho \otimes \tau_0) \) is irreducible. Clearly, we are not in the exceptional case of Theorem 1.1 (iii). So, \( \Upsilon = \text{Ind}_{P_j}^{O(V_{r+j})} (\chi \rho \otimes \tau) \) is irreducible. Now, we can continue as in Step 3 to see \( \Theta(\Pi, r + j) = \tilde{\Upsilon} \). Steps 3 and 4 prove Theorem 2.1 (iii).

**Step 5.** Assume \( \Pi \) is reducible and \( \rho \) is unitary. (Then \( \rho \cong \tilde{\rho} \).) Let us write \( \Pi \cong \pi_1 \oplus \pi_2 \), where \( \pi_1, \pi_2 \) are tempered mutually inequivalent irreducible representations. It follows that \( R_{Q_j} (\pi_i) = \rho \otimes \sigma, \ i = 1, 2, \) and

\[
(2.5) \quad R_{Q_j} (\Pi) \cong R_{Q_j} (\pi_1) \oplus R_{Q_j} (\pi_2).
\]

Let us prove that \( \tilde{\Upsilon} \) is also reducible. If not, then, as in Step 3, we see \( \Theta(\Upsilon, n + j) = \Pi \). Now, (2.5) implies that

\[
\rho \otimes \sigma \otimes \tilde{\Upsilon} \oplus \rho \otimes \sigma \otimes \tilde{\Upsilon}
\]

is a quotient of \( R_{Q_j} (\omega_{n+j,r+j}) \). This is a contradiction (cf. Proposition 2.1 (ii)). Hence, by Theorem 1.1, \( \tilde{\Upsilon} \) is a direct sum of two mutually non-equivalent tempered representations.

Next, taking \( \pi = \pi_i, \ i = 1, 2, \) in Step 1, we see

\[
(2.6) \quad \dim_{\mathbb{C}} \text{Hom}_{G} (\pi \otimes \chi \tilde{\rho} \otimes \tau) = 1.
\]

Since, by Step 1, \( \Theta(\pi_i, r + j) \) is a quotient of \( \tilde{\Upsilon} \) and \( R_{P_j} (\tilde{\Upsilon}) \) is semisimple, (2.6) implies that \( \Theta(\pi_i, r + j) \) is irreducible. Similarly, each irreducible subrepresentation \( \pi \) of \( \tilde{\Upsilon} \), satisfies \( \Theta(\pi, n + j) \in \{ \pi_1, \pi_2 \} \). So, we must have

\[
\tilde{\Upsilon} \cong \Theta(\pi_1, r + j) \oplus \Theta(\pi_2, r + j).
\]

Now, Theorem 2.1 (ii) follows from Remark 1.1 (ii). This completes the proof of Theorem 2.1. \( \square \)

**Proof of Theorem 2.2.** We will prove (i). The proof of (ii) is analogous. The second part of (i) (see (2.3)) follows from (2.1) and Proposition 2.1 (i). Now, to prove Theorem 2.2 (i) it is enough to prove the first part. To simplify formulae, put \( \Pi = \text{Ind}_{Q_i}^{Sp(n+1)} (\chi |m_r-n \otimes \sigma). \)
First, if $m_r - n \neq 0$, then, as in Step 4, we see $\Theta(\Pi, r + 1) \cong \text{Ind}_{P_1}^{O(V_{r+1})}(|m_r-n| \otimes \tau)$. This completes the proof in that case.

Again, the tempered case $m_r - n = 0$ is more difficult. First,

$$(2.7) \quad \text{Hom}_{Sp(n) \times O(V_{r+1})}(\omega_{n+1,r+1}, \Pi \otimes \text{Ind}_{P_1}^{O(V_{r+1})}(1 \otimes \tau)) \cong$$

and

$$(2.8) \quad \text{Hom}_{O(V_{r+1})}(\Theta(\Pi, r+1), \text{Ind}_{P_1}^{O(V_{r+1})}(1 \otimes \tau)).$$

Then, Proposition 2.1 (i) implies that the space in (2.8) is one dimensional. Take a non-trivial map from the space in (2.7). Then, the first part of the proof shows that its image is isomorphic to $\Theta(\sigma, r)$, further, let $\varphi$ be the corresponding map in (2.8). We claim that $V = \ker(\varphi)$ is the unique proper maximal submodule of $\Theta(\Pi, r + 1)$. If not, then there is a submodule $V'$, such that $\Theta(\Pi, r + 1) = V + V'$. Hence, $V/V \cap V'$ is a quotient of $\Theta(\Pi, r + 1)$. The filtration of $R_{Q_1}(\omega_{n+1,r+1})$ (cf. Section 3) and Lemma 3.2 imply that $R_{P_1}(V/V \cap V')$ has a quotient $1 \otimes \tau$. This gives a map in (2.8), which is not, up to a scalar, equal to $\varphi$. This is a contradiction. This completes the proof of Theorem 2.2. □

3. Proof of Proposition 2.1

First, we will recall filtrations of certain Jacquet modules of oscillator representations (cf. [Ku], [Ku1], or [MVW], Chapitre III, IV.5).

(a) $R_{P_j}(\omega_{n+j,r+j})$ has filtration by $I_{j,k}$, $0 \leq k \leq j$, where

$$I_{j0} = |\det|^{-m_r+n+i\frac{1}{2}} \otimes \omega_{n,j,r} \quad \text{(quotient)}$$

$$I_{jj} = \text{Ind}_{Q_j \times GL(j,F) \times O(V_j)}^{Sp(n+j) \times GL(j,F) \times O(V_j)}(\Sigma_j \otimes \omega_{n,r}) \quad \text{(subrepresentation)},$$

$$I_{jk} = \text{Ind}_{Q_k \times P'_{jk} \times O(V_j)}^{Sp(n+j) \times GL(j,F) \times O(V_j)}(\alpha_{jk} \otimes \Sigma_k \otimes \omega_{n-j-k,r}), \quad 0 < k < j.$$ 

Here $P'_{j,k}$ is the standard parabolic subgroups of $GL(j,F)$ which corresponds to the partition $(j-k,k)$, $\alpha_{jk} = |\det|^{-m_r+n+i\frac{1}{2}}$ is a character of $GL(j-k,F)$, and $\Sigma_k$ is the twist of the standard representation of $GL(k,F) \times GL(k,F)$ on smooth complex valued functions $C^\infty_c(GL(k,F))$: 

$$\Sigma_k(g_1, g_2) f(h) = |\det g_1|^{-m_r+i\frac{1}{2}} \chi(\det g_2) |\det g_2|^{m_r+i\frac{1}{2}} f(g_1^{-1}hg_2).$$

(Here the first $GL(k,F)$ (resp. the second) is a part of the Levi factor of $P'_{j,k}$ (resp. Levi factor of $Q_k$).)

(b) $R_{Q_j}(\omega_{n+j,r+j})$ has filtration by $J_{j,k}$, $0 \leq k \leq j$, where

$$J_{j0} = \chi|\det|^{-m_r-n+i\frac{1}{2}} \otimes \omega_{n,r+j} \quad \text{(quotient)}$$
\[ J_{jj} = \text{Ind}_{\text{Sp}(n) \times GL(j,F) \times O(V_{r+j})}^{\text{Sp}(n) \times GL(j,F) \times P_j} (\Sigma'_j \otimes \omega_{n,r}) \] (subrepresentation),

\[ J_{jk} = \text{Ind}_{\text{Sp}(n) \times Q'_j \times P_k}^{\text{Sp}(n) \times GL(j,F) \times O(V_{r+j})} (\beta_j \otimes \Sigma'_k \otimes \omega_{n,r+j-k}), \quad 0 < k < j. \]

Here \( Q'_{j,k} \) is the standard parabolic subgroup of \( GL(j,F) \) which corresponds to the partition \((j-k,k), \beta_{jk} = \chi | \text{det}^{m_r-n+i-k-1/2} \) is a character of \( GL(j-k,F) \), and \( \Sigma_k \) is the twist of the standard representation of \( GL(k,F) \times GL(k,F) \) on smooth complex valued functions \( C^\infty(GL(k,F)) \):

\[ \Sigma_k(g_1,g_2)f(h) = | \text{det} g_1 |^{m_r+j-k+1} \chi(| \text{det} g_1|) | \text{det} g_2 |^{-m_r-j+k+1} f(g_1 h g_2). \]

(Here the first \( GL(k,F) \) (resp. the second) is a part of the Levi factor of \( Q'_{j,k} \) (resp. Levi factor of \( P_k \)).)

**Remark 3.1.** If \( \pi \in \text{Irr}(GL(k,F)) \), then \( \Theta(\pi, \Sigma_k) \cong \chi \tilde{\pi} \). The same applies for \( \Sigma'_k \). Here we may consider \( \pi \) as a representation of \( GL(k,F) \times 1 \) or \( 1 \times GL(k,F) \).

We will prove only Proposition 2.1 (ii). The proof of (i) is analogous. To simplify notation, we will write

\[ M'_j = GL(j,F) \times \text{Sp}(n) \quad \text{and} \quad \rho' = \rho \otimes \sigma. \]

For \( s \in \mathbb{C} \), we put \( \rho'_s = | s \rho | \otimes \sigma \). Now, the filtration of \( R_{Q_j}(\omega_{n+j,r+j}) \) given in (b) immediately implies

**Lemma 3.1.** Assume that \( j > 1 \), and \( s \in \mathbb{C} \). Then

\[ \text{Hom}_{M'_j}(R_{Q_j}(\omega_{n+j,r+j}) / J_{jj}, \rho'_s) = 0. \]

Now, let us finish the proof of Proposition 2.1 (ii), assuming the following lemma.

**Lemma 3.2.** For each supercuspidal representation \( \rho' = \rho \otimes \sigma \in \text{Irr}(M'_j) \), \( \Theta(\rho', J_{jj}) \cong \text{Ind}_{P_j}^{O(V_{r+j})}(\chi \tilde{\rho} \otimes \tau) \).

First, recall that if \( V \in \mathcal{A}(M'_j) \), then we can decompose in the direct sum of smooth modules ([Be1], Chapter II, Proposition 26)

\[ V \cong V(\rho') \oplus V_{\rho'}, \]

such that each irreducible subquotient of \( V(\rho') \) is of the form \( \rho'_s \), for some \( s \in \mathbb{C} \), and \( V_{\rho'} \) does not have an irreducible subquotient of the form \( \rho'_s \). If \( V(\rho') \) is not zero, it has an irreducible quotient.

If \( j > 1 \), then, Lemma 3.1 implies \( R_{Q_j}(\omega_{n+j,r+j}) / J_{jj}(\rho') = 0. \) Now, it is not difficult to see

\[ R_{Q_j}(\omega_{n+j,r+j})(\rho') = J_{jj}(\rho'), \]
considering \( J_{jj} \subset \mathcal{R}_{Q_j}(\omega_{n+j,r+j}) \). Hence, the natural map

\[
\text{Hom}_{M'_j}(\mathcal{R}_{Q_j}(\omega_{n+j,r+j}), \rho') \to \text{Hom}_{M'_j}(J_{jj}, \rho')
\]

is an isomorphism of vector spaces. But, the map is also \( O(V_{r+j}) \)-equivariant. Hence, we have (cf. Lemma 1.1)

\[
\Theta(\mathcal{R}_{Q_j}(\omega_{n+j,r+j}), \rho') \cong \text{Hom}_{M'_j}(\mathcal{R}_{Q_j}(\omega_{n+j,r+j}), \rho') \cong \text{Hom}_{M'_j}(J_{jj}, \rho') \cong \Theta(J_{jj}, \rho').
\]

Now, Lemma 3.2 completes the proof of Proposition 2.1 (ii) in that case.

If \( j = 1 \), then \( \rho \) is a character of \( F' \times \mathbb{Z}/2\mathbb{Z}, \) which is, by the assumption of Proposition 2.1 (ii), different from the character that appears in \( J_{10} \) (cf. (b)). Now, \( \mathcal{R}_{Q_j}(\omega_{n+1,r+1}) \) has an obvious quotient

\[
J_{jj} = \rho' \otimes \Theta(\rho', J_{11}) \oplus J_{10}.
\]

Hence, one can see that the natural map (3.1) is an isomorphism. Finally, the proof of Proposition 2.1 (ii) can be completed as before.

It remains to prove Lemma 3.2. To achieve that, we need the following simple extension of a result of Bernstein [Be]:

**Lemma 3.3.** Assume that an \( l \)-group \( G' \) is the semidirect product \( G \rtimes \mathbb{Z}/2\mathbb{Z}, \) where \( G \) is a connected reductive \( F \)-group. Let \( P = MN \) be a parabolic subgroup of \( G, \) and let \( \overline{P} = M\overline{N} \) be the opposite parabolic subgroup of \( P. \) Assume that \( \mathbb{Z}/2\mathbb{Z} \) normalizes \( M, N \) and \( \overline{N}. \) Put \( M' = M \rtimes \mathbb{Z}/2\mathbb{Z}, P = M'N, \) and \( \overline{P} = M'\overline{N}. \) If \( \pi \in \mathcal{A}(M') \) and \( \Pi \in \mathcal{A}(G'), \) then we have an isomorphism \( \phi \mapsto \phi_0 \)

\[
\text{Hom}_{G'}(\text{Ind}_{\overline{P}}^G(\pi), \Pi) \cong \text{Hom}_{M'}(\pi, \mathcal{R}_{\overline{P}}(\Pi)),
\]

where \( \phi_0 \) is given by the composition of the natural inclusion (through a part of filtration that corresponds to a open orbit \( P'\overline{P}' \) in \( P' \setminus G' \))

\[
\pi \hookrightarrow \mathcal{R}_{\overline{P}}(\text{Ind}_{\overline{P}}^G(\pi)),
\]

and the natural map \( \phi : \mathcal{R}_{\overline{P}}(\text{Ind}_{\overline{P}}^G(\pi)) \to \mathcal{R}_{\overline{P}}(\Pi). \)

**Proof.** If \( G' \) is connected, then Bernstein has shown that the map \( \phi \mapsto \phi_0 \) is an isomorphism. Now, Lemma 3.3 follows, considering the restriction of all representations in question to \( G. \) \( \square \)

**Proof of Lemma 3.2.** Set \( \Pi = \Theta(\rho', J_{jj}) \) and \( \overline{P}_j = M_j\overline{N}_j. \) Clearly, \( \text{Ind}_{\overline{P}_j}^{O(V_{r+j})}(\chi_{\rho} \otimes \tau) \) is a quotient of \( \Pi. \) To prove the lemma, it is enough to see that \( \Pi \) is a quotient of that induced representation.

Let \( \varphi \) be the natural epimorphism of \( M'_j \times O(V_{r+j}) \)-modules \( J_{jj} \rightarrow \rho' \otimes \Pi. \) Then, as in Lemma 3.3, we can define a morphism

\[
\varphi_0 : \Sigma_j \otimes \omega_{n,r} \rightarrow \rho' \otimes \mathcal{R}_{\overline{P}_j}(\Pi).
\]
Remark 3.1 implies that the image of $\varphi_0$ is isomorphic to

\[(3.2) \quad \rho' \otimes \chi\tilde{\rho} \otimes \tau.\]

So, we can factor $\varphi_0 = \varphi'' \cdot \varphi'$, where $\varphi'$ is the natural projection from $\Sigma'_j \otimes \omega_{n,r}$ to the module given by (3.2), and $\varphi''$ is an inclusion. Write $\text{Ind}(\varphi')$ for the morphism of the corresponding induced modules. Let $\varphi_1$ be the morphism from Lemma 3.3, such that $(\varphi_1)_0 = \varphi''$. It is not difficult to see $(\varphi_1 \cdot \text{Ind}(\varphi'))_0 = \varphi_0$. Hence, by Lemma 3.3, $\varphi = \varphi_1 \cdot \text{Ind}(\varphi')$. This implies that the image of $\varphi$ is isomorphic to a quotient of $\rho' \otimes \text{Ind}^{O(V_{j+1})}_{P_j}(\chi\tilde{\rho} \otimes \tau)$. This completes the proof of Lemma 3.2. \qed

4. An example

In this section we will assume that the characteristic of $F$ is zero, because we will apply the reducibility results obtained by Shahidi [Sh].

We continue by recalling some results from [Sh]. Assume that $\rho$ is a unitary supercuspidal representation of $GL(j,F)$, $j > 1$. Then, following [Sh], we call $\rho$ a representation of symplectic type if $\text{Ind}^{\text{Sp}(j)}_{Q_j}(|\det|^{1/2}\rho)$ is reducible, and a representation of orthogonal type if $\text{Ind}^{\text{Sp}(j)}_{Q_j}(\rho)$ is reducible. In both cases $\rho \cong \tilde{\rho}$. Also, ([Sh], Lemma 3.6) implies that every selfcontragredient supercuspidal representation is exactly of one of the above types. (For more details see [Sh], and for later interpretation in terms of $K$–types we refer to [MR].) One would like to describe all reducibilities of induced representations induced in terms of reducibility discussed above for split classical groups or their inner forms. The theorem below gives new examples of reducibilities.

**Theorem 4.1.** Assume that $V_0$ is four dimensional anisotropic space. (Then $\chi = 1$.) Let $\rho \in \text{Irr}(GL(j,F))$ be a supercuspidal unitary representation, and let $s \in \mathbb{R}$. Put $I(s) = \text{Ind}^{SO(V_j)}_{P_j}(s \rho \otimes 1_{SO(V_0)})$. Then we have

(i) If $\rho \not\cong \tilde{\rho}$, then $I(s)$ is irreducible, for all $s \in \mathbb{R}$.

(ii) If $\rho$ has orthogonal type (hence $j > 1$), then $I(0)$ is reducible, and $I(s)$ is irreducible, for $s \neq 0$.

(iii) If $\rho$ has symplectic type (hence $j > 1$), then $I(s)$ is irreducible, for $s \neq \pm 1/2$, and $I(\pm 1/2)$ is reducible.

(iv) If $\rho$ is the trivial character of $GL(1,F)$, then $I(s)$ is irreducible, for $s \neq \pm 2$ and $I(\pm 2)$ is reducible. If $\rho$ is a nontrivial quadratic character of $GL(1,F)$, then $I(0)$ is reducible, and $I(s)$ is irreducible, for $s \neq 0$.

**Proof.** Note that $\Theta(1_{O(V_0)},0) = 1_{Sp(0)}$. Now, the theorem is a consequence of Theorems 1.1, 2.1, and 2.2, well-known reducibilities of induced representations in $SL(2) = Sp(1)$, and the fact that we can have at most one point of reducibility point for $s \geq 0$ (cf. [Si1]). \qed
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